About Optimization Techniques in Application to Symbolic-Numeric Optimal Control Seeking Aproach

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Abstract: The optimal control problem for nonlinear dynamic systems is considered. The proposed approach is based on the both partially analytical and partially numerical techniques of the optimal control problem solving. Using the maximum principle the system with the state and co-state variables can be determined and after closing up the initial optimal control problem, it can be reduced to unconstrained extremum problem. The extremum problem is related to seeking for the initial point for the co-state variables that would satisfy the boundaries. To solve the optimization problem, well-known global optimization techniques are suggested and compared. The settings of the algorithms were varied. Also, the new modified hybrid evolutionary strategies algorithm was compared to common techniques and in the current study it was more efficient.

1 INTRODUCTION

The optimal control problem for dynamic systems with one control input and integral functional is considered. Since the problem is old and it originates from the practical needs, there exist many techniques to solve the optimal control problem in different problem definitions and for different systems. But the developing of the modern technologies creates new optimal control problems that cannot be solved via well-known and classical approaches. The main problem is nonlinearity of the system model or the criterion. In general case, there is no universal analytical technique that guarantees the solution of nonlinear differential equation to be found. But using the maximum principle, we can always determine the characteristics of the function that is suspected to be the solution of the optimal control problem.

On the other hand, the numerical approaches are useful and efficient but only for some problems that they were designed for. Any control function approximation technique that is being used to determine the solution for the initial optimal control problem is related with reduction of the problem to extremum seeking on the real vector field. And the problem reduction uses a convolution of different objective functions and penalty functions for all the constraints, and it requires more computational resources and more efficient optimization algorithms. There is no doubt that the direct method based techniques are efficient, but increasing of accuracy of the function approximation leads to increasing of extremum problem dimension.

The indirect method of solving the optimal control problem is related with solving the extremely difficult boundary-value problem, but the found solution gives us the proper control function with the known structure. In the given study the shooting method is based on the modified evolutionary optimization algorithm.

It is important to highlight that there is a sufficient benefit of using the information science techniques of solving the complex optimization problems. The modern methods and algorithms from the fields of informatics, bioinformatics and cybernetics are reliable, flexible and highly efficient techniques. And it is possible to improve them for every distinct optimization problem with unique characteristics via modifying the schemes, operators or hybridizing the algorithms.

Many works on optimal control problem solving for nonlinear dynamic systems are about some specific tasks. Many works are about the approaches to solve optimal control problems for affine nonlinear systems, like the work mentioned before, for example, (Popescu and Dumitrache, 2005) and (Primbs et al., 1999). In the last article the studied

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problem is related to optimal control of nonlinear systems via usage of the Lyapunov functions, but only one boundary in problem definition is considered. Also, in article (Chen et al., 2003) the approach of predictive optimal control for nonlinear systems is considered. There are plenty of numerical techniques application examples, (Rao, 2009). Actually, since the problem is complex and there are many problems with unique features, and there are many different problem definitions for optimal control.

These techniques also require an analytical form of the system state and fit only the considered structures. And in our study, the proposed approach with implementation of some efficient global optimization technique is suggested to be applicable and reliable for solving many optimal control problems, as an effective analogue to shooting-based techniques.

In the study (Bertolazzi et al., 2005) symbolic-numeric indirect approach is considered, which is based on Newton Affine Invariant scheme for solving boundary value problem, which fits the considered systems and is being different technique of solution seeking. Following scheme can find also a local optimum.

The evolutionary strategies algorithm was used to solve the optimal control problem, but as a direct method. In the paper (Cruz and Torres, 2007) the control function was discretized and every part of it was optimized via evolutionary strategies algorithm. That means, that there a as many optimization variables, as many discrete points are approximating the control.

The method of semi-analytical and semi-numerical optimal control problem solving is considered. The first part of the method is based on the Pontryagin's maximum principle (Kirk, 1970), after determination of the Hamiltonian, the system with co-state variables can be used. For the new system that is a transformation of the initial problem it becomes possible to reduce the optimal control problem to extremum seeking on a real vectors' field. For the last problem the new optimization technique is applicable. The Pontryagin's principle allows to close up the system of equations and to determine the structure of the control function in terms of other variables. The dimension of the field is the same as the dimension of the system for the considered problems. There are many suitable algorithms for the proposed problem, based on random search and evolution search, but the every distinct optimal control problem origins its own unique structure of criterion with its own features.

2 ANALYTICAL-NUMERICAL APPROACH TO OPTIMAL CONTROL PROBLEM

Let the system be described with nonlinear differential equation

$$\frac{dx}{dt} = f(x,u,t), \tag{1}$$

where

$f(\cdot): R^n \times R \times R^+ \to R^n$ is a vector function of its arguments;

$x \in R^n$ is a vector of system state;

$u \in R$ is a continuous control function;

n is the system dimension.

We need to find a control function $u(t)$ that brings the system from the initial point $x(0) = x^0$ to the end point $x(T) = x^*$ within finite time T, that delivers the extremum to functional

$$I(x,u) = \int_0^T F(x,u)dt \to extr, \tag{2}$$

The Hamiltonian (Kirk, 1970), is defined by the equation

$$H(x,u,t) = -F(x,u) + p \cdot f(x,u,t), \tag{3}$$

therefore the system with co-state variables p can be determined with equations

$$\frac{dx}{dt} = f(x,u,t), \frac{dp}{dt} = \frac{-dH}{dx}. \tag{4}$$

The given system (4) is completed with system state and conjugated variables starting points $x(0) = x^0$ and $p(0) = p^0$, respectively. It means that the control function $u(t)$ can be determined with some p^0. Then, it is necessary to close up the system with the following condition

$$\frac{dH}{du} = 0. \tag{5}$$

Since the differentiation of the analytical expression is not a common problem, the forming of the system in the current study was not made automatically. Anyway, some mathematic software is able to operate with analytical problems, simplify expressions and differentiate them. The method can be implemented in future.

The structure of the control function is determined by equation (5). After using of the transversability conditions, by changing the starting point of the co-state variables, we change the control function and the solution of the optimal control problem.

Actually, it means that the proper vector of the co-state variables initial point, which provides the condition $x(T) = x^*$ would give us the solution for the whole problem, since the functional (2) and the differential equation (1) are forming the system (4). Initial point is the real vector and it could be searched with some optimization technique.

Since the main problem is reduced to optimization problem on the field R^n, let the $x(t), p(t)\big|_{p(0)=p^0}$ be the solution for the system (4) in case of $p(0) = p^0$. Now let us describe the functional

$$K(p^0) = \left\| x^* - \tilde{x}(T)\big|_{\tilde{p}(0)=p^0} \right\| \to \min_{p^0} . \qquad (6)$$

The proposed criterion is multimodal, complex function of its arguments. It is not known, in general, where any extremum is located. Moreover, if the initial system (1) or functional (2) that forms the Hamiltonian (3) is nonlinear, so there is no analytical solution for the given criterion (6) and it can be evaluated only numerically.

The given criterion (6) is being transformed into fitness function for the evolutionary algorithms

$$fitness(p^0) = \frac{1}{1 + K(p^0)},$$

so the fitness function is a mapping: $R^n \to [0,1]$. The greater fitness is, the better is current solution.

To prove the high complexity of the optimization problem let the system be defined by equation

$$f(x,t,u) = \begin{pmatrix} \ln(x_1) + \cos(x_0) \\ t \cdot \sin(x_1) + u(t) \end{pmatrix},\ T = 1,$$
$$x^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ x(T) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \qquad (7)$$

and the integrand function for the functional of the optimal control problem

$$F(x,u) = u^2 \to \min_u \qquad (8)$$

is considered. Then, it is necessary to define the extended system (4),

$$F_P(x,p,t) = \begin{pmatrix} \ln(x_1) + \cos(x_0) \\ t \cdot \sin(x_1) + \dfrac{p_1}{2} \\ \sin(x_0) \cdot p_0 - t \cdot \cos(x_0) \cdot p_1 \\ \dfrac{-p_0}{x_1} \end{pmatrix},$$

since we closed up the system with condition (5),

$$\frac{dH}{du} = 0 \to u(t) = \frac{p_1}{2}.$$

Now, having the system with co-state variables and the structure of the control function it is possible to form an optimization problem for initial point of co-state variables, so the end point of the system state would be achieved at time T.

As one can see, the nonlinear differential equation consists of logarithm function, trigonometric functions and the system itself is nonstationary.

The mapping (6) for the given problem is shown on the figure 1. The surface was made via evaluating numerically the nonlinear differential equation for extended system, varying the initial point of the co-state variables. As it can be shown on the current surface some extremum problems that are reduced from the optimal control problems have a lot of local maximums that are less than 1 and so do not satisfy two-point problem, and among them there could be closed sets or distinct points, that delivers extremum to criterion (6) and it equals 1.

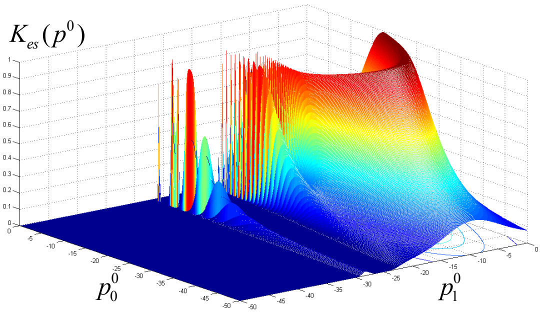

Figure 1: The surface of the criterion (6) for optimal control problem (7)-(8).

Let us describe the next optimal control problem for the plant with inverted pendulum, which movement is determined with system of nonlinear differential equation

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$$f(x,t,u)=\begin{pmatrix} x_1 \\ -x_1+\sin(x_0)+u(t)\cdot\cos(x_0)\end{pmatrix},$$
$$T=5,\ x^0=\begin{pmatrix}1\\1\end{pmatrix},\ x^T=\begin{pmatrix}0\\0\end{pmatrix}. \tag{9}$$

and functional

$$F(x,u)=u^2+x_0\to\min_{u,x_0} \tag{10}$$

to be considered. Then, it is necessary to define the extended system (4),

$$F_p(x,p,t)=\begin{pmatrix} x_0 \\ -x_1+\sin(x_0)+\dfrac{p_1\cdot\cos^2(x_0)}{2} \\ -2\cdot x_0+p_1\cdot\cos(x_0)-\dfrac{p_1\cdot\sin(2\cdot x_0)}{4} \\ -p_1+p_0\end{pmatrix},$$

because we closed up the system with condition (5),

$$\frac{dH}{du}=0\to u(t)=\frac{p_1\cdot\cos(x_0)}{2}.$$

The mapping (6) for current problem is shown on the figure 2.

Every optimal control problem reduced to extremum problem for vector function with unknown characteristics and behavior of the criterion.

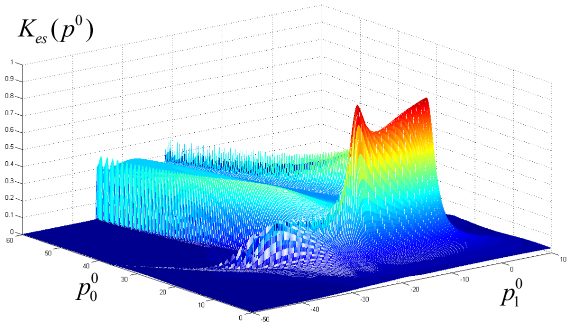

Figure 2: The surface of the criterion (6) for optimal control problem (9)-(10).

As it can be seen on figures, the problem is complex. Moreover, there is no any information about the location of the extremum.

To sum up, seeking for the solution of the reduced problem, in general, is associated with the optimization technique that works on the vector field without any constraints. In other words, the techniques of extremum seeking on R^n should be used. Anyway, it is possible to use the optimization techniques, which works on the compact, but then the special procedure of extending the compact or switching to different one should be implemented.

Since many optimization techniques are suitable for the considered problem and deal with its features, it was suggested to compare these well-known techniques: evolutionary strategies, differential evolution and particle swarm optimization.

To provide the efficiency growth evolutionary strategies algorithm was modified. The basis of the algorithms and modifications proposed is described below.

3 EXTREMUM SEEKING TECHNIQUES

As it has been mentioned before, the features of the problem lead using the efficient global optimization techniques that works on the real vector field. One of the most sufficient characteristic of the algorithms is that the search can proceed in any direction and without any without any constrains.

Thus, the optimal control problem was reduced to extremum problem for the real numbers and the objective function cannot be evaluated analytically. Now the techniques of extremum seeking are to be described.

The main principle of evolutionary strategies (ES) is described in (Schwefel, 1995). It was extended via adding the operations of selection, borrowed from the genetic algorithm. Proposed ES-based optimization algorithm besides the selection uses recombination, mutation and hybridized with local optimization technique. The number of parents which recombine to produce an offspring in the current investigation was set to 2. Then the offspring is mutating and the mutation operand is also modified. The population size is constant for all generations. Let every individual be represented with a tuple

$$Id_i^1=\left\langle op^i, sp^i, fitness(op^i)\right\rangle, i=\overline{1,N_p},$$

where

$fitness(op)=\dfrac{1}{1+K(op)}$ is the fitness function for criterion (6);

$op_j^i\in R,\ j=\overline{1,k}$ is the set of objective parameters;

$sp_j^i\in R^+,\ j=\overline{1,k}$ is the set of method strategic

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parameters;
N_p is the size of population.

There are a lot of different ways to transform the criterion of optimization problem to the evolutionary algorithm's fitness function, but since the criterion (6) is nonnegative the simple transformation can be used.

Now we have to modify the mutation operation for the ES adapting to the given problem. Let $m_p^1 \in [0,1]$ be the mutation probability for every gene and Z_1 be the Bernoulli distributed random value with $P(z_1 = 1) = m_p^1$. Then

$$op_i = op_i + z_1 \cdot N(0, sp_i), \forall op_i \in R, i = \overline{1, card(op)};$$

$$sp_i = |sp_i + z_1 \cdot N(0,1)|, i = \overline{1, card(sp)}.$$

Another modification of the evolutionary strategies algorithm suggested is the CMA-ES, which is described in (Hansen, 2006) and uses the covariance matrix adaptation.

As the next optimization technique the differential evolution (DE) algorithm is suggested, which main principle is described in (Storn, Price, 1997). Let every individual in the current algorithm be represented with a tuple

$$Id_i^2 = \left\langle op^i, fitness(op^i) \right\rangle, i = \overline{1, N_p},$$

where

$op_j^i \in R,\ j = \overline{1,k}$ are the variables to be optimized;

$fitness(op) = \dfrac{1}{1 + K(op)}$ is the fitness function for the criterion (6).

The Algorithm uses two settings, the first is $C_r \in [0,1]$ and the other is $F \in [0,2]$, as it is recommended. On every iteration for every individual three different individuals are to be randomly chosen, but they must differ from the current individual. Then the random number $N_r \in \{1, ..., n\}$ is generated, as the random vector with coordinates $rand_i = U(0,1),\ i = \overline{1,n}$. For every individual the trial vector is generated

$$op_i^{trial} = \begin{cases} op_i, f_{de} = 1, \\ op_i^a + F \cdot (op_i^b - op_i^c), f_{de} = 0, \end{cases}$$

where

$op_i^a, op_i^b, op_i^c, i = \overline{1,n}$ are the coordinates of the randomly chosen individuals;

$f_{de} = \begin{cases} 1, if\ rand_i < C_r\ or\ i = N_r, \\ 0, if\ rand_i > C_r\ and\ i \neq N_r, \end{cases}$ is the special indicator function.

If inequality $fitness(op^{trial}) > fitness(op^i)$ is satisfied, the individual changes to the trial $op^i = op^{trial}$.

The last considered method of extremum seeking is the partial swarm optimization (PSO), which is described in (Kennedy, Eberhart, 2001). Let every individual be represented with a tuple

$$Id_i^3 = \left\langle op^i, v^i, fitness(op^i) \right\rangle, i = \overline{1, N_p},$$

$op_j^i \in R,\ j = \overline{1,k}$ are the variables to be optimized;

$v_j^i \in R,\ j = \overline{1,k}$ are the velocities for each coordinate of an every particle;

$fitness(op) = \dfrac{1}{1 + K(op)}$ is the fitness function for the criterion (6).

After every algorithm's iteration the variable op^{best} is to be refreshed. It is the vector that has the highest fitness. There are also variables that store the best found position that every individual ever had, $\hat{op}$. For every individual, the random values $r_1, r_2 \sim U(0,1)$ are generated and the velocity and new individual's position, for $j = \overline{1,k}$:

$$v_j = \omega \cdot v_j + \varphi_1 \cdot r_1 \cdot (\hat{op}_j - op_j) + \varphi_2 \cdot r_2 \cdot (op_j^{best} - op_j),$$
$$op_j = op_j + v_j.$$

The random coordinate-wise real-valued genes optimization has been implemented for the algorithms performance improvement. The optimization is fulfilled in the following way. For every N_2 randomly chosen real-valued genes for N_1 randomly chosen individuals N_3 steps in random direction with step size h_l are executed.

For the numerical experiments in our study, the parameters of the hybridization for ES-based optimization procedure were set as followed: the recombination probability is 0.8, the mutation probability for every gene was set to $1/|sp|$. Local improvement parameters were set as $N_1 = 2 \cdot |op|$, $N_2 = |op|$ and $N_3 = 0.1$ with $h_l = 0.05$. The proposed algorithm performance has been evaluated on twenty test problems and was found to be promising.

4 OPTIMAL CONTROL PROBLEM AND ALGORITHMS EFFICIENCY INVESTIGATION

Unfortunately, it is impossible for the described initial optimal control problem to determine the most efficient settings of every algorithm, since every different control problem leads to different criterion (6). There is another important thing to be mentioned, that there can be no control functions available for some nonlinear system and boundaries.

Let us consider optimal control problems described earlier to make the efficiency investigation of optimization techniques listed above. To simplify the representation of the results the following names of algorithms are suggested: evolutionary strategies – ES; differential evolution – DE; particle swarm optimization – PSO; for hybrid evolutionary strategies algorithm - ES+LO; ES with covariance matrix adaptation – CMA-ES. For problems that were described above: (7)-(8), (9)-(10), we set the maximum numbers of criterion evaluation to 8000, and tested different setting of the given algorithms. The number of algorithms' iterations and the size of populations were varied too: 1600 and 5, 800 and 10, 400 and 20, 200 and 40, 100 and 80, respectively.

Since the proposed optimization techniques have different natures and their settings were varied regarding to the features of algorithms. For the evolutionary strategies techniques the selection was varied: proportional, rank, tournament; crossover operator was varied: intermediate, weighted intermediate and discrete; mutation: classical and modified, with mutation probability equals to $1/k$. For differential evolution technique the settings were chosen due to recommendation given: $C_r = 0.5$ and $F \in \{a_i = 0.2 \cdot i : i \leq 10, i \in N\}$. For the particle swarm optimization settings were taken from the followig sets: $\omega \in \{0.5 \cdot i : i \leq 4, i \in N\}$, $\varphi_{1,2} \in \{0.4 \cdot i : i \leq 6, i \in N\}$. The initial population was randomly generated, $op_i \in N(0,10)$, $sp_i \in N(0,1)$ and $v_i \in N(0,1)$. For the ES+LO technique, the settings for LO and the number of individuals and populations were chosen as the numbers, which sum is equal to maximum number evaluation. The settings for the CMA-ES algorithm were set as it is recommended in reference, for this technique the only numbers of populations and individuals were varied.

For the problem (9)-(10) sometimes are about to stagnate, the average of the fitness function for every population and the fitness of the best individual are shown on the figure 3.

On the figure 3 there are next measurements: dotted thin line is the best solution found by PSO at the current iteration; dotted thick line is the average fitness function value for the population; the same with dashed lines for differential evolution and dot dashed lines for ES. These lines are the averaging of the presented variables after 20 restarts of algorithms with the same settings. The horizontal axis is the number of iteration for every algorithm; vertical axis is the fitness function value. As one can see, the PSO technique efficiency is more dependent on its settings, because the whole population can fall to the best solution found fast enough without possibility to leave local extremum.

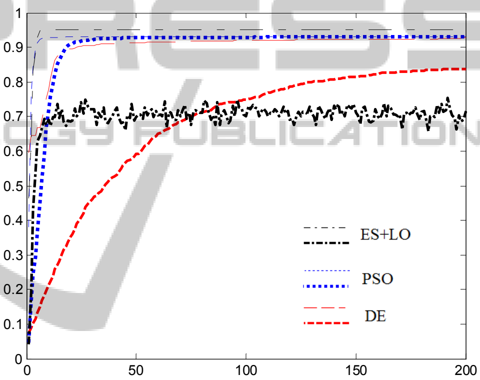

Figure 3: Behaviour of average fitness function value and fitness of the best individual. Thick lines are the average fitness and thin lines are the best fitness.

In table 1 the average values of the fitness function for the found solution are presented. In every column the efficiency with the best settings of every technique are shown.

Table 1: Average values of the fitness function for different techniques with the most efficient settings.

	Algorithm				
Problem	ES	DE	PSO	CMA-ES	ES+LO
(7)-(8)	0.97	0.98	0.95	0.99	0.99
(9)-(10)	0.93	0.95	0.96	0.94	0.97

In table 2 the probability estimation of $1 - fitness(op^*) < 0.05$ is considered, since that is the measurement of how the technique handle with the complexity of objective function for the problem (9)-(10).

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Table 2: The estimation of probability to return solution that is close to the global optimum.

	Algorithm				
	ES	DE	PSO	CMA-ES	ES+LO
Problem	0.3	0.5	0.6	0.35	0.65

In the current investigation, all the algorithms' settings and population, individual numbers were varied. Due to given problems, the hybrid evolutionary strategies algorithm was the most effective in searching the extremum and reliable in case of the problems with objective function that have a surface as it is shown on figure 2. After all the runs, the best settings were estimated as following ones: 20 individuals for 200 populations, tournament selection (10%), discrete crossover, modified mutation (mutation probability 0.75) and $N_1 = N_2 = N_I / 2$, $N_3 = 0.1$, with evaluations number limitation equal to 8000.

For every of the proposed algorithms the most efficient settings were estimated too. The balance between the number of populations and size of population differs from one method to another. For example, PSO shows better results with increasing the number of individuals, but DE does not.

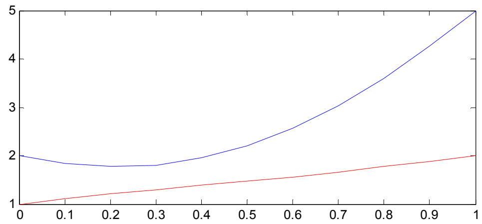

Figure 4: The system output for the control problem (7)-(8), $x_1(t), x_2(t)$.

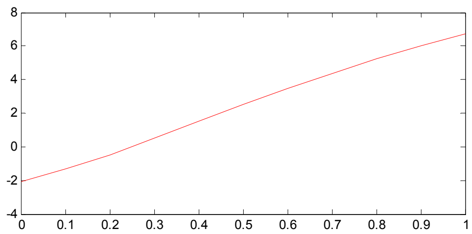

Figure 5: Function $u(t)$. Control problem (7)-(8).

Let us consider the optimal control problem (7)-(8) again. It has been shown earlier how we analytically transformed the initial problem and closed the system with state and co-state variables. The modified hybrid evolutionary strategies algorithm was applied for this control problem. After 20 runs of the algorithm, the best solution was taken. The state variables are shown on the Figure 4 and the control function is shown on the Figure 5. As one can see, system state with the given control reaches the desired end point at time *T*.

For the problem (9)-(10), another 20 runs of the ES+LO algorithm gave us solution that is shown on the Figures 6 and 7.

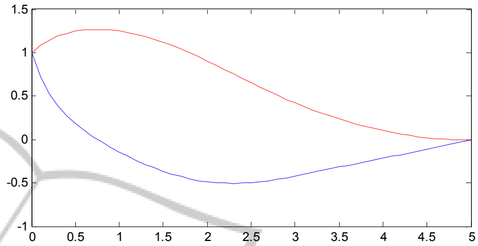

Figure 6: The system output for the control problem (9)-(10), $x_1(t), x_2(t)$.

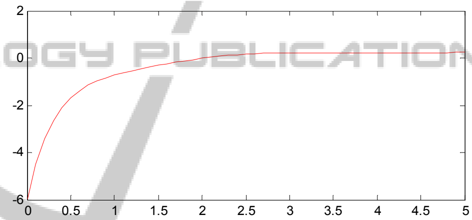

Figure 7: Function $u(t)$. Control problem (9)-(10).

As it is shown in the given examples, the proposed algorithm effectively solves the unconstrained optimal control problem for nonlinear dynamic systems. The initial point for co-state variables is to be found, because after closing up the extended system with state and co-state variables, regarding to the maximum principle, the control function is fully determined with them. As one can see, the proposed approach does not guarantee that any optimal control problem can be solved, but if the solution exists, there is a chance for it to be found. Also, the proposed approach fits only the optimal control problems that can be analytically transformed and the control function can be expressed via state and co-state variables after closing the system.

The proposed approach is suitable for the optimal control problems with different definitions, only criterion for extremum problem on the real vector field would be different.

5 CONCLUSIONS

In this study the analytically-numerical algorithm of

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the optimal control problem solving for nonlinear dynamic systems with one input and many outputs is considered. The initial two-point optimal control problem with integral functional was reduced to extremum problem on real vectors field. With all the transformations, listed above, the system with state and co-state variables can be closed and the control function structure can be defined in terms of the co-state variables and system state. The seeking of initial point for the co-state variables brings the solution for the optimal control problem and suggested to be based on stochastic or evolution unconstrained global optimization algorithms.

As the optimization techniques evolutionary strategies, differential evolution, particle swarm optimization and their modifications were suggested and investigated on the given optimal control problems set. It is suggested that optimization techniques with special features are required and future works are related with designing hybrid algorithms that allow both the extremum seeking and surface scouting.

Since, there are plenty of different optimal control problems, as the system equations and functional differs from task to task, the mapping to be optimized has different characteristics and there is no set of the problems that allows making a complete investigation of the optimization algorithms efficiency. All in all, the investigation of efficiency for heuristic techniques proceeds in the same way, techniques are testing on the well-known set of the objective functions. The proposed approach leads to investigate the efficiency of techniques with different extremum problem definition, and it seems to be the only way to find the most suitable algorithm by testing it on some set of the control problems.

In the current investigation the hybrid evolutionary strategies approach was the most effective.

The aim of the further investigation is to design the critique program agent, which will be the controller for the optimization technique, via analysing the data: fitness function set and its history, the topology of the individuals and its dynamics. The algorithms with implementation of this agent probably will be more efficient for global extremum problems.

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