Optimal Feedback Control for a Perimeter Traffic Flow at an Urban Region

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Abstract: Traffic flow control has motivated many researchers since early decades of the 19th century. Recently, the concept of a perimeter traffic control for an urban region has been strengthened by a series of works, which have shown that a perimeter controller, located at a region border, can manipulate the transfer flows across the border to maximize the total outflow of the region. The macroscopic fundamental diagram (MFD), that relates average flow with accumulation, is used to model the traffic flow dynamics in the region. Assuming that the control inputs of the cross-border flows are coupled, i.e. the border is always utilized over time for transferring flows by one of the two directions (from and towards the region), and that the urban region has two traffic flow demands generated inside the region with internal and external destinations, and a generated traffic flow outside the region with a destination to the region, the explicit formulation of the optimal feedback control policy and a proof of optimality are provided. The proof is based on the modified Krotov-Bellman sufficient conditions of optimality, where the upper and lower bounds of state variables are calculated.

1 INTRODUCTION

In the last decade, network traffic flow modeling with the Macroscopic Fundamental Diagram (MFD) representation has intensively attracted the traffic flow and control researchers. The MFD simplifies the modelling task of the traffic flow dynamics for large-scale urban networks, as it provides aggregate relationships between traffic variables at an urban region.

The MFD provides a unimodal, low-scatter relationship between network vehicle density $(\mathrm{veh/km})$ or accumulation (veh) and network space-mean flow (outflow) $(\mathrm{veh/hr})$ for different network regions, if congestion is roughly homogeneous in the region. The physical model of the MFD was initially proposed by (Godfrey, 1969), but the theoretical elements for the existence of the MFD were provided later by (Daganzo, 2007). The MFD was first observed with dynamic features in congested urban network in Yokohama by (Geroliminis and Daganzo, 2008), and investigated using empirical or simulated data by (Buisson and Ladier, 2009; Ji et al., 2010; Mazloumian et al., 2010; Daganzo et al., 2011; Gayah and Daganzo, 2011; Zhang et al., 2013; Mahmassani et al., 1987; Olszewski et al., 1995) and others.

Homogeneous networks with small variance of link densities have *a well-defined* MFD (as illustrated in Fig. 1(a)), i.e. low scatter of flows for the same densities (or accumulations), (Geroliminis and Sun, 2011b; Mazloumian et al., 2010; Daganzo et al., 2011; Knoop et al., 2013; Mahmassani et al., 2013). Note that heterogeneous networks might not have well-defined MFDs, mainly in the decreasing part of the MFD, as the scatter becomes higher as accumulation increases and hysteresis phenomena has been found to exist (Daganzo et al., 2011; Buisson and Ladier, 2009; Saberi and Mahmassani, 2012; Geroliminis and Sun, 2011a). As a solution, these networks might be partitioned into more homogeneous regions with small variances of link densities, (Ji and Geroliminis, 2012). Note that the network topology, the signal timing plans of the signalized intersections inside the region, and the infrastructure characteristics affect the shape of the MFD, see e.g. (Geroliminis and Boyacı, 2012).

The MFD concept has been utilized to introduce control policies that aim at improving mobility and decreasing delays in large urban networks, (Daganzo, 2007; Haddad and Geroliminis, 2012; Geroliminis et al., 2013; Hajiahmadi et al., 2013; Haddad et al., 2013; Aboudolas and Geroliminis, 2013; Keyvan-Ekbatani et al., 2012; Knoop et al., 2012; Zhang et al.,

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2013). E.g. perimeter control strategies, i.e. manipulating the transfer flows at the perimeter border of the urban region, have been introduced for single-region cities in (Daganzo, 2007; Keyvan-Ekbatani et al., 2012; Shraiber and Haddad, 2014), and for multi-region cities in (Geroliminis et al., 2013; Haddad et al., 2013; Hajiahmadi et al., 2013; Aboudolas and Geroliminis, 2013). In this paper, we deal with perimeter control for a single urban region modelled by an MFD.

Different control approaches have been proposed to solve perimeter control problems for single-region cities. (Daganzo, 2007) has presented a bang-bang control as an optimal control policy for an urban region. A Proportional-Integrator (PI) perimeter controller has been designed for an urban region in (Keyvan-Ekbatani et al., 2012). The formulated nonlinear system is linearized around a priori known set-point chosen carefully within a value range in the uncongested regime of the MFD having positive slope and close to the critical state (total time spent) of the MFD function. The work in (Keyvan-Ekbatani et al., 2012) aims at regulating the dynamic system around the desired chosen set-point, at which the system's total time spent is minimized, in other words, the state reference is the same as the set-point. Moreover, the work in (Keyvan-Ekbatani et al., 2012) do not allow direct consideration of the control constraints, but impose them after the design process, e.g. adjusting or fine-tuning the controller gains.

In (Shraiber and Haddad, 2014), a robust perimeter controller has been designed for an urban region with the MFD representation. The designed controller is a fixed PI-controller with proportional K_{P} and integrator K_{I} gains, which stabilizes the linearized system against MFD and parameter uncertainties. The robust controller is also designed to handle control constraints within the design level in a systematic way. The results showed that the controller has performed well for the whole state set, and not necessary for a value range nearby a set-point.

In this paper, the optimal feedback control for a perimeter traffic flow at an urban region is derived, and a proof of optimality is provided with the help of the modified Krotov-Bellman sufficient conditions of optimality. The region is assumed to be a homogeneous region having a well-defined MFD with two traffic flow demands generated inside the region with internal and external destinations, and a generated traffic flow outside the region with a destination to the region.

2 OPTIMAL PERIMETER CONTROL: PROBLEM DEFINITION

This paper deals with a perimeter control problem for a homogeneous urban region having a well-defined MFD, schematically shown in Fig. 1. The flow dynamic equations for a homogeneous urban region have been already formulated in (Shraiber and Haddad, 2014), and they are briefly presented as follows. There are two state variables denoted by $n_{11}(t)$ and $n_{12}(t)$ (veh), which respectively represent the number of vehicles traveling in the region with destination inside and outside the region at time t. The total accumulated number of the vehicles in the region is $n_1(t) = n_{11}(t) + n_{12}(t)$. The MFD links the *accumulation*, $n_1(t)$, and *trip completion flow*, defined as the output flow of the region. The MFD provides a low-scatter relationship, if congestion is roughly homogeneous in the region. The MFD is denoted by $G_1\big(n_1(t)\big)$ (veh/s), and it is assumed to be *Lipschitz*, *continuous*, *non-negative*, and *unimodal*. This assumption is based on many simulation and empirical results, e.g. in (Geroliminis and Daganzo, 2008). The MFD is defined as the trip completion flow for the region at $n_1(t)$: (i) the sum of a transfer flow, i.e. trips from the region with external destination (outside the region), plus (ii) an internal flow, i.e. trips from the region with internal destination (inside the region). The transfer flow is calculated corresponding to the ratio between accumulations, i.e. $n_{12}(t)/n_1(t) \cdot G_1\big(n_1(t)\big)$, while the internal flow is calculated by $n_{11}(t)/n_1(t) \cdot G_1\big(n_1(t)\big)$.

The traffic flow demands generated in the region with internal and external destinations are respectively denoted by $q_{11}(t)$ and $q_{12}(t)$ (veh/s), while $q_{21}(t)$ (veh/s) denotes a generated traffic flow outside the region with destination to the region, as schematically shown in Fig. 1(b). Following (Shraiber and Haddad, 2014), a perimeter control is introduced on the border of the urban region, where its inputs are coupled $u(t)$ $(-)$ and $1-u(t)$ and control the ratios of flows, $0 \leq u(t) \leq 1$, that cross the border from inside to outside and from outside to inside the region at time t, respectively, see Fig. 1(b). It is also assumed that the perimeter control will not change the shape of the MFDs. Note also that the internal flow cannot be controlled or restricted.

The vehicle-conservation equations in the urban regions are given as follows (same equations (1) and (2) in (Shraiber and Haddad, 2014)):

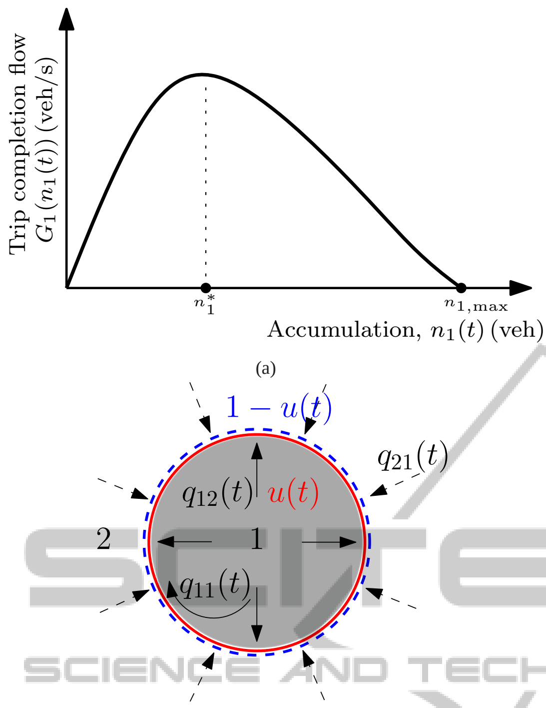

Figure 1: (a) A schematic MFD which is Lipschitz, continuous, non-negative, and unimodal function, (b) An urban region with three traffic demand $q_{11}(t)$, $q_{12}(t)$, $q_{21}(t)$, and a perimeter controller with inputs $u(t)$ and $1-u(t)$.

$$\frac{\mathrm{d}n_{11}(t)}{\mathrm{d}t} = q_{11}(t) + (1-u(t))\cdot q_{21}(t) - \frac{n_{11}(t)}{n_1(t)}\cdot G_1\big(n_1(t)\big)\,, \quad (1)$$

$$\frac{\mathrm{d}n_{12}(t)}{\mathrm{d}t} = q_{12}(t) - \frac{n_{12}(t)}{n_1(t)}\cdot G_1\big(n_1(t)\big)\cdot u(t)\,. \quad (2)$$

Let us now rewrite (1) to have a state equation corresponding to variable $n_1(t)$ instead of $n_{11}(t)$. The reason for that is only technical as this simplifies the mathematical proofs given later in Section 3. By summing (1) and (2) and substituting $n_{11}(t) = n_1(t) - n_{12}(t)$, one gets

$$\frac{\mathrm{d}n_1(t)}{\mathrm{d}t} = q_{11}(t) + q_{12}(t) + q_{21}(t) - \frac{n_1(t)-n_{12}(t)}{n_1(t)} \cdot G_1\big(n_1(t)\big) - \left(q_{21}(t) + \frac{n_{12}(t)}{n_1(t)}\cdot G_1\big(n_1(t)\big)\right)\cdot u(t)\,. \quad (3)$$

The optimal control problem aims at manipulating the control input $u(t)$ to optimize an objective J, subject to (2) and (3). There are a variety of criteria that can be chosen, e.g. the *throughput* of the transportation network and the total network *delay*. In this paper, the throughput of the transportation network in the region is chosen, which is defined as the total number of vehicles that complete their trips and reach their destination during the time interval $[t_0, t_{\mathrm{f}}]$, i.e.

$$J_1 = \int_{t_0}^{t_{\mathrm{f}}} G_1\big(n_1(t)\big)\,\mathrm{d}t\,, \quad (4)$$

where t_0 and t_{f} (s) are the starting and final times of the control process. Note that the defined problem here is an optimal control problem, while in (Shraiber and Haddad, 2014) the problem is defined as a regulator control problem.

3 OPTIMAL CONTROL DESIGN

The MFD function $G_1\big(n_1(t)\big)$ is assumed to be unimodal with a single maximum value at n_1^* (veh), see also Fig. 1(a), i.e.

$$n_1^* = \underset{n_1(t)}{\operatorname{argmax}}\, G_1\big(n_1(t)\big)\,. \quad (5)$$

Let us denote $u_{\mathrm{ss}}(t)$ $(-)$ as the steady-state control input, which corresponds to a steady-state condition $\mathrm{d}n_1/\mathrm{d}t = 0$, at $n_1(t) = n_1^*$, i.e. the steady state control input $u_{\mathrm{ss}}(t)$ is calculated from (3) with $\mathrm{d}n_1/\mathrm{d}t = 0$ and $n_1(t) = n_1^*$, as follows:

$$u_{\mathrm{ss}}(t) = \left[q_{11}(t) + q_{12}(t) + q_{21}(t) - \frac{n_1^* - n_{12}(t)}{n_1^*} \cdot G_1(n_1^*)\right] \Big/ \left[q_{21}(t) + \frac{n_{12}(t)}{n_1^*}\cdot G_1(n_1^*)\right]. \quad (6)$$

Note that $u_{\mathrm{ss}}(t)$ is a time dependent corresponding to the traffic demands $q_{11}(t)$, $q_{12}(t)$, and $q_{21}(t)$.

Theorem 3.1. *The optimal feedback control $u^*(n_1)$ for the problem P1: $\max_{u(t)} J_1$ subject to (2) and (3) is as follows:*
If $n_1(t) \neq n_1^*$, then

$$u^*(n_1) = \begin{cases} 0 & \forall n_1(t) < n_1^*\,, \\ 1 & \forall n_1(t) > n_1^*\,, \end{cases} \quad (7)$$

otherwise ($n_1(t) = n_1^*$),

$$u^*(n_1^*) = \begin{cases} u_{\mathrm{ss}}(t) & 0 \leq u_{\mathrm{ss}}(t) \leq 1\,, \\ 0 & u_{\mathrm{ss}}(t) < 0\,, \\ 1 & u_{\mathrm{ss}}(t) > 1\,. \end{cases} \quad (8)$$

Proof. The proof is based on the *unimodality* assumption of the function $G_1(n_1(t))$. The unimodal function $G_1(n_1(t))$ has the point-wise maximum for each point t at $n_1(t) = \overline{n}_1(t)$ if $n_1(t) \leq n_1^*$ and $\overline{n}_1(t)$ is an upper bound of $n_1(t)$. Similarly, $G_1(n_1(t))$ has the point-wise minimum for each point t at $n_1(t) = \underline{n}_1(t)$ if $n_1(t) \geq n_1^*$ and $\underline{n}_1(t)$ is a lower bound of $n_1(t)$. The upper bound $\overline{n}_1(t)$ and the lower bound $\underline{n}_1(t)$ have to be found with respect to the dynamical state equations (2) and (3) with initial conditions $n_1(t_0) = n_{1,0}$, $n_{12}(t_0) = n_{12,0}$ (see Lemmas 3.3 and 3.4). To complete the proof we shall use the sufficient global optimality conditions in the form of modified Krotov-Bellman conditions (see sections 3.1 and 3.2) and prove Lemmas 3.2, 3.3, and 3.4 in Section 3.3. □

According to (8), the optimal feedback control for $n_1(t) = n_1^*$ is $u^*(n_1^*) = u_{\text{ss}}(t)$, where $u_{\text{ss}}(t)$ has to satisfy the control constraint $0 \leq u_{\text{ss}}(t) \leq 1$. However, if $u_{\text{ss}}(t) < 0$ or $u_{\text{ss}}(t) > 1$, then the optimal feedback control is respectively $u^*(n_1^*) = 0$ or $u^*(n_1^*) = 1$. Note that in the latter two cases the state cannot be kept at n_1^*, since if $u_{\text{ss}}(t) < 0$ the variable $n_1(t)$ will decrease from $n_1(t) = n_1^*$ even with all feasible $u(t)$, but the minimum decrease is achieved by $u(t) = 0$, and if $u_{\text{ss}}(t) > 1$ then $n_1(t)$ will increase for all feasible $u(t)$, but the minimum increase is achieved by $u(t) = 1$. This is explained as follows. Let us first respectively denote $a(t)$ and $b(t)$ as follows:

$$a(t) = q_{11}(t) + q_{12}(t) + q_{21}(t) - \frac{n_1^* - n_{12}(t)}{n_1^*} \cdot G_1(n_1^*), \tag{9}$$

$$b(t) = q_{21}(t) + \frac{n_{12}(t)}{n_1^*} \cdot G_1(n_1^*), \tag{10}$$

then, (3) is rewritten as

$$\frac{\mathrm{d}n_1(t)}{\mathrm{d}t} = a(t) - b(t) \cdot u(t) \tag{11}$$

where $b(t) \geq 0$, and $a(t) - b(t) \cdot u_{\text{ss}}(t) = 0$. Therefore, if $u_{\text{ss}}(t) < 0 \leq u(t)$ then $\mathrm{d}n_1/\mathrm{d}t = a(t) - b(t) \cdot u(t) < a(t) - b(t) \cdot u_{\text{ss}}(t) = 0$, and if $u_{\text{ss}}(t) > 1 \geq u(t)$ then $\mathrm{d}n_1/\mathrm{d}t = a(t) - b(t) \cdot u(t) > a(t) - b(t) \cdot u_{\text{ss}}(t) = 0$.

3.1 Modified Krotov-Bellman Sufficient Conditions of Optimality

The Krotov-Bellman sufficient conditions of optimality are summarized as follows. The reader can refer to (Krotov, 1996) for further information. Given a dynamic system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \tag{12}$$

with state variables $\mathbf{x}(t)$, control inputs $\mathbf{u}(t)$, initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, and the following objective function

$$\min J = \int_{t_0}^{t_\text{f}} f_0(t, \mathbf{x}, \mathbf{u}) \mathrm{d}t, \tag{13}$$

one can construct a function $R(t, \mathbf{x}, \mathbf{u})$ as follows

$$R(t, \mathbf{x}, \mathbf{u}) = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) - f_0(t, \mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial t}, \tag{14}$$

where $V(t, \mathbf{x})$ is assumed to be a continuous and differentiable function. Taking into account that the full time derivative of $V(t, \mathbf{x})$ with respect to (12) is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(t, \mathbf{x}, \mathbf{u}) + \frac{\partial V}{\partial t}, \tag{15}$$

and substituting (15) and (14) in (13), one gets

$$J = V(t_\text{f}, \mathbf{x}(t_\text{f})) - V(t_0, \mathbf{x}_0) - \int_{t_0}^{t_\text{f}} R(t, \mathbf{x}, \mathbf{u}) \mathrm{d}t. \tag{16}$$

The sufficient Krotov-Bellman conditions of optimality are as follows: if there exists a pair $(\mathbf{x}^*, \mathbf{u}^*)$ such that $\mathbf{x}^*$ is the solution of the dynamic system

$$\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}t} = \mathbf{f}(t, \mathbf{x}^*, \mathbf{u}^*) \tag{17}$$

over the time interval $[t_0, t_\text{f}]$, and the following properties

$$\begin{aligned} \mathbf{u}^* &= \operatorname{argsup}_{\mathbf{u}} R(t, \mathbf{x}, \mathbf{u}), \\ R(t, \mathbf{x}, \mathbf{u}^*) &= \mu(t), \\ \Theta = V(t_\text{f}, \mathbf{x}(t_\text{f})) &= \text{Constant}, \end{aligned} \tag{18}$$

hold, then this pair $(\mathbf{x}^*, \mathbf{u}^*)$ is a global optimum solution. Note that $\mu(t)$ is any measurable bounded function of t. According to these sufficient conditions of optimality, the problem is reduced to a solution of the nonlinear Krotov-Bellman PDE for the function $V(t, \mathbf{x})$.

In this paper, the *modified* Krotov-Bellman conditions are proposed, where the maximization of $R(t, \mathbf{x}, \mathbf{u})$ over $\mathbf{u}$ in (18) is replaced by the maximization of $R(t, \mathbf{x}, \mathbf{u})$ over $\mathbf{x}$, as follows:

$$\begin{aligned} \mathbf{x}^* &= \operatorname{argsup}_{\mathbf{x}} R(t, \mathbf{x}, \mathbf{u}), \\ R(t, \mathbf{x}^*, \mathbf{u}) &= \mu(t), \\ \Theta = V(t_\text{f}, \mathbf{x}(t_\text{f})) &= \text{Constant}. \end{aligned} \tag{19}$$

Note that in both variants (18) and (19), the resulting function $R(t, \mathbf{x}, \mathbf{u})$ after maximization, i.e. $R(t, \mathbf{x}, \mathbf{u}^*)$ and $R(t, \mathbf{x}^*, \mathbf{u})$, respectively, will be a function of time t only.

3.2 Application to the Maximum throughput Objective

Applying the modified Krotov-Bellman conditions to the problem (2)–(4), one gets

$$\sup_{n_1(t),n_{12}(t)}\Bigg\{\frac{\partial V}{\partial n_1}\Bigg[q_{11}(t)+q_{12}(t)+q_{21}(t) - \frac{n_1(t)-n_{12}(t)}{n_1(t)}G_1(n_1(t)) - \left(q_{21}(t)+\frac{n_{12}(t)}{n_1(t)} \cdot G_1(n_1(t))\right)\cdot u(t)\Bigg] + \frac{\partial V}{\partial n_{12}(t)}\Bigg[q_{12}(t)-\frac{n_{12}(t)}{n_1(t)} \cdot G_1(n_1(t))\cdot u(t)\Bigg] + G_1(n_1(t)) + \frac{\partial V}{\partial t}\Bigg\} = \mu(t)\,, \tag{20}$$

$$\Theta = V(t_{\mathrm{f}}, n_1(t_{\mathrm{f}}), n_{12}(t_{\mathrm{f}})) = \text{Constant}\,. \tag{21}$$

Note that (20) and (21) are without taking into account the upper and lower bound constraints on state variable $n_1(t)$, i.e.

$$\overline{g}_1(t) = n_1(t) - \overline{n}_1(t) \le 0\,, \tag{22}$$

$$\underline{g}_1(t) = \underline{n}_1(t) - n_1(t) \le 0\,. \tag{23}$$

Let $\overline{\lambda}_1$ and $\underline{\lambda}_1$ be the Lagrange multipliers for (22) and (23), respectively. Now, let us choose $V(t,n_1(t),n_{12}(t)) = C$, where C is a constant. Then, imposing state constraints (22) and (23) on (20) and (21) with $V(t,n_1(t),n_{12}(t)) = C$, one gets

$$\sup_{n_1(t)}[G_1(n_1(t)) - \overline{\lambda}_1\cdot\overline{g}_1(t) - \underline{\lambda}_1\cdot\underline{g}_1(t)] = \mu(t)\,, \tag{24}$$

$$\Theta = C\,. \tag{25}$$

Note that Lagrange multipliers are taken with sign minus because of maximization of J_1. According to Karush-Kuhn-Tucker (KKT) conditions all Lagrange multipliers are non-negative and may have positive values only when the corresponding constraint is binding, i.e. non-redundant. From maximization of (24), one gets

$$\frac{\partial G_1}{\partial n_1} - \overline{\lambda}_1 + \underline{\lambda}_1 = 0\,. \tag{26}$$

This implies that $n_1(t) = \overline{n}_1(t)$ when $\frac{\partial G_1}{\partial n_1} > 0$, i.e. when $n_1(t) < n_1^*$ (because $G_1\big(n_1(t)\big)$ is assumed to be unimodal with a single maximum value at n_1^*), and $n_1(t) = \underline{n}_1(t)$ when $\frac{\partial G_1}{\partial n_1} < 0$, i.e. when $n_1(t) > n_1^*$. Note that Lagrange multipliers for upper and lower constraints cannot be non-zero simultaneously, and because of unimodality the only point when $\frac{\partial G_1}{\partial n_1} = 0$ is the point $n_1(t) = n_1^*$.

3.3 Upper and Lower Bounds of State Variable $n_1(t)$

Recall that to complete the proof of Theorem 3.1, the upper bound $\overline{n}_1(t)$ and the lower bound $\underline{n}_1(t)$ have to be found with respect to the dynamical state equations (2) and (3) with initial conditions $n_1(t_0) = n_{1,0}$, $n_{12}(t_0) = n_{12,0}$. In this section, it is explained how to achieve the upper bound $\overline{n}_1(t)$ and the lower bound $\underline{n}_1(t)$, according to Lemmas 3.3 and 3.4, respectively. But first Lemma 3.2 is presented, which is taken from (Krotov et al., 1971) and utilized to prove Lemmas 3.3 and 3.4.

Lemma 3.2. *Consider an ODE system* $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x},\mathbf{u},t)$, *where* $\mathbf{f}(\mathbf{x},\mathbf{u},t)$ *is Lipschitz and continuous vector-function,* $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$ *are the state variables, and* $\mathbf{u}(t)$ *are measurable bounded control inputs. The upper bound of the solution* $\mathbf{x}(t)$ *with initial conditions* $\mathbf{x}(t_0) = \mathbf{x}_0$ *is denoted as* $\overline{\mathbf{x}}(t)$. *Each component i of this bound can be calculated according to the following equation*

$$\frac{\mathrm{d}\overline{x}_i}{\mathrm{d}t} = \sup_{\mathbf{u},x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n} f_i(x_1,x_2,\ldots,x_{i-1},\overline{x}_i,x_{i+1},\ldots,x_n,\mathbf{u},t)\,, \tag{27}$$

with the initial condition $\overline{x}_i(t_0) = x_{i,0}$, *where* $f_i(\cdot)$ *is the i-component of the vector-function* $\mathbf{f}(\cdot)$. *The lower bound* $\underline{x}_i(t)$ *can be calculated in the same way by just replacing* sup *by* inf *in (27).*

Proof. The proof is a straightforward, however, it is not presented in this paper. The reader can refer to (Krotov et al., 1971) for an explicit proof. □

Lemma 3.3. *The upper bound for state variable* $n_1(t)$ *is achieved with control input* $u(t) = 0$.

Proof. Let us start with state variable $n_{12}(t)$. From (2), it is clear that the second term in the right-hand side is a non-positive and in particular it is equal to zero for $u(t) = 0$. Therefore, the supremum of the right-hand side will be achieved for $u(t) = 0$ and it depends only on t. According to Lemma 3.2, an upper bound $\overline{n}_{12}(t)$ is achieved with $u(t) = 0$. Now, let us consider state variable $n_1(t)$. From (3) one can see that the supremum over $u(t)$ is achieved for $u(t) = 0$, then the right-hand side is a function of t and the variables $n_1(t)$, $n_{12}(t)$. It has been shown that with $u(t) = 0$ from (2) it follows that $n_{12}(t) = \overline{n}_{12}(t)$. After substitution $n_{12}(t) = \overline{n}_{12}(t)$ in the right-hand side of (3), it follows according to Lemma 3.2 that an upper bound $n_1(t) = \overline{n}_1(t)$ is obtained for $u(t) = 0$. □

Lemma 3.4. *The lower bound for state variable* $n_1(t)$ *is achieved with control input* $u(t) = 1$.

Proof. The infimum over $u(t)$ of the right-hand side of (3) is achieved for $u(t) = 1$. Substituting $u(t) = 1$ into (3), one gets

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = q_{11}(t) + q_{12}(t) - G_1(n_1(t))\,. \tag{28}$$

It follows from Lemma 3.2 that the solution of (28) is a lower bound $\underline{n}_1(t)$. □

Lemmas 3.3 and 3.4 complete the proof of Theorem 3.1. One can add that if at some point in time $t = t_s$, the optimal trajectory will go left or right from the value $n_1(t_s) = n_1^*$, then we can split the problem into two pieces (namely from $t = t_0$ to $t = t_s$ and from $t = t_s$ to $t = t_f$) and build new upper or lower bounds respectively from the initial point $n_{12}(t_s)$ and $n_1(t_s) = n_1^*$.

4 CONCLUSIONS

The analytical solution for the optimal perimeter feedback control with the maximum throughput criterion in an urban region has been derived and described. The modified Krotov-Bellman sufficient conditions of optimality have been utilized for the proof of optimality. The resulting optimal control policy is oriented to keep the state variable, i.e. the total number of the moving vehicles in the region, as close as possible to the critical accumulation, n_1^*, where the MFD value is maximized. Though this optimal solution is intuitively expected here it is rigorously proven. The numerical simulations and comparison with existing practices will be done in consequent papers.

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REFERENCES

Aboudolas, K. and Geroliminis, N. (2013). Perimeter and boundary flow control in multi-reservoir heterogeneous networks. *Transportation Research Part B*, 55:265–281.

Buisson, C. and Ladier, C. (2009). Exploring the impact of homogeneity of traffic measurements on the existence of macroscopic fundamental diagrams. *Transportation Research Record*, 2124:127–136.

Daganzo, C. F. (2007). Urban gridlock: Macroscopic modeling and mitigation approaches. *Transportation Research Part B*, 41(1):49–62.

Daganzo, C. F., Gayah, V. V., and Gonzales, E. J. (2011). Macroscopic relations of urban traffic variables: Bifurcations, multivaluedness and instability. *Transportation Research Part B*, 45(1):278–288.

Gayah, V. V. and Daganzo, C. F. (2011). Clockwise hysteresis loops in the macroscopic fundamental diagram: An effect of network instability. *Transportation Research Part B*, 45(4):643–655.

Geroliminis, N. and Boyacı, B. (2012). The effect of variability of urban systems characteristics in the network capacity. *Transportation Research Part B*, 46(10):1607–1623.

Geroliminis, N. and Daganzo, C. F. (2008). Existence of urban-scale macroscopic fundamental diagrams: some experimental findings. *Transportation Research Part B*, 42(9):759–770.

Geroliminis, N., Haddad, J., and Ramezani, M. (2013). Optimal perimeter control for two urban regions with macroscopic fundamental diagrams: A model predictive approach. *IEEE Transactions on Intelligent Transportation Systems*, 14(1):348–359.

Geroliminis, N. and Sun, J. (2011a). Hysteresis phenomena of a macroscopic fundamental diagram in freeway networks. *Transportation Research Part A*, 45(9):966–979.

Geroliminis, N. and Sun, J. (2011b). Properties of a well-defined macroscopic fundamental diagram for urban traffic. *Transportation Research Part B*, 45(3):605–617.

Godfrey, J. W. (1969). The mechanism of a road network. *Traffic Engineering and Control*, 11(7):323–327.

Haddad, J. and Geroliminis, N. (2012). On the stability of traffic perimeter control in two-region urban cities. *Transportation Research Part B*, 46(1):1159–1176.

Haddad, J., Ramezani, M., and Geroliminis, N. (2013). Cooperative traffic control of a mixed network with two urban regions and a freeway. *Transportation Research Part B*, 54:17–36.

Hajiahmadi, M., Haddad, J., Schutter, B. D., and Geroliminis, N. (2013). Optimal hybrid macroscopic traffic control for urban regions: Perimeter and switching signal plans controllers. In *European Control Conference 13*.

Ji, Y., Daamen, W., Hoogendoorn, S., Hoogendoorn-Lanser, S., and Qian, X. (2010). Macroscopic fundamental diagram: Investigating its shape using simulation data. *Transportation Research Record*, 2161:42–48.

Ji, Y. and Geroliminis, N. (2012). On the spatial partitioning of urban transportation networks. *Transportation Research Part B*, 46(10):1639–1656.

Keyvan-Ekbatani, M., Kouvelas, A., Papamichail, I., and Papageorgiou, M. (2012). Exploiting the fundamnetal diagram of urban networks for feedback-based gating. *Transportation Research Part B*, 46(10):1393–1403.

Knoop, V., Hoogendoorn, S., and van Lint, H. (2013). The impact of traffic dynamics on the macroscopic fundamental diagram. In *92nd Annual Meeting of Transportation Research Board*, Washington D.C., USA.

Knoop, V. L., Hoogendoorn, S. P., and Van Lint, J. W. C. (2012). Routing strategies based on the macroscopic fundamental diagram. *Transportation Research Record*, 2315:1–10.

Krotov, V. F. (1996). *Global methods in optimal control theory*. M. Dekker, NY, USA.

Krotov, V. K., Burkreev, V. Z., and Gurman, V. I. (1971). *New Variational Methods In Flight Dynamics*. Coronet Books.

Mahmassani, H., Williams, J., and Herman, R. (1987). Performance of urban traffic networks. In Gartner, N. and Wilson, N., editors, *Proceedings of the 10th International Symposium on Transportation and Traffic Theory*, Amsterdam, The Netherlands. Elsevier.

Mahmassani, H. S., Saberi, M., and Zockaie, A. K. (2013). Network gridlock: Theory, characteristics, and dynamic. In *Procedia - Social and Behavioral Sciences*, volume 80, pages 79–98. doi: 10.1016/j.sbspro.2013.05.007 20th International Symposium on Transportation and Traffic Theory.

Mazloumian, A., Geroliminis, N., and Helbing, D. (2010). The spatial variability of vehicle densities as determinant of urban network capacity. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1928):4627–4647.

Olszewski, P., Fan, H. S. L., and Tan, Y.-W. (1995). Area-wide traffic speed-flow model for the Singapore CBD. *Transportation Research Part A*, 29A(4):273–281.

Saberi, M. and Mahmassani, H. (2012). Exploring properties of network-wide flow-density relations in a freeway network. In *Transportation Research Board 91st Annual Meeting*, Washington, D.C.

Shraiber, A. and Haddad, J. (2014). Robust control design for a perimeter traffic flow controller at an urban region. In *European Control Conference 14 (Accepted)*.

Zhang, L., Garoni, T., and de Gier, J. (2013). A comparative study of macroscopic fundamental diagrams of arterial road networks governed by adaptive traffic signal systems. *Transportation Research Part B*, 49:1–23.