A New Covariance-assignment State Estimator in the Presence of Intermittent Observation Losses

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Keywords: Covariance Assignment, Estimation, Networked Control System (NCS), Packet Loss, Riccati Difference Equation (RDE), Algebraic Riccati Equation (ARE).

Abstract: This paper introduces an improved linear state estimator which directly assigns the error covariance in an environment where the measured data are intermittently missing. Since this new estimator uses an additional information indicating whether each observation is successfully measured, represented as a bernoulli random variable in the measurement equation, it naturally outperforms the previous type of covariance-assignment estimators which do not rely upon such information. This fact is proved by comparing the magnitude of the state error covariances via the monotonicity of the Riccati difference equation, and demonstrated using a numerical example.

1 INTRODUCTION

Construction of recursive state estimatiors in the presence of intermittent noise-alone measurements can be traced back to the 1960s in tackling occasional data loss in target tracking problems in space (Nahi, 1969). The mostly used technique to cope with this data-loss problem in the estimation process is to model the measurement data loss using a bernoulli random variable taking one or zero with a probability in the measurement equation. For example, using a random variable $\gamma_k \in \{0,1\}$ whose distribution is described by

$$\Pr\{\gamma_k = 1\} = \overline{\gamma}, \tag{1a}$$

$$\Pr\{\gamma_k = 0\} = 1 - \overline{\gamma}, \tag{1b}$$

$$\mathbb{E}\{\gamma_k\} = \overline{\gamma}, \tag{1c}$$

in the state and measurement equation

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{v}_k, \tag{2a}$$

$$\mathbf{y}_k = \gamma_k \mathbf{C}\mathbf{x}_k + \mathbf{w}_k, \tag{2b}$$

a data-loss situation is expressed as $\mathbf{y}_k = \mathbf{w}_k$ with $\gamma_k = 0$ and when a data is successfully observed as $\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{w}_k$ now with $\gamma_k = 1$. Here $\mathbf{v}_k$ and $\mathbf{w}_k$ are the process and measurement noises, respectively.

In many cases such as the aforementioned tracking problem, the value of the random variable γ_k is not accessible in the estimation process of the state $\mathbf{x}_k$. Therefore the estimators used in previous research (Nahi, 1969) and (NaNacara and Yaz, 1997) were of the following form

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{y}_k - \overline{\gamma}\mathbf{C}\hat{\mathbf{x}}_k) \tag{3}$$

only using the expected value of γ_k. Nahi (1969) derived the minimum variance estimators and NaNacara & Yaz (1997) introduced covariance-assignment estimators using the form (3).

One of the recent application area where this intermittent data loss problem is important is the networked control systems (NCS) (Hespanha et al., 2007). In NCS, control and/or mesurement signals among sub-components within the system are trasferred via a commonly accessible network instead of using the componet-to-component connections. Naturally the problems such as data packet losses and time delays due to the network have been major research topics.

One of the major differences in NCS and the previous tracking problem is the accessibility of the information on the value of γ_k in the state estimation process. Both in (Sinopoli et al., 2004) and (Schenato et al., 2007), the optimal state estimator over lossy networks was derived, although the interim derivation processes were different, in which the estimators was of the form

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \gamma_k \mathbf{G}(\mathbf{y}_k - \mathbf{C}\hat{\mathbf{x}}_k). \tag{4}$$

Here we can use the value of γ_k for state estimation.

Whearas the previous type of estimators (3) uses only $\{\mathbf{y}_k\}_{k=1,2,3,...}$ for state estimation, the new form

Ko S., Kang S. and Cha J..
A New Covariance-assignment State Estimator in the Presence of Intermittent Observation Losses.
DOI: 10.5220/0004673902790284
In *Proceedings of the 3rd International Conference on Sensor Networks* (SENSORNETS-2014), pages 279-284
ISBN: 978-989-758-001-7

of estimators (4) utilize $\{\gamma_k\}$ as well as $\{\mathbf{y}_k\}$. Therefore, we may conjecture that the optimal estimator of the form (4) outperforms the optimal estimator of the form (3). In this paper, we formally prove this by comparing magnitude of the estimation error covariances.

In this context, we derive a new covariance-assignment estimtor for linear systems using the estimator form (4) in the next section and, then in Section 3, prove that the new estimator performs better than the old one (3) by comparing the magnitude of the state error covariances represented as the two different Riccati difference equations (RDE) resulted from (3) and (4). For this we use the monotonicity property of the RDE and demonstate the difference using a numerical example in Section 4. Finally Section 5 concludes this research.

In this paper, matrices will be denoted by upper case boldface (e.g., $\mathbf{A}$), column matrices (vectors) will be denoted by lower case boldface (e.g.,$\mathbf{x}$), and scalars will be denoted by lower case (e.g., y) or upper case (e.g., Y). For a matrix $\mathbf{A}$, $\mathbf{A}^T$ and $\mathrm{Tr}\{\mathbf{A}\}$ denote its transpose and trace, respectively. For a symmetric matrix $\mathbf{P} > \mathbf{0}$ or $\mathbf{P} \geq \mathbf{0}$ denotes the fact that $\mathbf{P}$ is positive definite or positive semi-definite, respectively. For a random vector $\mathbf{y}$, $\mathbb{E}\{\mathbf{x}\}$ denotes the expectation of $\mathbf{x}$.

2 A NEW COVARIANCE ASSIGNMENT STATE ESTIMATOR

Consider the state and measurement equations represented in (2) where $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{y}_k \in \mathbb{R}^p$. Here $\mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{w}_k \in \mathbb{R}^p$ are statistically independent zero-mean sequences representing the process and the measurement noises, respectively. The covariances of $\mathbf{v}_k$ and $\mathbf{w}_k$ are represented as $\mathbf{V} > \mathbf{0}$ and $\mathbf{W} > \mathbf{0}$ for all k, respectively. The noises are assumed to be mutually independent and also independent of the initial state $\mathbf{x}_0$ whose mean and covariance are $\bar{\mathbf{x}}_0$ and $\mathbf{X}_0$, respectively.

The random variable γ_k, indicating the information whether the measurement is successfully observed, has the distribution characterized by (1). For a simple development, we introduce a new random sequence $\tilde{\gamma}_k$ such that

$$\gamma_k = \bar{\gamma} + \tilde{\gamma}_k, \tag{5}$$

and thus

$$\mathbb{E}\{\tilde{\gamma}_k\} = 0, \tag{6a}$$

$$\sigma_{\tilde{\gamma}}^2 \triangleq \mathbb{E}[(\tilde{\gamma}_k - \mathbb{E}\{\tilde{\gamma}_k\})^2] = \bar{\gamma}(1-\bar{\gamma}). \tag{6b}$$

Therefore, the measurement equation (2b) can be written as

$$\mathbf{y}_k = (\bar{\gamma} + \tilde{\gamma}_k)\mathbf{C}\mathbf{x}_k + \mathbf{w}_k. \tag{7}$$

Differently from the estimator type used in the previous literature (NaNacara and Yaz, 1997) which does not access γ_k, here we employ a new estimator form (4) which uses the informaton on γ_k. Using the system equations (2) and the estimator (4) yields the state estimation error

$$\mathbf{e}_k \triangleq \mathbf{x}_k - \hat{\mathbf{x}}_k, \tag{8}$$

resulting in

$$\mathbf{e}_{k+1} = (\mathbf{A} - \gamma_k\mathbf{G}\mathbf{C})\mathbf{e}_k + \mathbf{v}_k + \gamma_k\mathbf{G}\mathbf{w}_k. \tag{9}$$

Note that the estimator (4) is unbiased if $\mathbb{E}\{\mathbf{e}_0\} = \mathbf{0}$.

The covariance matrix of the state estimation error vector is defined as

$$\mathbf{P}_k \triangleq \mathbb{E}\{\mathbf{e}_k\mathbf{e}_k^T\} \tag{10}$$

which propagates in time according to

$$\begin{aligned}\mathbf{P}_{k+1} &= (\mathbf{A} - \bar{\gamma}\mathbf{G}\mathbf{C})\mathbf{P}_k(\mathbf{A} - \bar{\gamma}\mathbf{G}\mathbf{C})^T \\ &\quad + \mathbf{G}(\sigma_{\tilde{\gamma}}^2\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \bar{\gamma}\mathbf{W})\mathbf{G}^T + \mathbf{V}.\end{aligned} \tag{11}$$

Rearranging and completing square yield

$$\begin{aligned}\mathbf{P}_{k+1} &= \mathbf{A}\mathbf{P}_k\mathbf{A}^T + \mathbf{V} - \bar{\gamma}\mathbf{G}_k^0(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \mathbf{W})\mathbf{G}_k^{0^T} \\ &\quad + \bar{\gamma}(\mathbf{G} - \mathbf{G}_k^0)(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \mathbf{W})(\mathbf{G} - \mathbf{G}_k^0)^T,\end{aligned} \tag{12}$$

where

$$\mathbf{G}_k^0 \triangleq \mathbf{A}\mathbf{P}_k\mathbf{C}^T(\mathbf{C}\mathbf{P}_k\mathbf{C}^T + \mathbf{W})^{-1}. \tag{13}$$

If the error covariance at steady state defined as

$$\mathbf{P} \triangleq \lim_{k\to\infty} \mathbb{E}\{\mathbf{e}_k\mathbf{e}_k^T\} \tag{14}$$

exists,

$$\begin{aligned}&\mathbf{P} - \mathbf{A}\mathbf{P}\mathbf{A}^T - \mathbf{V} + \bar{\gamma}\mathbf{G}^0(\mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{W})^{-1}\mathbf{G}^{0^T} \\ &\quad = \bar{\gamma}(\mathbf{G} - \mathbf{G}^0)(\mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{W})(\mathbf{G} - \mathbf{G}^0)^T \\ &\quad \triangleq \mathbf{L}\mathbf{L}^T,\end{aligned} \tag{15}$$

where $\mathbf{L} \in \mathbb{R}^{n\times p}$ and

$$\mathbf{G}^0 \triangleq \mathbf{A}\mathbf{P}\mathbf{C}^T(\mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{W})^{-1}. \tag{16}$$

Defining a new non-singular matrix $\mathbf{T}$ such that

$$\mathbf{T}\mathbf{T}^T = \bar{\gamma}(\mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{W}) \tag{17}$$

yields

$$\mathbf{L}\mathbf{L}^T = (\mathbf{G} - \mathbf{G}_0)\mathbf{T}\mathbf{T}^T(\mathbf{G} - \mathbf{G}_0)^T, \tag{18}$$

from which we obtain, for an arbitrary orthogonal matrix $\mathbf{U}$ with consistent size,

$$\mathbf{L}\mathbf{U} = (\mathbf{G} - \mathbf{G}_0)\mathbf{T}. \tag{19}$$

Finally the filter gain becomes

$$\begin{aligned}\mathbf{G} &= \mathbf{G}_0 + \mathbf{L}\mathbf{U}\mathbf{T}^{-1} \\ &= \mathbf{A}\mathbf{P}\mathbf{C}^T(\mathbf{C}\mathbf{P}\mathbf{C}^T + \mathbf{W})^{-1} + \mathbf{L}\mathbf{U}\mathbf{T}^{-1}.\end{aligned} \tag{20}$$

The developments above can be summarized as Theorem 1:

Theorem 1. *For the linear discrete-time stochastic system with intermittent observation losses expressed as* (2) *and the state estimator form* (4)*, a given steady-state covariance* $\mathbf{P}$ *of state estimation error is assignable if and only if the left-hand side of* (15) *is non-negative definite with maximum rank p. In this case, all filter gains assign this steady-state covariance* $\mathbf{P}$ *are expressed as* (20)*.*

It can be easily shown that among all the steady-state error covariances expressed as (15), the minimum error covariance is attained when the filter gain $\mathbf{G}$ is equal to $\mathbf{G}^0$:

Corollary 1. *The minimum error covariance attainable using the estimator* (4) *is expressed as the following algebraic Riccati equation (ARE)*

$$\begin{aligned}\mathbf{P} &= \mathbf{APA}^T + \mathbf{V} - \bar{\gamma}\mathbf{G}^0(\mathbf{CPC}^T + \mathbf{W})\mathbf{G}^{0^T} \\ &= \mathbf{APA}^T + \mathbf{V} - \bar{\gamma}\mathbf{APC}^T(\mathbf{CPC}^T + \mathbf{W})^{-1}\mathbf{CPA}^T \end{aligned} \tag{21}$$

with the filter gain

$$\mathbf{G} = \mathbf{G}^0 = \mathbf{APC}^T(\mathbf{CPC}^T + \mathbf{W})^{-1}. \tag{22}$$

The minimum error covariance given by (21) can also be modified to

$$\begin{aligned}\mathbf{P} &= \mathbf{APA}^T + \mathbf{V} \\ &- \bar{\gamma}^2\mathbf{APC}^T(\bar{\gamma}^2\mathbf{CPC}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CPC}^T + \bar{\gamma}\mathbf{W})^{-1}\mathbf{CPA}^T\end{aligned} \tag{23}$$

using the relation $\bar{\gamma}^2 + \sigma_{\tilde{\gamma}}^2 = \bar{\gamma}^2 + \bar{\gamma}(1-\bar{\gamma}) = \bar{\gamma}$.

3 PERFORMANCE COMPARISION OF COVARIANCE-ASSIGNMENT ESTIMATORS

This section compares the magnitude of state error covariance of the state estimator (4) newly introduced in the previous section with that of the past estimator (3) suggested in (NaNacara and Yaz, 1997). This estimator satisfies the following estimation error covariance equation at steady state

$$\begin{aligned}&\tilde{\mathbf{P}} - \mathbf{A}\tilde{\mathbf{P}}\mathbf{A}^T - \mathbf{V} \\ &= (\mathbf{K}-\mathbf{K}^0)(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CXC}^T + \mathbf{W})(\mathbf{K}-\mathbf{K}^0)^T \\ &\quad - \mathbf{K}^0(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CXC}^T + \mathbf{W})\mathbf{K}^{0^T}\end{aligned} \tag{24}$$

where

$$\mathbf{K}^0 = \bar{\gamma}\mathbf{A}\tilde{\mathbf{P}}\mathbf{C}^T(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CXC}^T + \mathbf{W})^{-1} \tag{25}$$

and $\mathbf{X} \triangleq \lim_{k\to\infty}\mathbf{X}_k = \lim_{k\to\infty}\mathbb{E}\{\mathbf{x}_k\mathbf{x}_k^T\}$ is the converging solution of the state covariance equation

$$\mathbf{X}_{k+1} = \mathbf{AX}_k\mathbf{A}^T + \mathbf{V}. \tag{26}$$

Similarly to Corollary 1, the minimum state error covariance at steady state is expressed as the ARE

$$\begin{aligned}\tilde{\mathbf{P}} &= \mathbf{A}\tilde{\mathbf{P}}\mathbf{A}^T + \mathbf{V} \\ &\quad - \mathbf{K}^0(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CXC}^T + \mathbf{W})\mathbf{K}^{0^T} \\ &= \mathbf{A}\tilde{\mathbf{P}}\mathbf{A}^T + \mathbf{V} \\ &- \bar{\gamma}^2\mathbf{A}\tilde{\mathbf{P}}\mathbf{C}^T(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CXC}^T + \mathbf{W})^{-1}\mathbf{C}\tilde{\mathbf{P}}\mathbf{A}^T\end{aligned} \tag{27}$$

which is obtained by plugging $\mathbf{K} = \mathbf{K}^0$ into (24).

Remark 1 (Convergence of Riccati difference equations). It is well known (Bitmead and Gevers, 1991) that if the matrix pair $[\mathbf{A}, \mathbf{C}]$ is stabilizable and $[\mathbf{A}, \mathbf{V}^{1/2}]$ is detectable, the solution of a RDE converges to the solution of the corresponding ARE. Therefore the solutions of the following RDEs

$$\begin{aligned}\tilde{\mathbf{P}}_{k+1} = &\mathbf{A}\tilde{\mathbf{P}}_k\mathbf{A}^T + \mathbf{V} - \bar{\gamma}^2\mathbf{A}\tilde{\mathbf{P}}_k\mathbf{C}^T \\ &\times(\bar{\gamma}^2\mathbf{C}\tilde{\mathbf{P}}_k\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CX}_k\mathbf{C}^T + \mathbf{W})^{-1}\mathbf{C}\tilde{\mathbf{P}}_k\mathbf{A}^T,\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{P}_{k+1} = &\mathbf{AP}_k\mathbf{A}^T + \mathbf{V} - \bar{\gamma}^2\mathbf{AP}_k\mathbf{C}^T \\ &\times(\bar{\gamma}^2\mathbf{CP}_k\mathbf{C}^T + \sigma_{\tilde{\gamma}}^2\mathbf{CP}_k\mathbf{C}^T + \bar{\gamma}\mathbf{W})^{-1}\mathbf{CP}_k\mathbf{A}^T\end{aligned} \tag{29}$$

converge to the solution of (27) and (23), respectively.

In order to compare the magnitude of $\mathbf{P}$ with $\tilde{\mathbf{P}}$, the followng monotonicity property of the Riccati equation (Bitmead and Gevers, 1991) is useful.

Lemma 1 (Monotonicity property #1 of the RDE (Bitmead and Gevers, 1991))**.** *Consider two Riccati Difference Equations with the same* $\mathbf{A}$, $\mathbf{C}$ *and* $\mathbf{R}$ *matrices but possibly different* $\mathbf{V}_1$ *and* $\mathbf{V}_2$*. Denote their solution matrices* $\mathbf{P}_k^1$ *and* $\mathbf{P}_k^2$*, respectively, of the following two Riccati difference equations*

$$\begin{aligned}\mathbf{P}_{k+1}^i &= \mathbf{AP}_k^i\mathbf{A}^T + \mathbf{V}_i \\ &- \mathbf{AP}_k^i\mathbf{C}^T(\mathbf{CP}_k^i\mathbf{C}^T + \mathbf{W})^{-1}\mathbf{CP}_k^i\mathbf{A}^T,\ i = 1, 2.\end{aligned} \tag{30}$$

Suppose that $\mathbf{V}_1 \geq \mathbf{V}_2$*, and, for some k we have* $\mathbf{P}_k^1 \geq \mathbf{P}_k^2$*, then for all* $j > 0$

$$\mathbf{P}_{k+j}^1 \geq \mathbf{P}_{k+j}^2. \tag{31}$$

Using this Lemma 1, a similar monotonicity property of the RDE now with different $\mathbf{W}_i$ but the same $\mathbf{V}$ matrix can be obtained.

Lemma 2 (Monotonicity property #2 of the RDE)**.** *Consider two Riccati Difference Equations with the same* $\mathbf{A}$, $\mathbf{C}$ *and* $\mathbf{V}$ *matrices but possibly different* $\mathbf{W}_1$ *and* $\mathbf{W}_2$*. Denote their solution matrices* $\mathbf{P}_k^1$ *and* $\mathbf{P}_k^2$*, respectively, of the following two Riccati difference equations*

$$\begin{aligned}\mathbf{P}_{k+1}^i &= \mathbf{AP}_k^i\mathbf{A}^T + \mathbf{V} \\ &- \mathbf{AP}_k^i\mathbf{C}^T(\mathbf{CP}_k^i\mathbf{C}^T + \mathbf{W}_i)^{-1}\mathbf{CP}_k^i\mathbf{A}^T,\ \ i = 1, 2.\end{aligned} \tag{32}$$

Suppose that $\mathbf{W}_1 \geq \mathbf{W}_2$*, and, for some k we have* $\mathbf{P}_k^1 \geq \mathbf{P}_k^2$*, then for all* $j > 0$

$$\mathbf{P}_{k+j}^1 \geq \mathbf{P}_{k+j}^2. \tag{33}$$

Proof 1. *See Appendix.* □

Since $\mathbf{X}_k \geq \mathbf{P}_k$ based on (26) and (29) and $\overline{\gamma} < 1$, we have

$$\sigma_{\overline{\gamma}}^2 \mathbf{C}\mathbf{X}_k\mathbf{C}^T + \mathbf{W} \geq \sigma_{\overline{\gamma}}^2 \mathbf{C}\mathbf{P}_k\mathbf{C}^T + \overline{\gamma}\mathbf{W}, \tag{34}$$

and the difference between the two terms above is

$$\begin{aligned}\Delta_k &\triangleq \sigma_{\overline{\gamma}}^2 \mathbf{C}(\mathbf{X}_k - \mathbf{P}_k)\mathbf{C}^T + (1-\overline{\gamma})\mathbf{W} \\ &= (1-\overline{\gamma})(\overline{\gamma}\nabla_k + \mathbf{W}) \geq \mathbf{0}\end{aligned} \tag{35}$$

with $\nabla_k \triangleq \mathbf{C}(\mathbf{X}_k - \mathbf{P}_k)\mathbf{C}^T \geq \mathbf{0}$.

Therefore, applying Lemma 2 to (28) and (29) together with (34) yields

$$\tilde{\mathbf{P}}_k \geq \mathbf{P}_k \quad \text{for all } k > 0, \tag{36}$$

provided that each of the initial covariances are the same.

Now if the sufficient conditions hold for the convergence of the RDE in Remark 1, the following theorem can be obtained from (36).

Theorem 2. *For the discrete-time stochastic system with intermittent observation losses expressed as* (2)*, the state estimators given by* (3) *and* (4) *yield the minimum covariances of state estimation error* (27) *and* (23)*, respectively. Furthermore,*

$$\tilde{\mathbf{P}} \geq \mathbf{P}, \tag{37}$$

i.e., the state error covariance of the old estimator (3) *is bigger than that of the current estimator* (4)*.*

Based on Theorem 2 and (35), we observe the followng:

(1) As we can expect, when $\overline{\gamma} = 1$, there is no performance difference between the two estimators, i.e., $\tilde{\mathbf{P}} = \mathbf{P}$;

(2) Whether Δ_k is increasing or decreasing with respect to $\overline{\gamma}$ is not straightforward because the derivative

$$\frac{\partial \Delta_k}{\partial \overline{\gamma}} = -(2\overline{\gamma}\nabla_k + \mathbf{W}) + \left[\nabla_k + \overline{\gamma}(1-\overline{\gamma})\frac{\partial \nabla_k}{\partial \overline{\gamma}}\right] \tag{38}$$

can be non-negative or non-positive matrix[1]. However, if $\overline{\gamma}$ and the noise covariance $\mathbf{W}$ are relatively big, the derivative may become a non-positive definite. In this case, the bigger $\overline{\gamma}$ the smaller the performance difference.

[1]Note $\frac{\partial \nabla_k}{\partial \overline{\gamma}} = -\frac{\partial P_k}{\partial \overline{\gamma}} \geq \mathbf{0}$ from the monotonicity property in Lemma 1.

Remark 2 (Connections to the Kalman filter with intermittent observation). The Kalman filter with intermittent observation derived in Refs. (Sinopoli et al., 2004) and (Schenato et al., 2007) iterates a Riccati difference equation:

$$\begin{aligned}\Sigma_{k+1} &= \mathbf{A}\Sigma_k\mathbf{A}^T + \mathbf{V} \\ &\quad - \gamma_k \mathbf{A}\Sigma_k\mathbf{C}^T(\mathbf{C}\Sigma_k\mathbf{C}^T + \mathbf{W})^{-1}\mathbf{C}\Sigma_k\mathbf{A}^T\end{aligned} \tag{39}$$

However, the covariance obtained by this equation is stochastic since it depends on γ_k so that it cannot be calculated offline. They suggested a deterministic upper bound of the expectation of Σ_k:

$$\mathbb{E}_\gamma\{\Sigma_k\} \leq \mathbf{P}_k \tag{40}$$

We observe that this upper bound is equal to the non-steady-state version of the algebraic Riccati equation (23) developed in this paper.

4 NUMERICAL EXAMPLE

In order to demonstrate the performance difference between the two estimators using covariance assignment as proved in the previous section, the same numerical example as in (NaNacara and Yaz, 1997) is used here:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0.90 & 0.02 \\ 0.01 & 0.84 \end{bmatrix}\mathbf{x}_k + \mathbf{v}_k \tag{41a}$$

$$y_k = \gamma_k \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}_k + w_k \tag{41b}$$

Here the zero-mean noise sequence $\mathbf{v}_k$ and w_k have gaussian distributions with covariaces, respectively,

$$\begin{aligned}\mathbf{V} &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \\ W &= 0.02.\end{aligned} \tag{42}$$

To see the effect of the observation-success probability on performance difference, two different values of $\overline{\gamma} = 0.9$ and 0.6 were tried. To confirm the result using the algebraic Riccati equations (27) and (23), a 10000-run Monte-carlo simulation was also conducted for each case. The followings summarize the result of each case.

- **$\overline{\gamma} = 0.9$ case:**

Based on the filter gains (22) and (25) for the new and old estimators respectively

$$\begin{aligned}\mathbf{G} = \mathbf{G}^0 &= \begin{bmatrix} 0.4348 \\ 0.0517 \end{bmatrix}, \\ \mathbf{K} = \mathbf{K}^0 &= \begin{bmatrix} 0.3880 \\ 0.0462 \end{bmatrix},\end{aligned} \tag{43}$$

the algebraic Riccati equations and the 10000-run Monte-Carlo simulation result in

$$\mathbf{P}_{\text{Riccati}} = \begin{bmatrix} 0.0186 & 0.0022 \\ 0.0022 & 0.0677 \end{bmatrix}, \quad \tilde{\mathbf{P}}_{\text{Riccati}} = \begin{bmatrix} 0.0194 & 0.0022 \\ 0.0022 & 0.0678 \end{bmatrix}, \tag{44}$$

and

$$\mathbf{P}_{\text{Monte}} = \begin{bmatrix} 0.0189 & 0.0016 \\ 0.0016 & 0.0702 \end{bmatrix}, \quad \tilde{\mathbf{P}}_{\text{Monte}} = \begin{bmatrix} 0.0201 & 0.0016 \\ 0.0016 & 0.0701 \end{bmatrix}. \tag{45}$$

- **$\overline{\gamma} = 0.6$ case:**

$$\mathbf{G}^0 = \begin{bmatrix} 0.4782 \\ 0.0573 \end{bmatrix}, \quad \mathbf{K}^0 = \begin{bmatrix} 0.3223 \\ 0.0388 \end{bmatrix}, \tag{46}$$

$$\mathbf{P}_{\text{Riccati}} = \begin{bmatrix} 0.0225 & 0.0026 \\ 0.0026 & 0.0678 \end{bmatrix}, \quad \tilde{\mathbf{P}}_{\text{Riccati}} = \begin{bmatrix} 0.0250 & 0.0029 \\ 0.0029 & 0.0678 \end{bmatrix}, \tag{47}$$

$$\mathbf{P}_{\text{Monte}} = \begin{bmatrix} 0.0225 & 0.0026 \\ 0.0026 & 0.0678 \end{bmatrix}, \quad \tilde{\mathbf{P}}_{\text{Monte}} = \begin{bmatrix} 0.0250 & 0.0029 \\ 0.0029 & 0.0678 \end{bmatrix}. \tag{48}$$

The numerical results shown above confirm the formal verification of Section 3 that the new estimator accessing the information on γ_k outperforms the old estimator in terms of the minimum error covariance. This simulation results are also shown in Figure 1 through Figure 4. Figure 1 and 2 compare the estimates of the first state and the estimation errors in terms of absolute errors for $\overline{\gamma} = 0.9$. Figure 3 and 4 correspond to $\overline{\gamma} = 0.6$ case. As shown the figures, the smaller $\overline{\gamma}$, the bigger the peformance difference.

5 CONCLUSIONS

This paper confirmed that in the presence of intermittent observation losses the new covariance assignment estimator which can access the information on observation-success or fail performs better than the previous covariance-assignment state estimator which does not use such an information. The performance difference was formally shown by comparing the magnitude of the error covariance matrices via the monotonicity properties of the Riccati equation.

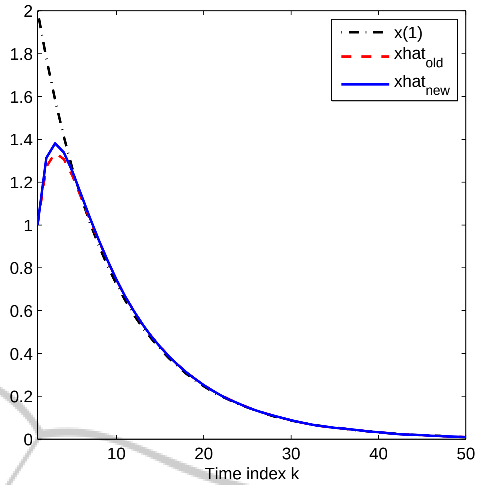

Figure 1: Comparison of state estimates with $\overline{\gamma} = 0.9$.

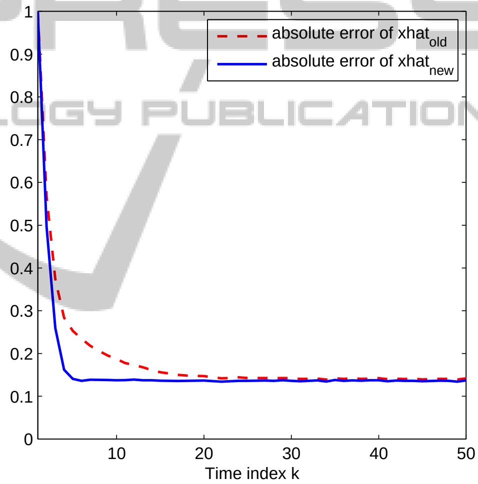

Figure 2: Comparison of estimation errors with $\overline{\gamma} = 0.9$.

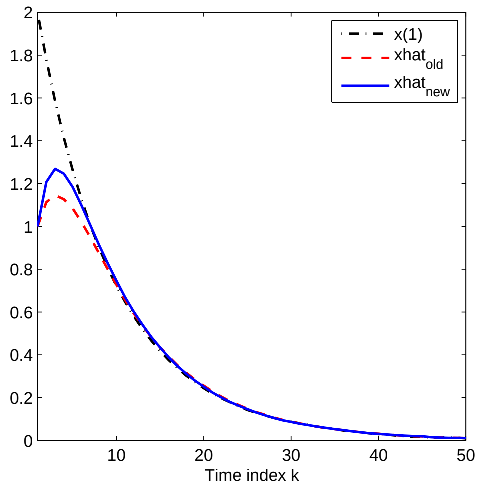

Figure 3: Comparison of state estimates with $\overline{\gamma} = 0.6$.

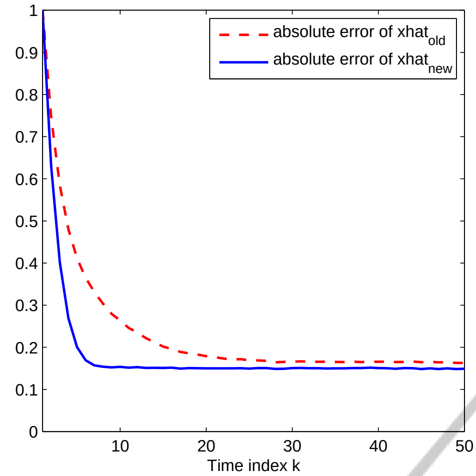

Figure 4: Comparison of estimation errors with $\bar{\gamma} = 0.6$.

This fact was then numerically demonstrated using a discrete-time linear stochastic system example.

ACKNOWLEDGEMENTS

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0013998).

REFERENCES

Bitmead, R. R. and Gevers, M. (1991). Riccati difference and differential equations: Convergence, monotonicity and stability. In Bittanti, S., Laub, A. J., and Willems, J. C., editors, *The Riccati equation*, pages 263–291. Springer-Verlag, Berlin, Germany.

Hespanha, J., Naghshtabrizi, P., and Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162.

Nahi, N. E. (1969). Optimal recursive estimation with uncertain observation. *IEEE Transactions on Information Theory*, IT-15(4):457–462.

NaNacara, W. and Yaz, E. E. (1997). Recursive estimator for linear and nonlinear systems with uncertain observations. *Signal Processing*, 62(15):215–228.

Schenato, L., Sinopoli, B., Franceschetti, M., Polla, K., and Sastry, S. (2007). Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1):163–187.

Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K., Jordan, M. I., and Sastry, S. S. (2004). Kalman filtering with intermittent observations. *IEEE Transactions on Automatic Control*, 49(9):1453–1464.

APPENDIX

Proof for Lemma 2

Let $\mathbf{W}_1 = \mathbf{W}_2 + \Delta\mathbf{W}$ with $\Delta\mathbf{W} \geq \mathbf{0}$. Then using the matrix inversion lemma yields

$$\begin{aligned}(\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_1)^{-1} &= (\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2 + \Delta\mathbf{W})^{-1} \\ &= (\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2)^{-1} - \mathbf{M}\end{aligned} \quad (49)$$

with

$$\begin{aligned}\mathbf{M} \triangleq\ & (\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2)^{-1}\Delta\mathbf{W}^{1/2} \\ & \times [\Delta\mathbf{W}^{1/2}(\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2)^{-1}\Delta\mathbf{W}^{1/2} + \mathbf{I}]^{-1} \\ & \times \Delta\mathbf{W}^{1/2}(\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2)^{-1} \geq \mathbf{0}.\end{aligned} \quad (50)$$

Then the first Riccati equation becomes

$$\begin{aligned}\mathbf{P}_{k+1}^1 &= \mathbf{AP}_k^1\mathbf{A}^T + \mathbf{V}_1 \\ &\quad - \mathbf{AP}_k^1\mathbf{C}^T(\mathbf{CP}_k^1\mathbf{C}^T + \mathbf{W}_2)^{-1}\mathbf{CP}_k^1\mathbf{A}^T\end{aligned} \quad (51)$$

with $\mathbf{V}_1 \triangleq \mathbf{V} + \mathbf{AP}_k^1\mathbf{C}^T\mathbf{MCP}_k^1\mathbf{A}^T \geq \mathbf{V}$ since $\mathbf{M} \geq \mathbf{0}$.
Comparing this with the second Riccati equation

$$\begin{aligned}\mathbf{P}_{k+1}^2 &= \mathbf{AP}_k^2\mathbf{A}^T + \mathbf{V} \\ &\quad - \mathbf{AP}_k^2\mathbf{C}^T(\mathbf{CP}_k^2\mathbf{C}^T + \mathbf{W}_2)^{-1}\mathbf{CP}_k^2\mathbf{A}^T\end{aligned} \quad (52)$$

yields

$$\mathbf{P}_{k+j}^1 \geq \mathbf{P}_{k+j}^2. \quad (53)$$

based on Lemma 1. □