Investment Lags
A Numerical Approach

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Abstract: In this paper we use a mixture of numerical methods including finite difference and body fitted co-ordinates to form a robust stable numerical scheme to solve the investment lag model presented in the paper by Bar-Ilan and Strange (1996). This allows us to apply our methodology to models with different stochastic processes that does not have analytic solutions.

1 INTRODUCTION

Most investment projects take a long time to become operational so there are often periods where a firm will incur losses before the project starts generating income. Such a period might be referred to as the "construction lag", "time to build " or "Investment Lag" (Costeniuc et al., 2008). These investment lags can be quite lengthy which can result in a serious cost for the investor, an example of which is described by (MacRae, 1989) where it could take up to 10 years to see the positive income when investing in a power generating plant – similar situations can be found in investment projects on natural resources. For example, when an oil company buys a license from a government, it takes time to search fields and estimate the fields' reserve quantity before the beginning of oil production. Thus, when evaluating a project such as this the "lag" should be taken into consideration. If the sale price of a firm's product is modelled by a stochastic process, then the lag brings added risk to the project since the price may rise or fall during this lag, resulting in a negative cash flow. This situation and its effect on an investment has been studied by (Gauthier and Morellec, 2000) and they implied that it a has significant consequences on investment decisions.

The use of option theory to value and assess investment decisions has a long history going back to (Myers, 1977), but it was (Brennan and Schwartz, 1985) that first allowed the project to be mothballed rather than abandoned so that it could be reopened at a later date. They showed that if there was a fixed cost to move between the states, the decision to start the project would happen at a price higher than the decision to mothball. Following on from this, (Bar-Ilan and Strange, 1996) applied investment lags on irreversible investments and they found that a lag can reduce the effects of uncertainty in an investment, since the investor has more time to act on an unexpected fall in the price or changes in the investment. In order to generate the results for their model, they present an analytic technique,(see Brekke and Øksendal, 1994, for more details. The method as described by (Bar-Ilan and Strange, 1996) is flawed in that it relies on the particular form of the process, so they can only solve the problem with a simple geometric Brownian motion. The contribution of this paper is to apply a more generic numerical approach which can be extended to many classes of stochastic processes. We present a robust numerical technique for solving generic problems of this type.

2 MODEL FRAMEWORK

We follow the general framework as laid down by (Bar-Ilan and Strange, 1996) in valuing a firm that can pay (on delivery) $k \geq 0$ units to exercise an irreversible option to produce and sell 1 unit of product per unit time forever. The marginal cost of production is ω per unit, and both the future revenues and costs are discounted at the rate of ρ. The project can later be abandoned at a cost of $l \geq 0$. The price of the product P_t follows a standard geometric Brownian motion

$$\frac{dP}{P} = \mu dt + \sigma dz. \quad (1)$$

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where

- μ is the rate of return of the product P_t,
- σ is the volatility of the product price P_t,
- dz is the increment of the standard Wiener process.

When the investment starts, in many cases, it takes time from the decision to invest until the time the project begins to generate revenue. For example, if a firm wishes to build an oil refinery it will make that decision depending on today's oil price but it usually takes around $6-7$ years to start producing oil (Senate, 2002). We shall denote $h \geq 0$ to be the investment lag in our model.

As a result the firm at any one time may be in three different states. The states are characterised as

- $V_0(P)$ Inactive firm.

  In this stage there is no money invested and no revenue.
- $V_2(P,t)$ the firm in the process of construction where t is the clock that starts after the decision is made ($0 \leq t \leq h$).

  In this stage the firm has made the decision to invest and is waiting until time h when the firm will pay k and start production.
- $V_1(P)$ Active firm and generating revenue.

  In this state we have invested the amount k and it is working and generating the amount of P for each unit produced.

Our goal is to find at what price of the product should we invest and for what price should we leave the project. We shall denote P_H as the price at which it is high enough to start construction at a cost of $ke^{-\rho h}$ (discounted value of the payment at time $t=0$), and P_L as the price of product which is low enough to abandon the project for cost of l.

2.1 Calculating the Firm's Value at Different Situations

Suppose it is not optimal to invest at an infinitesimal period of dt, then

$$V_0(P_t) = e^{-\rho dt}\mathrm{E}_t\left[V_0(P_{t+dt})\right], \tag{2}$$

where P_t is the price at time t. Using Ito's Lemma we can write

$$\frac{\sigma^2}{2}P^2V_0''(P) + \mu PV_0'(P) - \rho V_0(P) = 0. \tag{3}$$

The boundary condition for $P=0$ is simply

$$\lim_{P\to 0} V_0(P) = 0, \tag{4}$$

and given that we optimally decide to invest the following must hold

$$V_0(P_H) = V_2(P_H) - ke^{-\rho h} \tag{5}$$

$$V_0'(P_H) = V_2'(P_H). \tag{6}$$

General solutions to the ODE in (3) can be found of the form

$$V_0(P) = BP^{\beta} \tag{7}$$

where B is a constant and β is the positive solution of the characteristic equation of

$$\frac{\sigma^2}{2}\xi(\xi-1) + \mu\xi - \rho = 0. \tag{8}$$

For $V_1(P)$, if we assume it is optimal to sell products over the next small period in time we have

$$V_1(P_t) = e^{-\rho dt}\mathrm{E}_t\left[V_1(P_{t+dt})\right] + \mathrm{E}_t\left[\int_t^{t+dt}(P_t-\omega)e^{-\rho(\tau-t)}d\tau\right],$$

where the extra term here is the total amount of profit from selling at the rate one product per unit time. We calculate the value of the active firm in the same manner as we did in the inactive case to arrive at

$$\frac{\sigma^2}{2}P^2V_1''(P) + \mu PV_1'(P) - \rho V_1(P) = \omega - P. \tag{9}$$

The boundary condition as $P\to\infty$ takes the form

$$\lim_{P\to\infty} V_1(P) = \lim_{P\to\infty}\frac{P}{\rho-\mu} - \frac{\omega}{\rho}. \tag{10}$$

Therefore, solutions of equation (9) can be written

$$V_1(P) = AP^{\alpha} + \frac{P}{\rho-\mu} - \frac{w}{\rho}, \tag{11}$$

where A is yet to be determined and α is the negative solution of equation (8). Since we can optimally decide to shut down operations we also have

$$V_1(P_L) = V_0(P_L) - l \tag{12}$$

$$V_1'(P_L) = V_0'(P_L). \tag{13}$$

These form the solution of the investment and disinvestment problem in Pindyck and Dixit (1996), where the time to build is not considered. For the investment lag problem, we must now consider the extra state of the firm $V_2(P,t)$ during the lag. Giving that we are waiting for production to start we can write

$$V_2(P_t,t) = e^{-\rho dt}\mathrm{E}_t\left[V_2(P_{t+dt},t+dt)\right], \tag{14}$$

and following standard procedure we obtain

$$\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma^2P^2\frac{\partial^2 V_2}{\partial P^2} + \mu P\frac{\partial V_2}{\partial P} - \rho V_2 = 0 \tag{15}$$

where the boundary conditions are

$$\lim_{P\to 0} V_2(P,t) = -le^{-\rho(h-t)}, \quad (16)$$

$$\lim_{P\to\infty} V_2(P,t) = \lim_{P\to\infty} \frac{Pe^{(\rho-\mu)(h-t)}}{\rho-\mu} - \frac{\omega e^{-\rho(h-t)}}{\rho}, \quad (17)$$

and

$$\lim_{t\to h} V_2(P,t) = \begin{cases} AP^{\alpha} + \frac{P}{\rho-\mu} - \frac{\omega}{\rho} & \text{if } P \geq P_L \\ BP^{\beta} - l & \text{if } \quad P < P_L \end{cases}.$$

These equations describe the three states of the firm, which all need to be solved to determine P_L and P_H.

3 NUMERICAL APPROACH

Although this problem has been solved in (Bar-Ilan and Strange, 1996) and later again by (Sødal, 2006), in this paper we present a new methodology which gives more flexibility to the practitioner. The idea is to solve the problem using finite differences with body-fitted co-ordinates to quickly solve for P_H and P_L. To simplify the algebra in the method we first apply a log transformation to the ODEs (3) and (9) and the PDE (15).

3.1 Derivation

To solve the problem we define two grids of x and y each of which have $m+1$ points. We apply log transforms to the equations involving V_1 and V_2 by setting

$$P = P_L e^y \Longrightarrow y = \ln\left(\frac{P}{P_L}\right), \quad (18)$$

The grid itself is generated from

$$\Delta y = \frac{y_{min} - y_{max}}{m}. \quad (19)$$

using the parameters

$$y_{min} = 0 \text{ and } y_{max} = 10\sigma\sqrt{h}. \quad (20)$$

Likewise, for equation involving V_0 we apply a log transform

$$P = P_H e^x \Longrightarrow x = \ln\left(\frac{P}{P_H}\right). \quad (21)$$

The grid becomes

$$\Delta x = \frac{x_{min} - x_{max}}{m}. \quad (22)$$

such that

$$x_{min} = -10\sigma\sqrt{h} \text{ and } x_{max} = 0. \quad (23)$$

First we can apply the log transform (18) to the equation of the active firm value (9) to get

$$\frac{\sigma^2}{2}\frac{\partial^2 V_1}{\partial y^2} + (\mu - \frac{\sigma^2}{2})\frac{\partial V_1}{\partial y} - \rho V_1 = \omega - P_L e^y. \quad (24)$$

Using the notation

$$v_1^i = V_1(P = P_L e^{i\Delta y}) = V_1(e^{y_i}) \quad (25)$$

we apply standard finite differencing and a Newton linearisation with

$$v_1^{i,k+1} \simeq v_1^{i,k} + \delta v_1^i \text{ and } P_L^{k+1} \simeq P_L^k + \delta P_L \quad (26)$$

where k is the number of iterations. The resulting scheme is given by

$$\left(\frac{\sigma^2}{2(\Delta y)^2} - \frac{2\mu - \sigma^2}{4\Delta y}\right)\delta v_1^{i-1} + \left(-\frac{\sigma^2}{2(\Delta y)^2} - \rho\right)\delta v_1^i \quad (27)$$

$$+\left(\frac{\sigma^2}{2\Delta y} + \frac{2\mu - \sigma^2}{4\Delta y}\right)\delta v_1^{i+1} + e^{y_i}\delta P_L = \mathcal{F}(P) \quad (28)$$

where

$$\mathcal{F}(P) = -\frac{1}{2}\sigma^2\frac{v_1^{i-1} - 2v_1^i + v_1^{i+1}}{2(\Delta y)^2} \quad (29)$$

$$-(\mu - \frac{1}{2}\sigma^2)\frac{v_1^{i+1} - v_1^{i-1}}{2\Delta y} + \rho v_1^i - P_L^i e^{y_i} + \omega.$$

Now for the smooth pasting boundary conditions we use a one sided difference of the form

$$V_1'(P = P_L) = \frac{-3(v_1^0 + \delta v_1^0) + 4(v_1^1 + \delta v_1^1) - (v_1^2 + \delta v_1^2)}{2\Delta y P_L} \quad (30)$$

and to calculate $V_0'(P_L)$ we use central differencing

$$V_0'(P_L) = \frac{V_0(P_L(1+\Delta y)) - V_0(P_L(1-\Delta y))}{2\Delta y P_L} \quad (31)$$

where the values of $V_0(P_L(1+\Delta y))$ and $V_0(P_L(1-\Delta y))$ must be interpolated. For the contact boundary condition at $P = P_L$ we expand with a Taylor series to get

$$\delta v_1^0 - V_0'(P_L)\delta P_L = V_0(P_L) - v_1^0 \quad (32)$$

Similarly, for the inactive or mothballed firm $V_0(P)$ defined in equation(3), we apply a log transformation on P_H with standard differencing and a Newton linearisation. The result is the same left hand side as in (28) with δv_1^i and δP_L replaced by δv_0^i and δP_H, and the right hand side is now given by

$$\mathcal{F}(P) = -\frac{1}{2}\sigma^2\frac{v_0^{i-1} - 2v_0^i + v_0^{i+1}}{2(\Delta x)^2} - \quad (33)$$

$$(\mu - \frac{1}{2}\sigma^2)\frac{v_0^{i+1} - v_0^{i-1}}{2\Delta x} + \rho v_0^i.$$

As before the boundary conditions become

$$\delta v_0^m - V_2'(P_H)\delta P_H = V_2(P_H) - ke^{-\rho h} - v_0^m \tag{34}$$

and

$$\frac{3\delta v_0^m - 4\delta v_0^{m-1} + \delta v_0^{m-2}}{2\Delta x P_H} = V_2'(P_H) - \frac{3v_0^m - 4v_0^{m-1} + v_0^{m-2}}{2\Delta x P_H} \tag{35}$$

where $V_2(P_H)$ here is $V_2(P_H, t=0)$ which must be calculated from (15).

We can use either quadrature integration (Andricopoulos et al., 2003) or finite difference to solve for V_2 in (15). For any point x of the n points on $V_2(x, t=0)$ we have

$$V_2(x,t=0) = A(x)\int_{-\infty}^{\infty} B(x,y)V_2(y,t=h)dy, \tag{36}$$

then we calculate the value of V_2 using

$$A(x) = \frac{1}{\sqrt{2\sigma^2\pi h}}e^{-\frac{1}{2}kx-\frac{1}{8}\sigma^2k^2h-\rho h}, \tag{37}$$

and

$$B(x,y) = e^{-(x-y)^2/2\sigma^2 h + 1/ky}, \tag{38}$$

and

$$k = \frac{2(\rho-d)}{\sigma^2} - 1 \tag{39}$$

where d is the dividends ($d = \rho - \mu$). The reason we calculate $V_2(y, t=0)$ using $V_2(y, t=h)$ that is because we solve the problem backwards in time where

$$V_2(y,t=h) = \begin{cases} V_1(y) & \text{if } e^y > 1 (P > P_L) \\ V_0(y) - l & \text{if } e^y \leq 1 (P \leq P_L) \end{cases}. \tag{40}$$

Given the fact we have applied a different transformations on V_0 and V_1 we must interpolate V_0 to get values in the y grid points using the relation

$$x = y_i + log(\frac{P_H}{P_L}). \tag{41}$$

We may use the asymptotic form of the solution to fill in the gaps outside the grid, then we write

$$V_2(x,t=0) = A(x)(\int_{y_{min}}^{y_{max}} B(x,y)V_2(y,t=h)dy + I_1 + I_3) \tag{42}$$

where

$$I_1 = -\int_{-\infty}^{y_{min}} B(x,y)le^h dy \tag{43}$$

and

$$I_3 = \int_{y_{max}}^{\infty} B(x,y)(\frac{e^{y-(\rho-\mu)h}}{\rho-\mu} - \frac{we^{-\rho h}}{\rho})dy \tag{44}$$

4 Cox-Ingersoll-Ross MODEL

In a novel extension to the problem, we set the process followed by the sale price as a Cox-Ingersoll-Ross (CIR) process. These sort of processes are often appropriate when modelling commodity prices as the price tends to a mean value over a long time scale. We can write the new price process as

$$dP = \kappa(\Phi - P)dt + \sigma\sqrt{P}dz \tag{45}$$

such that

- κ is the speed of reversion
- Φ : is the long term mean level

and σ and P are as defined previously. Now the equations of $V_0(P)$, $V_1(P)$ and $V_2(P)$ will become

$$\frac{\sigma^2}{2}PV_0''(P) + \kappa(\Phi - P)V_0'(P) - \rho V_0(P) = 0, \tag{46}$$

$$\frac{\sigma^2}{2}PV_1''(P) + \kappa(\Phi - P)V_1'(P) - \rho V_1(P) = w - P. \tag{47}$$

and

$$\frac{\partial V_2}{\partial t} + \frac{P}{2}\sigma^2\frac{\partial^2 V_2}{\partial P^2} + \kappa(\Phi - P)\frac{\partial V_2}{\partial P} - \rho V_2 = 0. \tag{48}$$

At $P = 0$ we solve the degenerate ODE for V_0

$$\kappa\Phi V_0' - \rho V_0 = 0 \tag{49}$$

and the degenerate PDE for V_2

$$\frac{\partial V_2}{\partial t} + \kappa\Phi\frac{\partial V_2}{\partial P} - \rho V_2 = 0 \tag{50}$$

For large P we set

$$V_1 \sim P^{-\frac{\rho}{\kappa}} + \frac{P}{\rho+\kappa} + \frac{\kappa\Phi}{\rho(\rho+\kappa)} - \frac{\omega}{\rho}, \text{ as } P \to \infty, \tag{51}$$

and assume a linear solution for V_2 so solve

$$\frac{\partial V_2}{\partial t} + \kappa\Phi\frac{\partial V_2}{\partial P} - \rho V_2 = 0. \tag{52}$$

The terminal condition for V_2 is as before given by

$$V_2(P,t=h) = \begin{cases} V_1(P) & \text{if } P \geq P_L \\ V_0(P) - l & \text{if } P < P_L \end{cases}$$

The smooth pasting conditions are the same as those defined in equations (5), (6) and (12), (13). We now transform the P-grid with a linear stretch get x and y grids

$$P = yP_L \text{ for } V_1(P) = V_1(yP_L)$$
$$P = xP_H \text{ for } V_0(P) = V_0(xP_H). \tag{53}$$

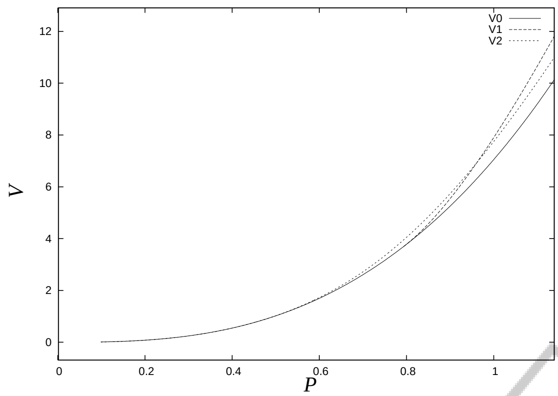

Figure 1: the Prices of V_0, V_1 and V_2 with $\sigma^2 = 0.1, \rho = 0.025, \mu = 0, \omega = 1., l = 0, k = 1, h = 6$ years grid size of 1000.

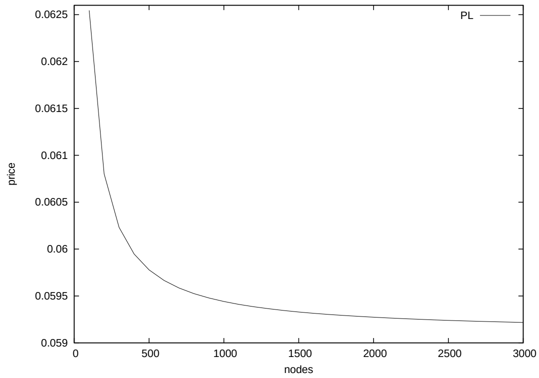

Figure 2: The convergence of P_L as we increase the nodes to calculate P_L using the numerical method with $\rho = 0.035, \mu = 0.5, \omega = 2., l = 3, k = 2, h = 4, \sigma^2 = 0.02$.

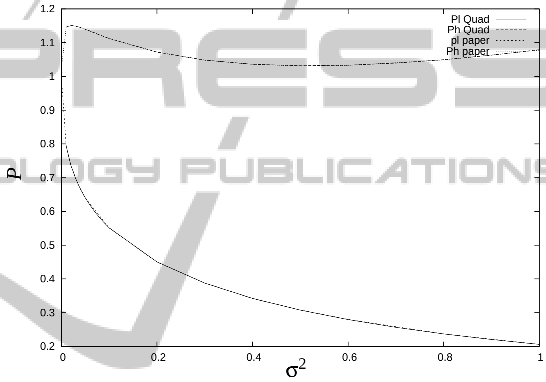

Figure 3: The values of P_L of P_H with $\mu = 0, \rho = .025, l = 0, k = 1, \omega = 1, \sigma^2 = 0.01$ using Algebraic equations vs the numerical method.

and

$$x \in [0,1]$$
$$y \in [1, y_{max}]. \tag{54}$$

Therefore, equation (47) will be transformed to

$$\frac{1}{2}\sigma^2 y V_1'' + \kappa(\Phi - yP_L)V_1' - \rho P_L V_1 = P_L(\omega - yP_L) \tag{55}$$

and (46) will be transformed to

$$\frac{1}{2}\sigma^2 x V_0'' + \kappa(\Phi - xP_H)V_0' - \rho P_H V_0 = 0. \tag{56}$$

We can follow the same method using a finite difference scheme with Newton linearisation. To calculate V_2, we must now solve the PDE using a Crank-Nicolson scheme since the kernel does not exist for this price process.

5 RESULTS

In figure 1, the switch from closed to in-construction ($V_0 \to V_2 - ke^{-\rho h}$) happens when $P_H = 1.14632$ which is the optimal price to start construction, while at $P_L = 0.793442$ the price is so low that it is not worth continuing production ($V_1 \to V_0 - l$). We can notice in this figure that P_H is higher than P_L, which is to be expected since we should only invest if the price is higher than the abandon price. Now we compare our results to those of (Sødal, 2006) in figure 3, and we find that our method generates values of P_L and P_H that are very close to the previous method. To demonstrate the integrity of our scheme, in figure 2 we plot the value of P_L for an increasing number of nodes. The convergence of the scheme can be shown empirically to be second order which matches with the finite difference methods used.

Next we show results for when we apply our method to the CIR process. In figure 4, we plot the values of P_L and P_H for varying values of h. We have noticed that the change in the values of P_H and P_L are not high as we increase h and that is because of the behaviour of the CIR process, since no matter how long is the lag period, the price will always return to the mean therefore the effect of P_L and P_H does not change significantly as we increase h, where both P_L and P_H decreases slowly, since as we increase h it is more likely to reach the mean. On the other hand, the effect of σ we can see in figure 5 on the prices of P_H and P_L is relatively higher than h, since as we increase the volatility, the price increases. Consequently the values P_L and P_H decreases, where as we increase σ significently, both P_L and P_H goes to zero because of the CIR property ($\sigma\sqrt{P}dt$). However, we believe that to model mean reversion process like some commodities, the price of the product P will return to the mean eventually. Therefore, the investor must not start the investment once it reaches the price

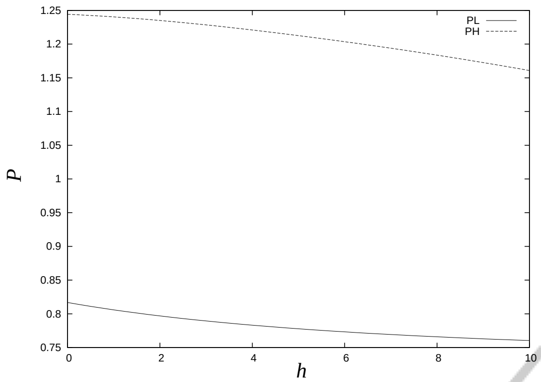

Figure 4: The values of P_L of P_H on CIR process using for different values of h with $\kappa = 0.01, \Phi = 1., \rho = .025, l = 0, k = 1, \omega = 1, \sigma = 0.1$.

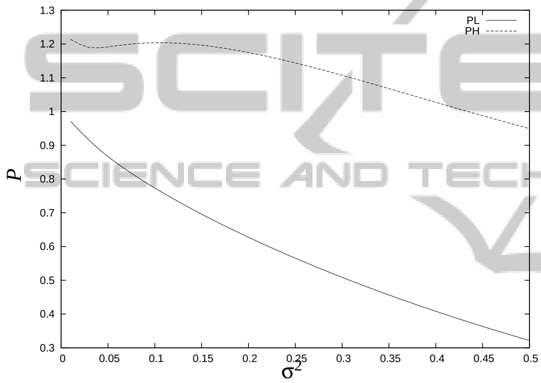

Figure 5: The values of P_L of P_H on CIR process using for different values of σ^2 with $\kappa = 0.01, \Phi = 1., \rho = .025, l = 0, k = 1, \omega = 1, h = 6$.

of P_H, before the price of the product stays around this price for a longer period. In otherwords, the investment lag model with mean reversion process should be modelled as Parisian option rather than European option as we did in this paper. The application of the Parisian option will be studied in future time.

6 CONCLUSIONS

In conclusion, we solved the problem for we have showed a new methodology using a mixture of Quadrature method and finite difference method with a body-fitted co-ordinate algorithm to solve an investment lag problem presented in (Bar-Ilan and Strange, 1996) and with a very high convergence rate and an acceptable speed of computing. Additionally, we have shown the the results presented in this paper are as accurate as the results presented in (Sødal, 2006) for the GBM process. Moreover, we have applied this model on other stochastic process such as CIR mean reversion process and have shown the results.

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