Efficient Characteristic 3 Galois Field Operations for Elliptic Curve Cryptographic Applications

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Abstract: Galois fields of characteristic 3, where the number of field elements is a power of 3, have a distinctive application in building high-security elliptic curve cryptosystems. However, they are not typically used because of their relative inefficiency in computing polynomial operations when compared to conventional prime or binary Galois fields. The purpose of this research was to design and implement characteristic 3 Galois field arithmetic algorithms with greater overall efficiency than those presented in current literature, and to evaluate their applicability to elliptic curve cryptography. The algorithms designed were tested in a C++ program and using a mapping of field element logarithms, were able to simplify the operations of polynomial multiplication, division, cubing, and modular reduction to that of basic integer operations. They thus significantly outperformed the best characteristic 3 algorithms presented in literature and showed a distinct applicability to elliptic curve cryptosystems. In conclusion, this research presents a novel method of optimizing the performance of characteristic 3 Galois fields and has major implications for the field of elliptic curve cryptography.

1 INTRODUCTION

Galois fields are one of the most important concepts in abstract algebra and have a wide variety of applications towards public-key cryptography algorithms. In essence, a Galois field is an algebraic structure with established operations for addition, subtraction, multiplication, and division that satisfy the requirements for an Abelian group. This means that operations follow the five axioms of an Abelian group: closure, associativity, commutativity, having an identity element and an inverse element. Most importantly, Galois fields have a finite number of elements in them (Lidl and Niederreiter, 1997).

The most efficient and secure cryptographic system in use today is known as elliptic curve cryptography (ECC) and is based on the concept of elliptic curves built over Galois fields (Koblitz, 1987). Our research in particular investigates elliptic curves built over Galois fields of characteristic 3. This essentially means that the number of elements in the field is a power of 3, allowing the Galois field to be notated as *GF(*3^k*)*, where k represents the degree of the field. In Galois fields of characteristic 3, elements of the field are represented as polynomials modulo a primitive polynomial *p(x)*, where coefficients are either 0, 1, or 2 (Lidl and Niederreiter, 1994). A primitive polynomial is an irreducible polynomial of degree $k - 1$ that can generate all elements of the field. After the research of Galbraith (2001), it is well accepted that characteristic 3 curves provide more security and bandwidth efficiency than conventional binary or prime curves. In addition, they are highly applicable towards building pairing-based cryptosystems, an attractive option for identity-based cryptographic algorithms (Boneh and Franklin, 2001). However, according to the canonical research of Harrison, Page and Smart (2002), they are not efficient enough despite their potential. This is mainly because characteristic 3 polynomial arithmetic operations rely on base 3 arithmetic (Figure 1) and are much slower compared to prime and binary Galois fields, which utilize the computer's inherent hardware arithmetic.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

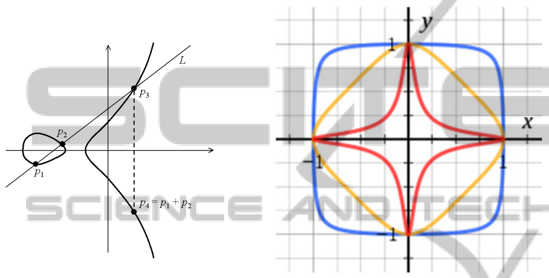
−	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

÷	0	1	2
0	u	0	0
1	u	1	2
2	u	2	1

Figure 1: Base 3 Arithmetic.

Elliptic curves are a type of equation of the form $y^2 = x^3 + ax + b$, where a and b represent integer coefficients. When elliptic curves are built over a Galois field, the points on the curve themselves form an Abelian group making it possible for operations to be done on points on the curve such as addition of two points, where the result is a third point on the curve, as shown on the left of Figure 2 (Hankerson et al.). This form of elliptic curve is known as the Weierstrass equation and is the most standard form of elliptic curve used in number theory (Koblitz, 1994). Another form of elliptic curve that is popular is the Edwards equation (right side of Figure 2) of the form $x^2 + y^2 = 1 + dx^2y^2$, where d represents a coefficient. Our research works primarily with the Edwards form of elliptic curves due to the lack of characteristic 3 Edwards research in the past.



Weierstrasss: $y^2 = x^3 + ax + b$ Edwards: $x^2 + y^2 = 1 + dx^2y^2$

Figure 2: Geometric Representation of Weierstrass Addition ("What is Diffie-Hellman", n.d) and Graphical Representation of Multiple Edwards Curves ("Edwards Curve", n.d).

Given the fact that operations can be performed on points on an elliptic curve, it is possible to design cryptographic algorithms based on difficult number-theoretic problems within this group (Silverman, 2006). For ECC this difficult problem is the Elliptic Curve Discrete Logarithm Problem (ECDLP), which states that it is difficult to find a point P and integer k, given their product $P \bullet k$. This operation of multiplying a point by an integer is referred to as scalar multiplication. Scalar multiplication not only dominates the execution time of ECC algorithms, but is also essential to the security of these systems.

1.1 Related Work

The research of Harrison, Page and Smart is regarded as the canonical paper on software implementation of characteristic 3 Galois fields for ECC applications. Their research uses conventional algorithms for polynomial arithmetic, and then provides software optimization. Research by Iyengar has developed efficient scalar multiplication algorithms. Three of these algorithms are use extensively in this research: the Binary Double-Add Algorithm, the Ternary Expansion Algorithm, and the Balanced Ternary Expansion Algorithm.

1.2 Research Goals

This research has two main goals:

1. To design and implement characteristic 3 Galois field operations with greater overall efficiency than conventional state-of-the-art algorithms.
2. To analyze this new method's applicability to elliptic curve cryptography.

Overall efficiency is evaluated as a combination of a comparison of implementation speeds, and time-space tradeoffs. If a new and more efficient method for characteristic 3 Galois field operations can be developed, it would be a major advancement for elliptic curve cryptography and Internet security in general.

2 CONVENTIONAL ALGORITHMS

Algorithms 1 – 6 are the characteristic 3 Galois field algorithms as presented in the research of Harrison, Page and Smart, 2002. They are widely considered the most efficient characteristic 3 algorithms in literature.

Algorithm 1: Characteristic 3 Polynomial Addition/Subtraction

```
INPUT: Polynomials f(x) = [$a_n…a_1$, $a_0$]
and g(x) = [$b_m…b_1$, $b_0$]
OUTPUT: f(x) + g(x)
        1.  For i from 0 to n if n > m,
            from 0 to m if m > n
                a.  $P_i$ = ($a_i$ +/- $b_i$ ) % 3
        2.  Return P(x)
```

Algorithm 3: Conventional Characteristic 3 Polynomial Multiplication

```
INPUT: Polynomials f(x) = [$a_n…a_1$, $a_0$]
and g(x) = [$b_m…b_1$, $b_0$]
OUTPUT: f(x) *g(x)
        1.  For i from 0 to n
             a. For j from 0 to m
                   i. $P_{i+j}$ = ($a_i$ * $b_j$) % 3
        2.  Return P(x)
```

Algorithm 4: Conventional Characteristic 3 Polynomial Cubing

```
INPUT: Polynomial f(x) = [$a_n…a_1$, $a_0$]
OUTPUT: f(x) ^ 3
        1.  For i from 0 to n
                a.  $P_{i*3}$ = ($a_i$ * 3) % 3
```

Conventional characteristic 3 polynomial cubing takes the degrees of all terms of f(x), and multiplies them by 3. The resulting polynomial is then reduced modulo the primitive polynomial of the system.

Algorithm 5: Conventional Characteristic 3 Polynomial Division

```
INPUT: Polynomials f(x) = [$a_n$…$a_1$, $a_0$]
and g(x) = [$b_m$…$b_1$, $b_0$]
OUTPUT: f(x) / g(x)
        1. h(x) = Extended Euclidean
           Algorithm Inverse of g(x)
        2. Return h(x) * f(x)
```

Conventional characteristic 3 polynomial division is a complex operation requiring multiple steps. Firstly, the inverse of the divisor is taken using Algorithm 6. This is then multiplied by the dividend using Algorithm 3.

Algorithm 6: Extended Euclidean Algorithm

```
INPUT: Polynomial f(x) = [$a_n$…$a_1$, $a_0$],
Primitive Polynomial p(x)
OUTPUT: $f^{-1}$(x)
   1. remainder[1] = p(x);
      remainder[2] = f(x)
   2. inverse[1] = 0; inverse[2] = 1
   3. i = 2
   4. while remainder[i] > 2
      a. i = i + 1
      b. remainder[i] = remainder[i-2]
         mod remainder[i-1]
      c. quotient[i] = remainder[i-2] /
         remainder[i-1]
      d. If(inverse[i] == 2) inverse[i]
         = 2(-quotient[i] *
   inverse[i-1] + inverse[i-2])
      e. Else (inverse[i] = -
   quotient[i] * inverse[i-1] +
   inverse[i-2])
      f. Return inverse[i]
```

3 KEY RESEARCH CONCEPTS

The key idea of this research was to map the polynomials to a simpler representation more conducive to efficient arithmetic. The algorithms designed were inspired by the concept of Zech's logarithms presented in the work of Lidl and Neiderreiter, 1997. The algorithms designed include the following: logarithm table generation, polynomial multiplication, polynomial division, and polynomial cubing.

3.1 Our Novel Contributions

This research designed and developed a new and highly efficient way of doing characteristic 3 Galois field operations using a logarithm-table approach. Furthermore, this research explored and analyzed Edwards curves over characteristic 3 fields. Finally, scalar multiplication was implemented using binary, ternary, and balanced-ternary algorithms.

Algorithm 7: Logarithm Table Generation

```
INPUT: Primitive Polynomial P(x)
OUTPUT: Mapped table of field elements
and logarithms
    1. LogTable[0] = x
    2. For i from 1 to field size do
       a. LogTable[i] =
    LogTable[i-1] * x
         b. if degree of LogTable[i]=
       degree of P(x) do
    LogTable[i] modulo P(x)or
    substitution reduction
    3. Return LogTable
```

The logarithm table generation algorithm aims to create a table of field element logarithms, mapped from a power representation. This is done by repeatedly multiplying successive terms in the table by the value *x*, and then reducing these values modulo the primitive polynomial of the system. This algorithm also utilizes the concept of substitution reduction to simplify polynomial modular reduction. Substitution reduction basically substitutes, during the computation phase, an identity previously computed in the table, in order to simplify modular reduction. To better illustrate this concept, we have created a small example of logarithm table generation as shown in Figure 3.

Figure 3: Example of Logarithm Table Generation and Use

The following example shows the creation of a log table for the **Galois Field 3^3** over the primitive polynomial **$P(x) = x^2 + 2x + 2$**

Power Rep	Galois Field Rep	Operation Done
1	1	
x	x	$1 \bullet x$
x^2	x + 1	$x^2 \bmod x^2 + 2x + 2$
x^3	2x + 1	$x^2 + x = 2x + 1$
x^4	2	$2x^2 + x = 2$
x^5	2x	$2 \bullet x$
x^6	2x + 2	$2x^2 = 2x + 2$
x^7	x + 2	$2x^2 + 2x = x + 2$
x^8	1	$x^2 + 2x = 1$

This example computes the table in 8 multiplications by x, and just 1 modular reduction using repeated substitution with the identity $x^2 = x + 1$.

Once created, the logarithm table can then be used to perform the following operations very efficiently:

Polynomial multiplication and modular reduction: Addition of power representation exponents – ***EX***: $(x + 1) \bullet (2x + 1) = x^2 \bullet x^3 = x^5 = 2x$

Polynomial division and modular reduction: Subtraction of power representation exponents – ***EX:*** $(2x + 2) / 2 = x^6 / x^4 = x^2 = x + 1$

Polynomial exponentiation and modular reduction: Multiplication of power representation exponent by the desired exponent – ***EX:*** $(2x + 1)^2 = (x^3)^2 = x^{3 \bullet 2} = x^6 = 2x + 2$

3.2 Implementation

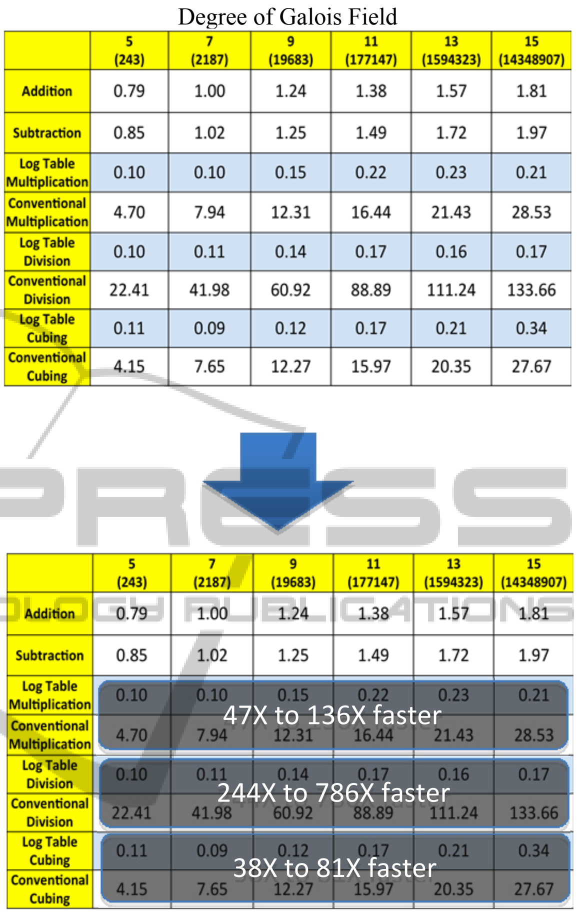
The main instrument used in this research was a Windows 7 computer with a 2.10 GHz Intel Core i3 processor installed with a Microsoft Visual Studio compiler. The main program was written in C++. An open source implementation for a Galois Field of characteristic 2 (Partow, 2006) was used as the starting point for the programming part of the research. The algorithms for the Galois Field of characteristic 3 were designed independently and then implemented into the program for testing. Primitive polynomials were generated for each Galois field size using the open source software Primpoly (O'Connor, 2013).

4 RESULTS AND DISCUSSION

4.1 Galois Field Operations

The first goal of this research was to design and implement characteristic 3 Galois field operations more efficient than conventional algorithms. These algorithms were tested by performing operations on a wide range of values within Galois fields of six different sizes: 3^5, 3^7, 3^9, 3^{11}, 3^{13}, $3^{15.}$ These operations were measured in terms of processor cycle counts, and finally averaged out as an indication of the algorithms' overall efficiencies. Table 1 compares the average speed of these operations using both the logarithm table method designed in this research and the conventional methods.

Table 1: Comparison of Performance of Galois Field Operations (Processor Clock Cycles).



Degree of Galois Field

	5 (243)	7 (2187)	9 (19683)	11 (177147)	13 (1594323)	15 (14348907)
Addition	0.79	1.00	1.24	1.38	1.57	1.81
Subtraction	0.85	1.02	1.25	1.49	1.72	1.97
Log Table Multiplication	0.10	0.10	0.15	0.22	0.23	0.21
Conventional Multiplication	4.70	7.94	12.31	16.44	21.43	28.53
Log Table Division	0.10	0.11	0.14	0.17	0.16	0.17
Conventional Division	22.41	41.98	60.92	88.89	111.24	133.66
Log Table Cubing	0.11	0.09	0.12	0.17	0.21	0.34
Conventional Cubing	4.15	7.65	12.27	15.97	20.35	27.67

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Conventional Cubing	4.15	7.65	12.27	15.97	20.35	27.67

As shown in Table 1, the logarithm table methods of doing basic characteristic 3 Galois field operations were orders of magnitude faster than their conventional counterparts. To conclude, research goal 1 was met.

4.2 Elliptic Curve Analysis

Research goal 2 was to evaluate the applicability of the logarithm table method towards elliptic curve cryptography. The underlying Galois Field was thus implemented, tested, and verified over Edwards elliptic curves. This basically involved timing scalar multiplication operations using the Binary Double-Add, the Ternary Expansion, and the Balanced Ternary Expansion algorithms for Edwards curves using six different Galois field sizes: 3^5, 3^7, 3^9, 3^{11}, 3^{13}, 3^{15}. This procedure was first done using the conventional polynomial arithmetic operations.

Legend for Figures 4 and 5

- Binary Double-Add
- Ternary Expansion
- Balanced Ternary Expansion

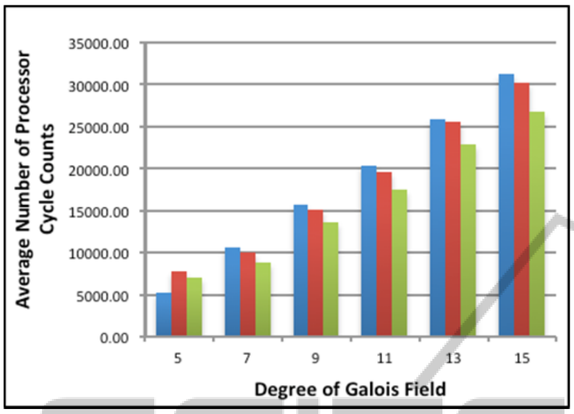

Figure 4: Elliptic Curve Scalar Multiplication - Conventional Polynomial.

Figure 4 shows the average performance of scalar multiplication algorithms over 6 Galois field sizes using the conventional characteristic 3 arithmetic algorithms. The Balanced Ternary Expansion algorithm is the most efficient for all Galois field sizes except 3^5. Furthermore, as the size of the underlying Galois field increases, the efficiency decreases in a linear manner.

This same scalar multiplication testing procedure was applied with the logarithm table method for polynomial arithmetic.

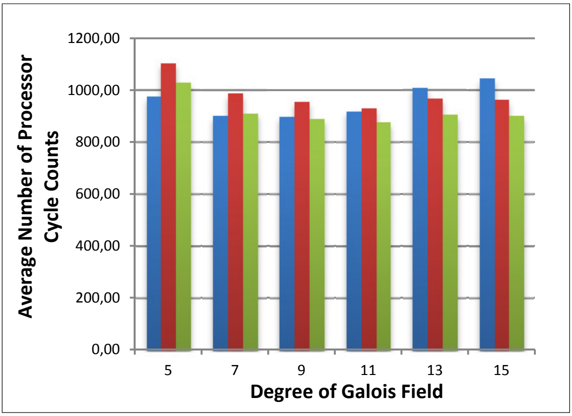

Figure 5: Elliptic Curve Scalar Multiplication – Logarithm Table.

Figure 5 shows the average time for scalar multiplication operations over the 6 different Galois fields with the 3 different scalar multiplication algorithms using logarithm table polynomial operations. It is clear that the Balanced-Ternary algorithm is generally the fastest, and as the size of the field increases, the speed of all algorithms remains constant. Most importantly, these operations are significantly faster in comparison to their conventional counterparts shown in Figure 4. Specifically, the logarithm table method for scalar multiplication ranges from ~5X faster for a field of degree 5, to ~30X faster for a field of degree 15.

4.3 Next Steps

A very attractive option for future research is developing a hybrid method that utilizes both logarithm table and conventional arithmetic, reducing the cost upfront and the storage needed, while also taking advantage of the speed provided by logarithm table-based arithmetic. This could be done by using a logarithm table method for a small subfield, and then extending this field to a larger power. Also, next steps include testing larger numbers such as NIST (National Institute for Standards in Technology) size elliptic curves in order to evaluate the scalability of these algorithms.

5 CONCLUSIONS

In this paper, we present a novel and efficient method for characteristic 3 Galois field operations and analyze this method's distinctive applications to elliptic curve cryptography. We thus meet both research goals. The findings of this research have a wide significance towards the findings of other researchers such as Harrison et al. and those at NIST who have disregarded characteristic 3 Galois fields for elliptic curve cryptographic applications. This research shows that in fact characteristic 3 can be a feasible option for some ECC applications.

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