On the Use of Copulas in Joint Chance-constrained Programming

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Abstract: In this paper, we investigate the problem of linear joint probabilistic constraints with normally distributed constraints. We assume that the rows of the constraint matrix are dependent, the dependence is driven by a convenient Archimedean copula. We describe main properties of the problem and show how dependence modeled through copulas translates to the model formulation. We also develop an approximation scheme for this class of stochastic programming problems based on second-order cone programming.

1 INTRODUCTION

Consider an *uncertain linear optimization problem*

$$\min c^T x \text{ subject to } \Xi x \leq h,\ x \in X \tag{1}$$

where $x \in X \subset \mathcal{R}^n$ is a decision vector of the problem, $\Xi \in \mathcal{R}^K \times \mathcal{R}^n$ is an uncertain (unknown) data matrix, $c \in \mathcal{R}^n$, $h = (h_1, \ldots, h_K)^T \in \mathcal{R}^K$ are fixed deterministic vectors, dimensions n, K are structural elements of the optimization problem (1). If a realization of the data element Ξ is known and fixed in advance (before a decision is taken), we can solve the problem (1) as classical linear optimization problem. This situation is rarely the case. More often, we have to consider *uncertainty of the data* as natural element of the modeling phase.

During the history of mathematical optimization, various methods were developed to deal with the uncertainty: ex-post sensitivity analysis, parametric programming, or robust optimization. In our paper, we concentrate on the *stochastic programming* approach assuming that the data matrix Ξ is a random matrix whose probabilistic characteristics are known in advance. For example, if the constraints of (1) are required to be satisfied with a prescribed sufficiently high probability $p \in [0;1]$, then the problem (1) can be reformulated as

$$\min c^T x \text{ subject to } \mathbb{P}\{\Xi x \leq h\} \geq p,\ x \in X \tag{2}$$

where $p \in [0;1]$ is a prescribed probability level. The problem (2) is known as *probabilistically* (or *chance*) *constrained linear optimization problem*. The problem was treated many times in literature; for a thorough review of methods and bibliography we refer to the classical book (Prékopa, 1995) and recent chapters (Prékopa, 2003) and (Dentcheva, 2009).

The chance constrained optimization problems are very challenging in their general (linear or nonlinear) form. Two main issues of the stochastic optimization theory concerning these problems are the *convexity* of the set of feasible solutions, and a very high *computational effort* to be accomplished. In detail: even for the "nice" linear program (2) the feasible set may be nonconvex, and the probability $\mathbb{P}$ can result in an intractable computation of multivariate integrals.

In our paper, we restrict our consideration to a *problem with linear normally distributed constraint rows*, namely, the rows Ξ_k^T of Ξ follow n-dimensional normal distributions with means μ_k and positive definite covariance matrices Σ_k. To further simplify the situation we assume that $X = \mathcal{R}^n$ (only the probabilistic constraints are in question). Denote

$$X(p) := \left\{x \in \mathcal{R}^n \mid \mathbb{P}\{\Xi x \leq h\} \geq p\right\}. \tag{3}$$

We are interested in an equivalent formulation of the set $X(p)$ convenient for numerical purposes. To this end, we first present a result for the set

$$M(p) := \left\{x \in \mathcal{R}^n \mid \mathbb{P}\{g_k(x) \geq \xi_k,\ k = 1, \ldots, K\} \geq p\right\}, \tag{4}$$

where $\xi := (\xi_1, \ldots, \xi_K)$ is an absolutely continuous random vector and $g_k(x)$ are continuous functions. $M(p)$ is usually referred to as the set of feasible solutions for a continuous *chance-constrained problem with random right-hand side*.

The convexity of the sets $X(p)$ and $M(p)$ is treated several times in the literature; we mention (Miller and Wagner, 1965), (Prékopa, 1971), (Jagannathan, 1974) as the first classical results, and (Henrion, 2007),

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(Henrion and Strugarek, 2008), (Prékopa et al., 2011) as recently published papers. These results are simplified either by restricting consideration to one-row problem only, or by assuming independence of matrix rows. In our paper we demonstrate the use of *copula theory* to deal with dependence of rows in (2). This was done first by (Henrion and Strugarek, 2011) for the set $M(p)$ using a class of so-called logexp-concave copulas. We extend their results to another large, more usual class of copulas and formulate an equivalent description of the problem (2) convenient to be solved by methods of second-order cone programming.

2 DEPENDENCE

2.1 Basic Facts about Copulas

Theory of copulas is well known for the people of probability theory and mathematical statistics but, to our knowledge, was not used up to these days in stochastic programming to describe the structure of the problem. In this section, we mention only some basic facts about copulas necessary for our following investigation. Most of the notions here (up to Proposition 2.7) were taken from the book (Nelsen, 2006).

Definition 2.1. A *copula* is the distribution function $C : [0;1]^K \to [0;1]$ of some K-dimensional random vector whose marginals are uniformly distributed on [0;1].

Proposition 2.2 (Sklar's Theorem)**.** *For any K-dimensional distribution function $F : \mathcal{R}^K \to [0;1]$ with marginals $F_1,\dots,F_K$, there exists a copula C such that*

$$\forall z \in \mathcal{R}^K \quad F(z) = C(F_1(z_1),\dots,F_K(z_K)). \qquad (5)$$

If, moreover, F_k are continuous, then C is uniquely given by

$$C(u) = F(F_1^{-1}(u_1),\dots,F_K^{-1}(u_K)). \qquad (6)$$

Otherwise, C is uniquely determined on $\operatorname{range} F_1 \times \dots \times \operatorname{range} F_K$.

Through Sklar's Theorem, we have in hand an efficient general tool for handling an arbitrary dependence structure. First, if we know the marginal distributions F_k together with the copula representing the dependence we can unambiguously determine the joint distribution. On the other hand, the copula can be uniquely derived from the knowledge of the joint and all marginal distributions. Our first example is the *independent (product) copula* which is nothing else than the independence formula for distribution functions:

$$C_\Pi(u) = \prod_k u_k. \qquad (7)$$

The second important example is the *Gaussian copula* which is given by Sklar's Theorem applied to a joint normal distribution and its normally distributed marginals:

$$C_\Sigma(u) = \Phi^\Sigma(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_K)) \qquad (8)$$

where Φ^Σ is the distribution function of the multivariate normal distribution with zero mean, unit variance and covariance matrix Σ, and $\Phi^{-1}(u_k)$ are standard one-dimensional normal quantiles. For illustration purposes, we provide a set of figures (Figures 1–5) of some popular copulas. From the left-hand side, the reader can always find the distribution function of the copula (i. e., the copula itself), its density, and the density of the distribution given by the copula applied to the standard normal marginals. Figure 1 represents the independent copula; compare it to the Gaussian copula in Figure 2. Note that the Gaussian copula is the *only* copula that can represent the joint normal distribution.

The following proposition provides the limits in which the copulas can be located.

Proposition 2.3 (The Fréchet-Hoeffding bounds)**.** *Every copula C satisfies the inequalities*

$$W(u) \le C(u) \le C_M(u) \qquad (9)$$

where

$$W(u) := \max\left\{\sum u_k - K + 1, 0\right\},$$
$$C_M(u) := \min_k\{u_k\}.$$

The function W represents the completely negative dependence between marginal distributions, but it is known *not* to be a copula if $K > 2$. C_M represents the completely positive dependence and it is known under the name of the *comonotone (maximum) copula*. These functions together with the independent copula are often found to be limiting cases of some other classes of copulas.

The Gaussian copula has a rather complicated structure (even it is not analytic) to be treated directly in our optimization problems. Instead, we need a different, simpler class of copulas, which we found in so-called Archimedean copulas.

Definition 2.4. A copula C is called *Archimedean* if there exists a continuous strictly decreasing function $\psi : [0;1] \to [0;+\infty]$, called *generator of C*, such that $\psi(1) = 0$ and

$$C(u) = \psi^{-1}\left(\sum_{i=1}^n \psi(u_i)\right). \qquad (10)$$

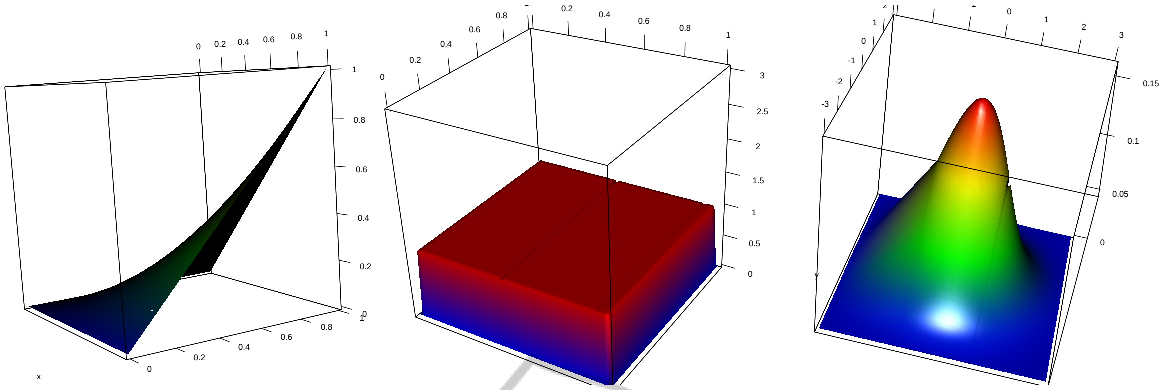

Figure 1: Independent copula: distribution, density, and density with standard normal marginals.

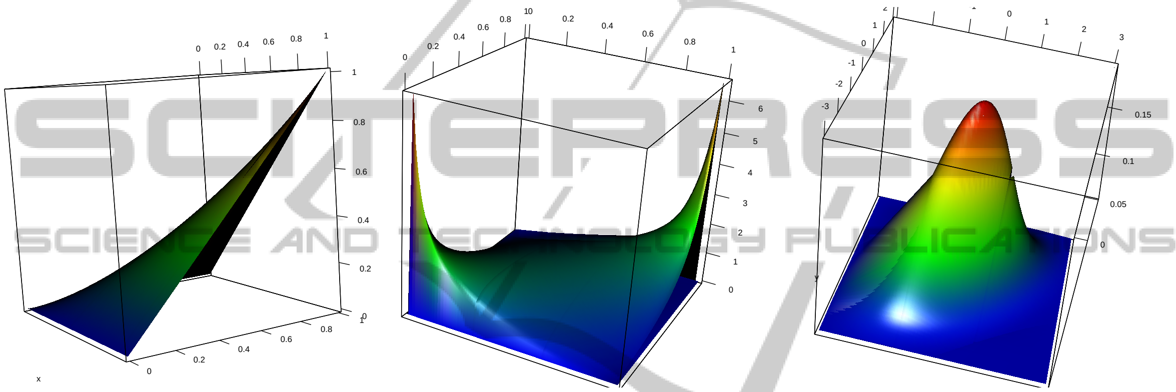

Figure 2: Gaussian copula ($\rho = 0.55$): distribution, density, and density with standard normal marginals.

If $\lim_{t\to 0}\psi(t) = +\infty$ then C is called a *strict Archimedean copula* and ψ is called a *strict generator*.

The inverse ψ^{-1} of a generator function is continuous and strictly decreasing on $[0;\psi(0)]$ (the value of $\psi(0)$ is defined as $+\infty$ if the copula is strict). Sometimes, ψ^{-1} is defined as the *generalized inverse* on the whole positive half-line $[0;+\infty)$ by setting $\psi^{-1}(s) = 0$ for $s \geq \psi(0)$ but such a definition is not needed through the context of our paper. To determine if some continuous strictly decreasing function ψ is a copula generator we introduce the following notion.

Definition 2.5. A real function $f : \mathcal{R} \to \mathcal{R}$ is called *completely monotonic* on an open interval $I \subseteq \mathcal{R}$ if it is nondecreasing, differentiable for each order k, and its derivatives alternate in sign, i. e.

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) \geq 0 \;\forall k = 0,1,\dots, \text{ and } \forall t \in I. \quad (11)$$

Proposition 2.6. *Let $\psi : [0;1] \to \mathcal{R}_+$ be a strictly decreasing function with $\psi(1) = 0$ and $\lim_{t\to 0}\psi(t) = +\infty$. Then it is a generator of a strict Archimedean copula for each dimension $K \geq 2$ if and only if ψ^{-1} is completely monotonic on $(0;+\infty)$.*

The extension of Proposition 2.6 given by (McNeil and Nešlehová, 2009) has the following corollary:

Proposition 2.7. *Any copula generator is convex.*

The Archimedean copulas are considered as a favorable and useful class of copulas due to their possibly simple formulation by a simple analytic function ψ and a small number of parameters (usually one or two). Many families adapted to concrete problem settings were already given in the literature; for example, the book (Nelsen, 2006) provides a table of 22 one-parameter families of Archimedean copulas. We give some examples in Table 1 and Figures 3–5. The Gumbel-Hougaard and Joe copulas are asymmetric (in the sense of density contours for normal marginals) stressing the dependence of positive random variables; the Clayton copula is in a similar view useful to model the positive dependence of negative random variables. The Frank copulas have symmetric density contours for normal marginals.

The Archimedean copulas provide an easy equivalent formulation for feasible sets (3) and (4). We start with the set $M(p)$; assume (for each $k = 1,\dots,K$) that the elements ξ_k of ξ have continuous distribution functions F_k, and the whole vector ξ has the joint distribution induced by a copula C. With these assump-

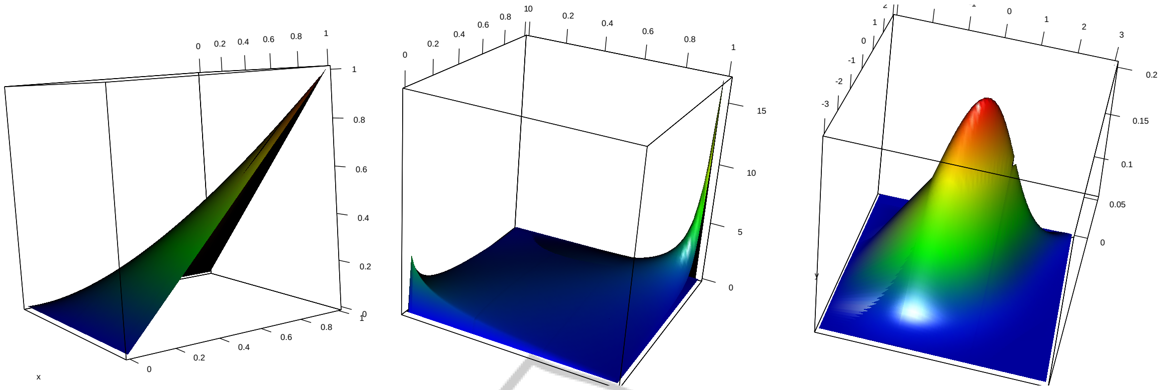

Figure 3: Gumbel-Hougaard copula ($\theta = 1.6$): distribution, density, and density with standard normal marginals.

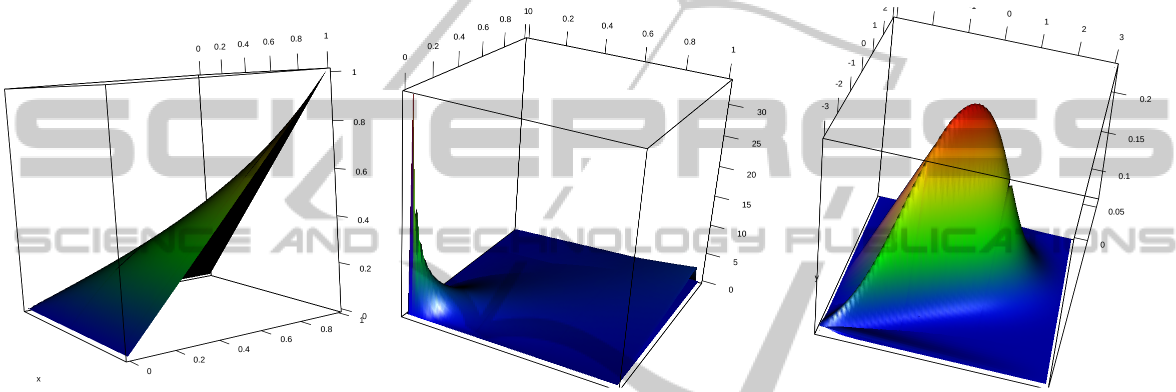

Figure 4: Clayton copula ($\theta = 1.8$): distribution, density, and density with standard normal marginals.

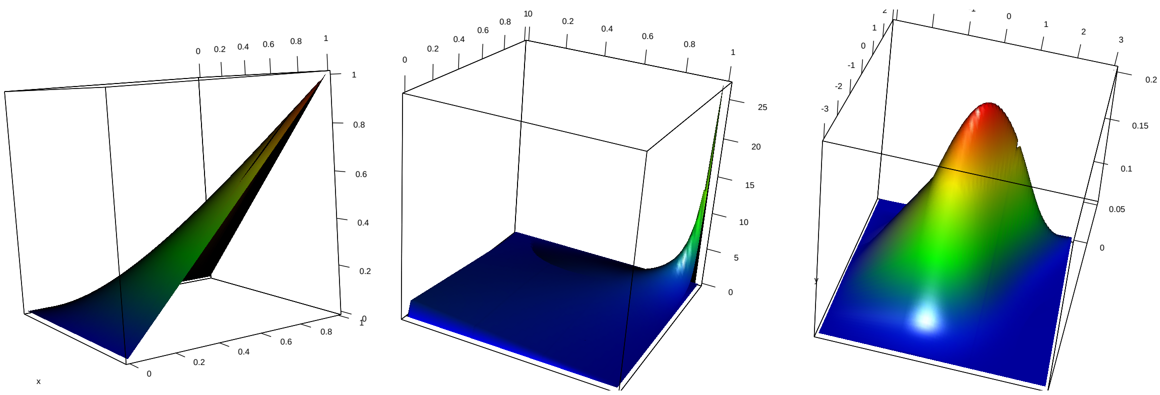

Figure 5: Joe copula ($\theta = 2.1$): distribution, density, and density with standard normal marginals.

tions, we can rewrite the set $M(p)$ as

$$M(p) = \{x \mid C(F_1(g_1(x)),\dots,F_K(g_K(x))) \geq p\} \quad (12)$$

and prove the following lemma.

Lemma 2.8. *If the copula C is Archimedean with a (strict or non-strict) generator ψ then*

$$M(p) = \Big\{x \mid \exists y_k \geq 0 : \psi[F_k(g_k(x))] \leq \psi(p) y_k \forall k,\ \sum_{k=1}^{K} y_k = 1\Big\}. \quad (13)$$

Proof. From basic properties of ψ it is easily seen that

$$\begin{aligned} M(p) &= \Big\{x \mid \psi^{-1}\Big(\sum_{k=1}^{K} \psi[F_k(g_k(x))]\Big) \geq p\Big\} \\ &= \Big\{x \in \mathcal{R}^n \mid \sum_{k=1}^{K} \psi[F_k(g_k(x))] \leq \psi(p)\Big\}. \end{aligned} \quad (14)$$

Assume that there exists nonnegative variables $y = (y_1,\dots,y_K)$ with $\sum_k y_k = 1$ such that (13) holds. Then the inequality in (14) can be easily obtained by sum-

Table 1: Selected Archimedean copulas with completely monotonic inverse generators.

Copula family	Param. θ	Gen. $\psi_\theta(t)$
Independent (product)	–	$-\ln t$
Gumbel-Hougaard	$\theta \geq 1$	$(-\ln t)^\theta$
Clayton	$\theta > 0$	$\frac{1}{\theta}(t^{-\theta} - 1)$
Joe	$\theta \geq 1$	$-\ln[1-(1-t)^\theta]$
Frank	$\theta > 0$	$-\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$

ming up all the inequalities in (13). The existence of such vector y for the case $p = 1$ is obvious; hence assume $p < 1$ and define

$$y_k := \frac{\psi\left[F_k(g_k(x))\right]}{\psi(p)} \text{ for } k = 1,\ldots,K-1,$$

$$y_K := 1 - \sum_{k=1}^{K-1} y_k.$$

It is now easy to verify that such definition of y_k satisfies (13). □

2.2 Introducing Normal Distribution

Return our consideration to the set $X(p)$ of the linear chance constrained problem defined by (3). Assume that the constraint rows Ξ_k^T have n-variate normal distributions with means μ_k and covariance matrices Σ_k. For $x \neq 0$ define

$$\xi_k(x) := \frac{\Xi_k^T x - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}, \quad g_k(x) := \frac{h_k - \mu_k^T x}{\sqrt{x^T \Sigma_k x}}. \quad (15)$$

The random variable $\xi_k(x)$ has one-dimensional standard normal distribution (in particular, this distribution is independent of x). Therefore the feasible set can be written as

$$X(p) = \left\{x \mid \mathbb{P}\left[\xi_k(x) \leq g_k(x)\ \forall k\right] \geq p\right\}. \quad (16)$$

If $K = 1$ (i. e., there is only one row constraint), the feasible set can be simply rewritten as

$$X(p) = \left\{x \mid \mu_1^T x + \Phi^{-1}(p)\sqrt{x^T \Sigma_1 x} \leq h_1\right\}. \quad (17)$$

where, again, Φ^{-1} is the one-dimensional standard normal quantile function. Introducing auxiliary variables y_k and applying Lemma 2.8, we derive the following lemma which gives us an equivalent description of the set $X(p)$ using the copula notion.

Lemma 2.9. *Suppose, in* (3)*, that* $\Xi_k^T \sim N(\mu_k, \Sigma_k)$ *(with appropriate dimensions) where* $\Sigma_k \succ 0$. *Then the feasible set of the problem* (2) *can be equivalently written as*

$$X(p) = \Big\{x \mid \exists y_k \geq 0 : \sum_k y_k = 1,$$

$$\mu_k{}^T x + \Phi^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \leq h_k\ \forall k\Big\} \quad (18)$$

where Φ *is the distribution function of a standard normal distribution and* ψ *is the generator of an Archimedean copula describing the dependence properties of the rows of the matrix* Ξ.

Proof. Straightforward using the arguments given above. The remaining case $x = 0$ is obvious. □

2.3 Convexity

It is not easy to show the convexity of the sets $M(p)$ and $X(p)$. We drop this question out of this paper and refer to (Houda and Lisser, 2013) where the convexity of both sets was proven under the conditions that

1. p is sufficiently high, and
2. ψ^{-1} is completely monotonic.

The proof was based on the theory presented in (Henrion and Strugarek, 2008) for the case of independence, and (Henrion and Strugarek, 2011) for the case of dependence modeled via logexp-concave copulas. Our approach is different and makes direct use of the convexity property of Archimedean generators. See the references above for details.

3 MAIN RESULT

3.1 Convex Reformulation

In Lemma 2.9 we have already stated an equivalent formulation of the feasible set $X(p)$. Together with the previous notice on convexity we can formulate the following theorem.

Theorem 3.1. *Consider the problem* (2) *where*

1. *the matrix* Ξ *has normally distributed rows* Ξ_k^T *with means* μ_k *and positive definite covariance matrices* Σ_k*;*
2. *the joint distribution function of* $\xi_k(x)$ *given by* (15) *is driven by an Archimedean copula with the generator* ψ.

Then the problem (2) *can be equivalently written as*

$$\begin{aligned} &\min c^T x \text{ subject to} \\ &\quad \mu_k{}^T x + \Phi^{-1}\left(\psi^{-1}(y_k \psi(p))\right)\sqrt{x^T \Sigma_k x} \leq h_k, \\ &\quad \sum_k y_k = 1 \\ &\quad x \in X,\ y_k \geq 0 \text{ with } k = 1,\ldots,K. \end{aligned} \quad (19)$$

Moreover, if

3. *the function* ψ^{-1} *is completely monotonic;*
4. $p > p^* := \Phi\left(\max\{\sqrt{3}, 4\lambda_{max}^{(k)}[\lambda_{min}^{(k)}]^{-3/2}\|\mu_k\|\}\right)$, *where* $\lambda_{max}^{(k)}, \lambda_{min}^{(k)}$ *are the largest and lowest eigenvalues of the matrices* Σ_k, *and* Φ *is the one-dimensional standard normal distribution function,*

then the problem is convex.

The value of the minimal probability level p^* was given by (Henrion and Strugarek, 2008) and it does not change for our dependent case.

3.2 SOCP Approximation

Second-order cone programming (SOCP) is a subclass of convex optimization in which the problem constraint set is the intersection of an affine linear manifold and the Cartesian product of second-order (Lorentz) cones (Alizadeh and Goldfarb, 2003). Formally, a constraint of the form

$$\|Ax+b\|_2 \le e^T x + f$$

is a second-order cone constraint as the affine function $(Ax+b, e^T x + f)$ is required to lie in the second-order cone $\{(y,t) \mid \|y\|_2 \le t\}$. The linear and convex quadratic constraints are nominal examples of second-order cone constraints. It is easy to see that the constraint (17) is SOCP constraint with $A := \Sigma_1$, $b := 0$, $e := -\frac{1}{\Phi^{-1}(p)}\mu_1$, and $f := \frac{1}{\Phi^{-1}(p)}h_1$ provided $p \ge \frac{1}{2}$. For a details about SOCP methodology we refer the reader to (Alizadeh and Goldfarb, 2003), and to the monograph (Boyd and Vandenberghe, 2004).

Theorem 3.1 provides us an equivalent nonlinear convex reformulation of the linear chance-constrained problem (2). Due to the decision variables y_k appearing as arguments to the (nonlinear) quantile function Φ^{-1}, it is not still a second-order cone formulation. To resolve this computational issue, we formulate a lower and upper approximation to the problem (19) using the favorable convexity property of the Archimedean generator. We first formulate an auxiliary convexity lemma which gives us a possibility to find these approximations.

Lemma 3.2. *If* $p > p^*$ *(given in Theorem 3.1), and* ψ *is a generator of an Archimedean copula, then the function*

$$y \mapsto H(y) := \Phi^{-1}\left(\psi^{-1}(y\psi(p))\right) \tag{20}$$

is convex on $[0;1]$.

Proof. The function $\psi^{-1}(\cdot)$ is a strictly decreasing convex function on $[0;\psi(0)]$ with values in $[p;1]$; the function $\Phi^{-1}(\cdot)$ is non-decreasing convex on $(p^*;1]$. Hence, the function $H(y)$ is convex. □

The proposed approximation technique follows the outline appearing in (Cheng and Lisser, 2012) and (Cheng et al., 2012). For both the approximations that follow, we consider a partition of the interval $(0;1]$ in the form $0 < y_{k1} < \ldots < y_{kJ} \le 1$ (for each variable y_k).

Remark 3.3. The number J of partition points can differ for each row index k but, to simplify the notation and without loss of generality, we consider this number to be the same for each index k.

3.2.1 Lower Bound: Piecewise Tangent Approximation

We approximate the function $H(y_k)$ using the first order Taylor approximation at each of the partition points; the calculated Taylor coefficients a_{kj}, b_{kj} translate into the formulation of the problem (19) as the linear and SOCP constraints with additional auxiliary nonnegative variables z^k and w^k. The convexity of $H(\cdot)$ ensures that the resulting optimal solution is a lower bound for the original problem.

Theorem 3.4. *Given the partition points* y_{kj}, *consider the problem*

$$\begin{aligned} \min c^T x & \text{ subject to} \\ & \mu_k{}^T x + \sqrt{z^{kT}\Sigma_k z^k} \le h_k, \\ & z^k \ge a_{kj}x + b_{kj}w^k \ (\forall k, \forall j) \\ & \sum_k w^k = x, \\ & w^k \ge 0,\ z^k \ge 0 \ (\forall k), \end{aligned} \tag{21}$$

where

$$a_{kj} := H(y_{kj}) - b_{kj}y_{kj},$$
$$b_{kj} := \frac{\psi(p)}{\phi(H(y_{kj}))\psi'\left(\psi^{-1}(y_{kj}\psi(p))\right)},$$

and ϕ *be the standard normal density. Then the optimal value of the problem* (21) *is a lower bound for the optimal value of the problem* (2).

Remark 3.5. The linear functions $a_{kj} + b_{kj}y$ are tangent to the (quantile) function H_k at the partition points; hence the origin of the name *tangent approximation*. This approximation leads to an outer bound for feasible solution set $X(p)$.

3.2.2 Upper Bound: Piecewise Linear Approximation

The line passing through the two successive partition points with their corresponding values $H(y_{kj})$ is an upper linear approximation of $H(y_k)$ between these two successive points. Taking pointwise maximum of these linear functions we arrive to an upper approximation of the function H, hence to an upper bound for the optimal value of the original problem.

Theorem 3.6. *Given partition points y_{kj}, consider the problem*

$$\begin{aligned} \min c^T x & \text{ subject to} \\ & {\mu_k}^T x + \sqrt{z^{kT}\Sigma_k z^k} \leq h_k, \\ & z^k \geq a_{kj}x + b_{kj}w^k \ (\forall k, j < J) \\ & \sum_k w^k = x, \\ & w^k \geq 0,\ z^k \geq 0\ (\forall k) \end{aligned} \tag{22}$$

where

$$a_{kj} := H(y_{kj}) - b_{kj}y_{kj},$$
$$b_{kj} := \frac{H(y_{k,j+1}) - H(y_{kj})}{y_{k,j+1} - y_{kj}}.$$

Then the optimal value of the problem (22) *is an upper bound for the optimal value of the problem* (2).

The last two problems are second-order cone programming problems and they are solvable by standard algorithms of SOCP. We do not provide further details in our paper; some promising numerical experiments were done by (Cheng and Lisser, 2012) for the problem with independent rows. If the dependence level is not too high (for example, if the parameter θ of the Gumbel-Hougaard copula approaches to one) the resulting approximation bounds are comparable to this independent case.

The second-order cone programming approach to solve chance-constrained programming problems opens a great variety of ways how to solve real-life problems. Many applications are modeled through chance-constrained programming: among them we can choose for example

- applications from finance: asset liability management, portfolio selection (covering necessary payments through an investment period with high probability),
- engineering applications in energy and other industrial areas (dealing with uncertainties in energy markets and/or weather conditions),
- water management (designing reservoir systems with uncertain stream inflows),
- applications in supply chain management, production planning, etc.

We refer the reader to the book (Wallace and Ziemba, 2005) for a diversified set of applications from these (and other) areas and for ideas how uncertainty is incorporated into the models by the stochastic programming approach. The method proposed in this paper shifts the research and open new possibilities as the constraint dependence is in fact a natural property of constraints involved in all mentioned domains.

4 CONCLUSIONS

In our paper, we have presented a way how copulas can be used to translate a known result for chance-constrained optimization problems with independent constraint rows to the case where the constraints exhibit some dependence. More specifically: if we assume that the dependence can be represented by a strict Archimedean copula with the completely monotonic generator inverse, the convexity of the feasible set is assured for sufficiently high values of p, and an equivalent deterministic formulation can be given. Furthermore, a lower and an upper bound for the optimal value of the problem can be calculated by introducing the piecewise tangent and piecewise linear approximations of the quantile function and by solving the associated second-order cone programming problems.

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