Control of a Pedaled, Self-balanced Unicycle with Adaptation Capability

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Abstract: In this paper, a pedaled, self-balanced, personal mobility vehicle is developed. The vehicle is structurally similar to a pedaled unicycle but uses a brushless DC (BLDC) hub motor as its main driving wheel. In order to reduce the interference and provide a better human-machine interaction, a novel balancing controller with adaptation capability is proposed. This balancing controller, when working together with a specially designed low-level BLDC driver, can adapt to the uncertain center of gravity of the vehicle frame plus the rider, and the amount of motor torque that fights against the pedaling torque can be reduced to minimum. The performance of the control system is validated by simulations.

1 INTRODUCTION

In densely populated urban environments, EVs for personal mobility purposes are increasingly accepted as capable, even appealing, forms of transportation. Following the recent advances in digital computers, senor and actuator technology, and control theory, researchers now have the opportunity to radically contemplate new concepts for the personal mobility EVs of next generation. One of the innovative products is Segway, the personal transporter invented by Dean Kamen(Kamen, 2001). Segway is basically a two-wheel mobile inverted pendulum. The equipped control system regulates the total and the differential torques on the motored wheels to keep the transporter balanced, moving forward and backward, and making turns based on the posture of the rider standing on the chassis. Since the release of Segway, several vehicles with the structure of a mobile wheeled inverted pendulum have been developed. Examples include Toyota's winglet(Toyota, 2008), Honda's U3-X(Honda, 2009), Enicycle(Polutnik, 2006), and so on.

In this paper, a novel, motor-driven, personal mobility vehicle, named Legway, is proposed. As shown in Fig. 1, Legway is structurally similar to a pedaled unicycle but with two small auxiliary wheels on the sides of the main driving wheel. The main driving wheel is itself a brushless DC (BLDC) hub motor. The rider can drive Legway forward either electrically by electrical throttle or manually by pedaling it as the conventional bicycle. The electric driving and the manual driving can be exercised simultaneously for the sake of saving the man power or the electric power. The steering of Legway is achieved by braking the left auxiliary wheel (for turning left) or the right auxiliary wheel (for turning right).

Due to the unique driving method, the control design for Legway requires a special attention. When the rider pedals, not only the pedalling torque from the rider but also the controlled torque from the motor are applied simultaneously on the vehicle. To ensure efficient operation, it is crucial that these two torques achieve coordination that they do not interfere with or fight against each other. In order to reduce the interference and provide a better human-machine interaction, a novel balancing controller with adaptation capability is proposed. This balancing controller, when working together with a specially designed BLDC motor driver, can adapt to the uncertain center of gravity of the vehicle plus the rider during the pedalling, and automatically slews the vehicle frame to a balancing posture depending on the amount of assistive motor torque demanded. In such a posture, the amount of motor torque that fights against the pedaling torque is reduced to minimum.

This paper investigates the modeling and control issues of Legway and is organized as follows: Section 2 shows the system model which includes the mechanism dynamics and the motor dynamics. A motor driver is particularly introduced to properly switch the modes of operation of the BLDC motor so that a consistent representation of the motor dynamics can be achieved. In Section 3, a new balancing controller is proposed and a stability theorem is given to design

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the control matrices in the controller. Simulations are performed in Section 4 to verify the usefulness of the control system. Finally, conclusions are given in Section 5.

Figure 1: Photo of Legway.

2 SYSTEM DESCRIPTION

2.1 Model of Legway

Legway's movement is mainly dictated by its motion in pitch direction. The pitch dynamics can be characterized by a WIP model in Fig.2. In the model shown, m_w and I_w respectively denote the mass and inertia of the wheel, m_b and I_b respectively denote the mass and inertia of the pendulum body (which includes the vehicle frame and the rider all together), r is the wheel radius, l is the distance from the wheel axle to the center of gravity (COG) of the pendulum body, θ_w is the absolute rotation angle of the wheel, θ_g is the absolute inclination angle of the COG of the pendulum body, θ_b is absolute inclination angle of a fixed spot on the vehicle frame where the tilt sensor is attached and $\phi \equiv \theta_b - \theta_g$ is the angular difference between θ_b and θ_g, τ_w is the relative mechanical torque between the pendulum body and the wheel that τ_w consists of the motor torque τ_m, and the rider's pedaling torque τ_p. It should be noticed that ϕ could be a time-varying quantity due to the rider's posture change during the riding.

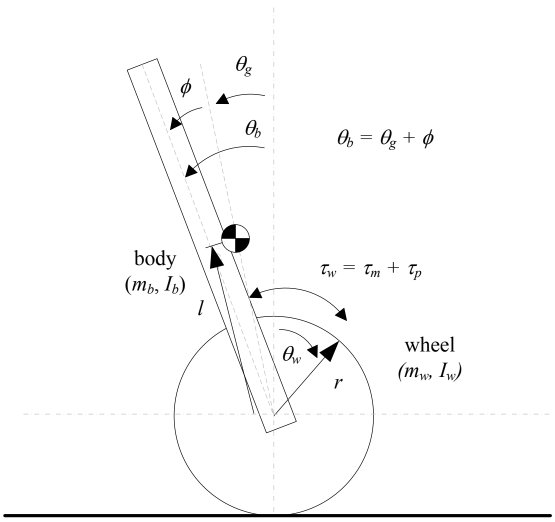

Figure 2: Schematics of the wheeled inverted pendulum (WIP) model.

Using the Lagrange formulation, one can derive the dynamic equation as:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\dot{\mathbf{q}},\mathbf{q})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau \tag{1}$$

where
$\mathbf{q} = \begin{bmatrix} \theta_g & \theta_w \end{bmatrix}^T$,
$\tau = \begin{bmatrix} (\tau_m + \tau_p) & (\tau_m + \tau_p) \end{bmatrix}^T$,
$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} I_b + m_b l^2 & -m_b lrc_g \\ -m_b lrc_g & I_w + (m_w + m_b)r^2 \end{bmatrix}$,
$\mathbf{C}(\dot{\mathbf{q}},\mathbf{q}) = \begin{bmatrix} 0 & 0 \\ m_b lrs_g \dot{\theta}_g & 0 \end{bmatrix}$, and
$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} -m_b gls_g & 0 \end{bmatrix}^T$. Notice that in the expressions for $\mathbf{H}(\mathbf{q})$, $\mathbf{C}(\dot{\mathbf{q}},\mathbf{q})$, and $\mathbf{g}(\mathbf{q})$, s_g and c_g are the abbreviations for $\sin\theta_g$ and $\cos\theta_g$ respectively.

The dynamic equation indicates that the minimum number of states required to describe the system is three and the state vector is chosen as $\begin{bmatrix} \theta_g & \dot{\theta}_g & \dot{\theta}_w \end{bmatrix}^T$. Equation (1) can be linearized around the equilibrium state $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ with $\tau_m = \tau_p = 0$, and the linearized state equation is given by:

$$\frac{d}{dt}\begin{bmatrix} \delta\theta_g \\ \delta\dot{\theta}_g \\ \delta\dot{\theta}_w \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\alpha\lambda}{\eta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\theta_g \\ \delta\dot{\theta}_g \\ \delta\dot{\theta}_w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\alpha+\gamma}{\eta} \\ \frac{\beta+\gamma}{\eta} \end{bmatrix} (\delta\tau_m + \delta\tau_p) \tag{2}$$

where, $\alpha = I_w + (m_w + m_b)r^2$, $\beta = I_b + m_b l^2$, $\gamma = m_b lr$, $\lambda = m_b gl$, $\eta = \alpha\beta - \gamma^2$. In (2), δ represents the perturbation around the equilibrium state. For simplicity, the symbol δ will be ignored in the subsequent discussion.

2.2 Motor Dynamics and Motor Driver

As revealed by the eigenvalues of the system matrix in (2), the pitch dynamics of Legway is open-loop unstable. Moreover, the state equation therein can be used to verify that the system is controllable under the motor torque τ_m. Therefore, τ_m can be used to not only stabilize/balance the vehicle but also act as an alternative source of propulsion to the rider's pedaling torque.

The motor torque, which is generated by a BLDC hub motor, is regulated by controlling the motor current via a motor driver. Since both positive torque or negative torque for a wide range of rotational speed is needed for balancing, accelerating or decelerating the vehicle, the motor driver devised for this research can automatically switch between the motoring mode and the regeneration mode for the sake of energy efficiency. The motor driver also provides a simple, consistent motor dynamics regardless of the mode switching.

The switching module in the motor driver as proposed in (Wu and Yeh, 2013) is adopted here. This module determines which mode the motor should be switched to for proper operation. Furthermore, to facilitate the subsequent control design, it also performs an input transformation so that a single 1st order differential equation in the form of

$$L\frac{di}{dt}+Ri=u \tag{3}$$

can describe the current dynamics for all the four modes. According to (3), the input to the differential equation, consequently the switching module is u. How the mode of operation is selected and how the duty ratios for PWM switching are computed from the input u, the measured coil current (i), and the rotational speed (ω) are given in (Wu and Yeh, 2013).

3 BALANCING CONTROL

In Legway, the COG of the vehicle body is dictated by the rider's posture and is uncertain to the control system. The successful operation of Legway requires the control system to adaptively estimate the uncertain COG, slew it to a balanced position and then maintain the motor torque at a commanded value. When the torque command is zero, the COG is slewed to the top of the wheel axle so that the rider feels minimum interference from the motor as he/she pedals the vehicle. The rider also does not have to worry about the uncontrollable acceleration of the vehicle even if he/she has yet learned to properly place his/her COG when he first rides the vehicle. In the case that the commanded torque is nonzero, the motor produces an assistive torque for climbing hills, or stopping the vehicle via regenerative braking.

3.1 Controller Design

The state equation and the output equation for the controller design are in the form of

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}+\mathbf{B}_\tau\tau_p \tag{4}$$

$$\mathbf{y}=\mathbf{x}+\mathbf{C}\varphi \tag{5}$$

where $\mathbf{x},\mathbf{u},\mathbf{y}$ are respectively the state, the input, and the output vectors, φ represents the output disturbance, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are system matrices. The state equation, which uses $\mathbf{x}=\begin{bmatrix}\theta_g & \dot{\theta}_g & i\end{bmatrix}^T$ as the state vector and $\mathbf{u}=u$, is obtained by concatenating the dynamics of θ_g and $\dot{\theta}_g$ in (2) with the current dynamics in (3) via $\tau_m=k_m i$, so $\mathbf{A}=\begin{bmatrix}0 & 1 & 0\\ \frac{\alpha\lambda}{\eta} & 0 & \frac{\alpha+\gamma}{\eta}k_m\\ 0 & 0 & -\frac{R}{L}\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}0\\0\\\frac{1}{L}\end{bmatrix}$, $\mathbf{B}_\tau=\begin{bmatrix}0\\\frac{\alpha+\gamma}{\eta}\\0\end{bmatrix}$. Notice that because it is up to the rider to apply the pedaling torque or the current command to control the wheel speed, in the controller design $\dot{\theta}_w$ is not considered as a state variable. As for the the output $\mathbf{y}$, it consists of the measured pitch angle (θ_b) and the pitch rate ($\dot{\theta}_b$) from the inclinometer and the rate gyro installed on the vehicle frame[1], and the motor current (i) from the hall sensors in the motor. It is assume that the the rider's posture remains relatively fixed to the vehicle frame, so $\phi=\theta_b-\theta_g$ is constant and $\dot{\phi}=\dot{\theta}_b-\dot{\theta}_g=0$. This gives $\varphi=\phi$,and $\mathbf{C}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$.

The control objective is to use the feedback from $\mathbf{y}$ to devise a control law for $\mathbf{u}$ to make $\mathbf{x}$ asymptotically converge to a reference state $\mathbf{x}_d$. The major challenge here is that the output is contaminated by unknown output disturbance φ, so an adaptive scheme is required for the control system to on-line estimate and then cancel φ. In the following investigation, we will temporarily ignore τ_p, and pose the control problem in a more general setting for $\mathbf{x}\in R^n$, $\mathbf{u}\in R^p$, and $\varphi\in R^q$ which correspond to $\mathbf{A}\in R^{n\times n}$, $\mathbf{B}\in R^{n\times p}$, and $\mathbf{C}\in R^{n\times q}$. The controller design for the general problem is given in the following theorem.

Theorem 1: Given the system in (4)(5) with $\tau_p=0$, and φ being an unknown, constant output disturbance, assume that $\mathbf{A}$ is invertible, $(\mathbf{A},\mathbf{B})$ controllable, and $\mathbf{C}$ has full rank. The control system containing

$$\mathbf{u}=-\mathbf{K}(\hat{\mathbf{x}}-\mathbf{x}_d)+\mathbf{u}_d \tag{6}$$

[1] $\dot{\theta}_b$ is measured directly from the rate gyro. However, to increase the sensing bandwidth, the measurement for θ_b is obtained by merging the outputs of the rate gyro and the inclinometer using a complementary filter(T.-J. Yeh and Wang, 2005).

as the control law, and

$$\dot{\hat{\varphi}} = -\mathbf{K}_\varphi(\hat{\mathbf{x}} - \mathbf{x}_d) \tag{7}$$

as the estimation law can make $\mathbf{x}$ asymptically converge to a reference state $\mathbf{x}_d$. In (6) and (7), $\hat{\varphi}$ is the estimation for φ, $\hat{\mathbf{x}}$ is the estimated state vector and is computed by $\mathbf{y} - \mathbf{C}\hat{\varphi}$, $\mathbf{x}_d$ is a constant reference command and $\mathbf{u}_d$ is the corresponding feedforward control and both of which should satisfy the structural constraint

$$\mathbf{A}\mathbf{x}_d + \mathbf{B}\mathbf{u}_d = 0, \tag{8}$$

$\mathbf{K}$ is the state feedback matrix which makes $\mathbf{A}_c$, where $\mathbf{A}_c = \mathbf{A} - \mathbf{BK}$ is the nominal closed-loop system matrix, a stable matrix, and finally $\mathbf{K}_\varphi$ is the estimator gain matrix which is computed via the following procedure:

1. Choose $\mathbf{Q}_1 = \mathbf{Q}_1^T > 0$, $\mathbf{Q}_2 = \mathbf{Q}_2^T > 0$. Compute the solution $\mathbf{R} = \mathbf{R}^T > 0$ to the Lyapunov matrix equation

$$\mathbf{R}\mathbf{A_c} + \mathbf{A}_c^T\mathbf{R} = -\mathbf{Q}_1 < 0 \tag{9}$$

and then the matrix

$$\mathbf{S} = (\mathbf{Q}_2\mathbf{A}^{-1})^T + \mathbf{RBK}. \tag{10}$$

Make sure $\mathbf{S}$ is invertible by choosing a "sufficiently small" $\mathbf{Q}_1$ or "sufficiently large" $\mathbf{Q}_2$[2].

2. Define

$$\mathbf{D} = \mathbf{I}_{n\times n} - \mathbf{SC}(\mathbf{C}^T\mathbf{A}^T\mathbf{SC})^{-1}\mathbf{C}^T\mathbf{A}^T \tag{11}$$

$$\mathbf{W} = \mathbf{C}(\mathbf{C}^T\mathbf{A}^T\mathbf{SC})^{-1}\mathbf{C}^T\mathbf{A}^T \tag{12}$$

and choose $\mathbf{Q}_3 = \mathbf{Q}_3^T > 0$. Compute $\mathbf{P}$, the solution to the following Riccati equation:

$$\mathbf{A}_c^T(\mathbf{D}^T + \mathbf{W}^T\mathbf{Q}_2\mathbf{A}^{-1})^T\mathbf{P} + \mathbf{P}(\mathbf{D}^T + \mathbf{W}^T\mathbf{Q}_2\mathbf{A}^{-1})\mathbf{A_c} + \mathbf{P}(\mathbf{W} + \mathbf{W}^T)\mathbf{P} = -\mathbf{Q}_3 < 0. \tag{13}$$

3. $\mathbf{K}_\varphi$ is computed by

$$\mathbf{K}_\varphi = (\mathbf{C}^T\mathbf{A}^T\mathbf{SC})^{-1}\mathbf{C}^T\mathbf{A}^T\mathbf{P} \in \mathbb{R}^{q\times n} \tag{14}$$

Proof. Using a Lyapunov function given by

$$V = (\hat{\mathbf{x}} - \mathbf{x}_d)^T\mathbf{P}(\hat{\mathbf{x}} - \mathbf{x_d}) + \tilde{\varphi}^T\mathbf{C}^T\mathbf{Q}_2\mathbf{C}\tilde{\varphi} + \dot{\tilde{\mathbf{x}}}^T\mathbf{R}\dot{\tilde{\mathbf{x}}} \tag{15}$$

one can prove that $\tilde{\mathbf{x}} \to \mathbf{0}$, and $\tilde{\varphi} \to \mathbf{0}$. □

4 SIMULATIONS

The proposed balancing controller is designed based on the linearized model of Legway. It is then applied to the nonlinear model consisting of the mechanism dynamics in (1) and motor dynamics in (3), and the control performance is examined using simulations.

Table 1: Parameters of Legway model used in the simulation.

	Mechanical part		Motor part
m_w	$6\ kg$	R	$0.37\ \Omega$
I_w	$0.0927\ kg\text{-}m^2$	L	$0.503\ mH$
r	$0.25\ m$	k_m	$1.11186\ N\text{-}m/A$
m_b	$88\ kg$	l	$0.85\ m$
I_b	$17.6983\ kg\text{-}m^2$		

The system parameters for simulations are listed in Table 1.

The control gain $\mathbf{K}$, which is designed using LQR method, is given by $\mathbf{K} = [\ 419.538 \quad 67.835 \quad 0.0229\ ]$. By choosing $\mathbf{Q}_1 = 1\times10^{-8}\mathbf{I}_3$, $\mathbf{Q}_2 = \mathbf{I}_3$ and $\mathbf{Q}_3 = 7\times10^{-5}\mathbf{I}_3$, $\mathbf{R}$ and $\mathbf{P}$ are solved, and the estimator gain matrix is given by $\mathbf{K}_\varphi = [\ 15.9391 \quad 2.7042 \quad 9.5750\times10^{-4}\ ]$[3]. The closed-loop poles for this design are -3.577, $-8.648+9.089i$, $-8.648-9.089i$, and -744.34 .

In the simulation, the performance of the proposed controller is compared to two other controllers respectively referred to as Controller A and Controller B. Controller A uses the measured, biased states for feedback so that $u = -\mathbf{Ky}$. Controller B is the one mentioned in Remark 5 and is given by $u = -\mathbf{K}'\mathbf{y} - k_I\int\tilde{i}dt$ in which the controller gains $\mathbf{K}' = [\ 472.827 \quad 67.835 \quad 0.0149\ ]$ and $k_I = 5.86$ are chosen to make the closed-loop poles identical to those of the proposed controller. The first simulation assumes that initially $\mathbf{x} = [\ \theta_g \quad \dot{\theta}_g \quad i\ ]^T = \mathbf{0}$. It is desired to maintain $\mathbf{x} = \mathbf{0}$ (by letting $\mathbf{x}_d = \mathbf{0}$) under a constant output disturbance $\phi = 5°$. Fig.3 shows the responses associated with the states (θ_g, $\dot{\theta}_g$, i) and the control (u) for the three controllers. All three controllers can stabilize the system, but Controller A, because the lack of estimation capability, exhibits steady errors in $\theta_g(\approx -5.994°)$ and $i(\approx 18.53A)$. On the other hand, the integral action allows both Controller B and the proposed controller to reject the output disturbance so the state errors all settle to zero within $1sec$. It should be noted that for Controller A, the steady state error in i results in a constant motor torque acting on the wheel which in turn causes an undesired acceleration to the vehicle.

To make further performance comparison between Controller B and the proposed controller, a constant bias of $1A$ is injected to the current measurement. As

[2]The magnitude of $Q_{(\bullet)}$ is quantified in terms of the matrix norm.

[3]Due to the fact that the magnitude of i in *Amps* is much larger than the magnitudes of θ_g and $\dot{\theta}_g$ respectively in *radians* and *radians per second*, the third component of $\mathbf{K}_\varphi$ is much smaller than its first two components.

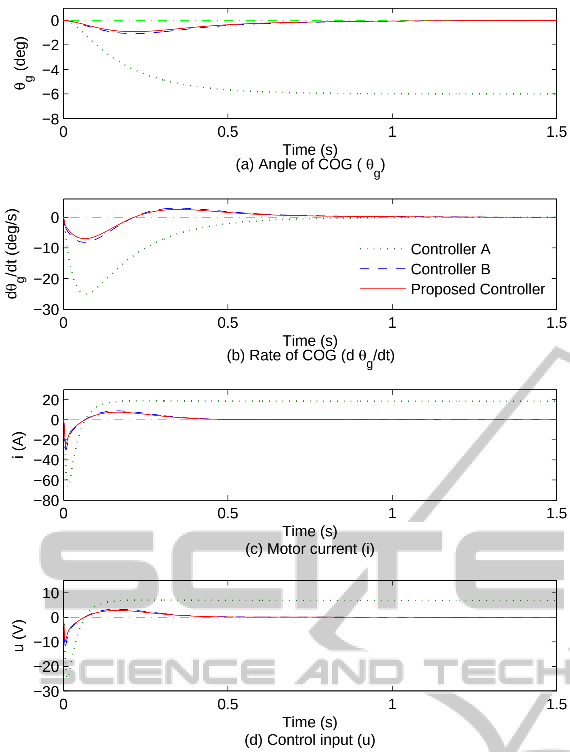

Figure 3: Responses of θ_b, i, and vehicle speed for the placement and removal of an $8kg$ deadweight.

shown in Fig.4, Controller B. which integrates i solely for integral control, is sensitive to the measurement bias in i and exhibits steady errors in $\theta_g(\approx 0.324°)$ and $i(\approx -1.0A)$. Although the proposed controller does not explicitly consider the current bias in the design process (i.e., $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$), the integration over all the states in the estimator does provide robustness to not only the measurement bias in θ_g but also the bias in i, and all the states reach zero at steady state.

During manual pedaling, τ_p exists and ϕ could be a time-varying quantity. They induce a motor torque even if $i_d = 0$. It is crucial that the control system keeps the induced motor torque small to avoid its interference to the pedalling action. To investigate the influence of τ_p and ϕ on the motor torque for the proposed controller, the closed-loop magnitude responses of the motor torque ($k_m i$) with respect to τ_p and ϕ for the linear model are plotted in Fig.5. It can be seen that both plots exhibit a high-pass filtering nature, and in particular, the motor torque response due to τ_p has a 0dB crossover frequency around $0.4Hz$. Therefore, as long as τ_p and ϕ are varied slowly, the induced motor torque is kept small, so the rider can pedal the vehicle with little interference from the motor.

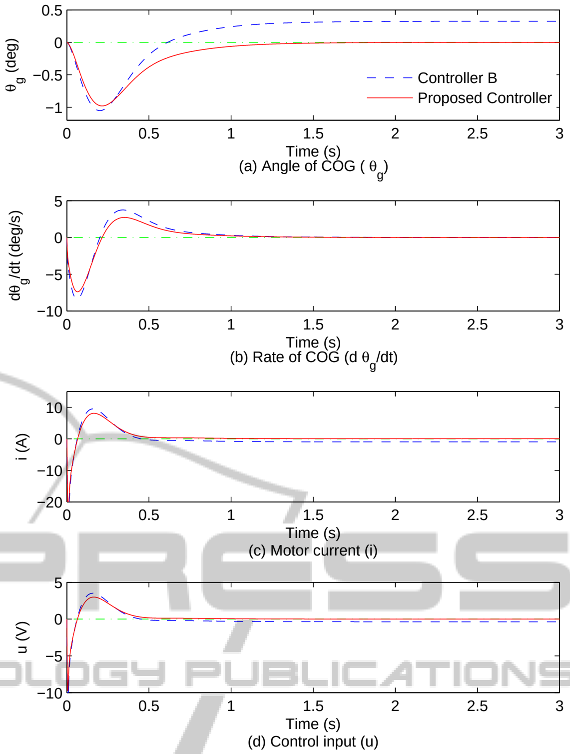

Figure 4: Responses of θ_b, i, and vehicle speed for the placement and removal of an $8kg$ deadweight.

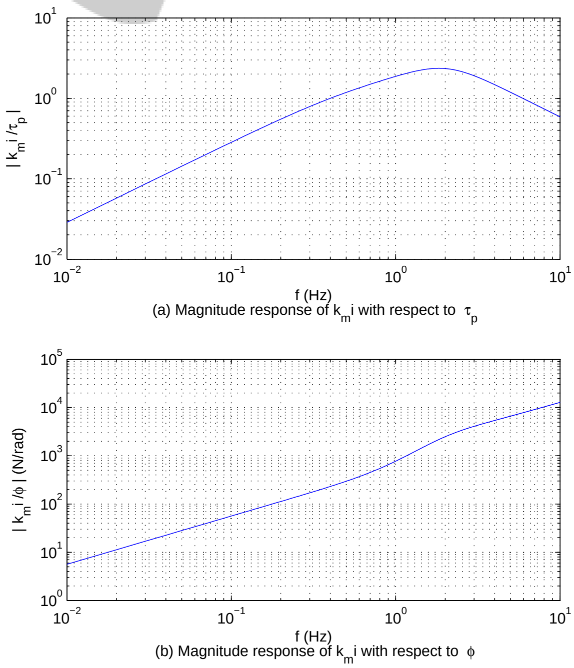

Figure 5: Responses of θ_b, i, and vehicle speed for the placement and removal of an $8kg$ deadweight.

5 CONCLUSIONS

In this paper, a pedaled, self-balanced, personal mobility vehicle, Legway, is developed. The vehicle is intended to be used in densely populated urban environments that one can integrate it with the public transportation system to increase the commuting range. In order to reduce the interference and provide a better human-machine interaction, a novel balancing controller with adaptation capability is proposed. This balancing controller, when working together with a specially designed low-level BLDC driver, can adapt to the uncertain center of gravity of the vehicle frame plus the rider, and the amount of motor torque that fights against the pedaling torque can be reduced to minimum. The performance of the control system is validated by simulation.

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APPENDIX

This is the proof for the negative definiteness of $\dot{V}$.

Taking time derivative on the Lyapunov function candidate V in (15) yields

$$\begin{aligned}\dot V &= 2\dot{\hat{\mathbf{x}}}^T\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d}) + 2\tilde{\varphi}^{\mathbf{T}}\mathbf{C}^{\mathbf{T}}\mathbf{Q_2C}\dot{\tilde{\varphi}} + 2\dot{\tilde{\mathbf{x}}}^{\mathbf{T}}\mathbf{R}\ddot{\tilde{\mathbf{x}}}\\ &= 2\dot{\mathbf{x}}^T\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d}) + 2\left[-\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d}) + \mathbf{Q_2C}\tilde{\varphi}\right]^T\mathbf{C}\dot{\tilde{\varphi}}\\ &\quad + 2\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{RBKC}\dot{\tilde{\varphi}} + 2\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\end{aligned}$$

in which the second equality is due to $\dot{\hat{\mathbf{x}}} = \dot{\mathbf{y}} - \mathbf{C}\dot{\hat{\varphi}} = \dot{\mathbf{x}} - \mathbf{C}\dot{\tilde{\varphi}}$ and differentiating the closed-loop dynamic equation for the expression of $\ddot{\tilde{\mathbf{x}}}$. Notice that

$$\begin{aligned}\dot{\mathbf{x}} - \mathbf{A}_c(\hat{\mathbf{x}}-\mathbf{x_d}) &= \dot{\tilde{\mathbf{x}}} - \mathbf{A}_c(\tilde{\mathbf{x}} - \mathbf{C}\tilde{\varphi})\\ &= \mathbf{A}_c\tilde{\mathbf{x}} + \mathbf{BKC}\tilde{\varphi} - \mathbf{A}_c(\tilde{\mathbf{x}} - \mathbf{C}\tilde{\varphi})\\ &= (\mathbf{BK}+\mathbf{A}_c)\mathbf{C}\tilde{\varphi} = \mathbf{AC}\tilde{\varphi},\end{aligned} \tag{16}$$

so $\mathbf{C}\tilde{\varphi} = \mathbf{A}^{-1}\left[\dot{\mathbf{x}} - \mathbf{A}_c(\hat{\mathbf{x}}-\mathbf{x_d})\right]$. Replacing $\mathbf{C}\tilde{\varphi}$ by such an expression, $\dot V$ becomes

$$\begin{aligned}\dot V &= 2\dot{\mathbf{x}}^T\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d})\\ &\quad + \mathbf{2}\left[\mathbf{Q_2A}^{-1}\dot{\mathbf{x}} - (\mathbf{Q_2A^{-1}A}_c + \mathbf{P})(\hat{\mathbf{x}}-\mathbf{x_d})\right]^T\mathbf{C}\dot{\tilde{\varphi}}\\ &\quad + 2\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{RBKC}\dot{\tilde{\varphi}} + 2\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\\ &= 2\dot{\mathbf{x}}^T\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d})\\ &\quad + 2\dot{\mathbf{x}}^{\mathbf{T}}\left[\left(\mathbf{Q_2A}^{-1}\right)^T + \mathbf{RBK}\right]\mathbf{C}\dot{\tilde{\varphi}}\\ &\quad - 2(\hat{\mathbf{x}}-\mathbf{x_d})^T(\mathbf{Q_2A^{-1}A}_c + \mathbf{P})^T\mathbf{C}\dot{\tilde{\varphi}} + 2\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\\ &= 2\dot{\mathbf{x}}^T\left[\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d}) + \mathbf{SC}\dot{\tilde{\varphi}}\right]\\ &\quad - 2(\hat{\mathbf{x}}-\mathbf{x_d})^T(\mathbf{Q_2A^{-1}A}_c + \mathbf{P})^T\mathbf{C}\dot{\tilde{\varphi}} + 2\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\end{aligned}$$

where the third equality is due to the definition of $\mathbf{S}$ in (10).

By the estimation law in (7), the definition of $\mathbf{K}_\varphi$ in (14), and φ being constant, the dynamics associated with $\tilde{\varphi}$ is given by

$$\dot{\tilde{\varphi}} = \dot{\hat{\varphi}} - \dot{\varphi} = \dot{\hat{\varphi}} = -(\mathbf{C^TA^TSC})^{-1}\mathbf{C}^T\mathbf{A}^T\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d}). \tag{17}$$

Replacing $\dot{\tilde{\varphi}}$ in $\dot V$ by the expression in (17), and then using the definitions of $\mathbf{D}$ and $\mathbf{W}$ respectively in (11) and (12), $\dot V$ can be written as

$$\begin{aligned}\dot V &= \mathbf{2}\dot{\mathbf{x}}^T\mathbf{DP}(\hat{\mathbf{x}}-\mathbf{x_d})\\ &\quad + \mathbf{2}(\hat{\mathbf{x}}-\mathbf{x_d})^T(\mathbf{Q_2A^{-1}A}_c + \mathbf{P})^T\mathbf{WP}(\hat{\mathbf{x}}-\mathbf{x_d})\\ &\quad + \mathbf{2}\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\\ &= \mathbf{2}(\hat{\mathbf{x}}-\mathbf{x_d})^T\left[\mathbf{A}_c^T\mathbf{D} + (\mathbf{Q_2A^{-1}A}_c + \mathbf{P})^T\mathbf{W}\right]\mathbf{P}(\hat{\mathbf{x}}-\mathbf{x_d})\\ &\quad + \mathbf{2}\tilde{\varphi}^T\mathbf{C}^T\mathbf{A}^T\mathbf{DP}(\hat{\mathbf{x}}-\mathbf{x_d}) + \mathbf{2}\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\end{aligned}$$

where the second equality is obtained by substituting the relation $\dot{\mathbf{x}} = \mathbf{A}_c(\hat{\mathbf{x}}-\mathbf{x_d}) + \mathbf{AC}\tilde{\varphi}$ as implied in (16).

Since

$$\begin{aligned}\mathbf{C}^T\mathbf{A}^T\mathbf{D} &= \mathbf{C}^T\mathbf{A}^T - \mathbf{C}^T\mathbf{A}^T\mathbf{SC}(\mathbf{C}^T\mathbf{A}^T\mathbf{SC})^{-1}\mathbf{C}^T\mathbf{A}^T\\ &= \mathbf{C}^T\mathbf{A}^T - \mathbf{C}^T\mathbf{A}^T = \mathbf{0}\end{aligned}$$

$\dot V$ can be further reduced to

$$\begin{aligned}\dot V &= \mathbf{2}(\hat{\mathbf{x}}-\mathbf{x_d})^T\left[\mathbf{A}_c^T(\mathbf{D}^T + \mathbf{W}^T\mathbf{Q_2A^{-1}})^T\mathbf{P}\right.\\ &\quad \left. + \mathbf{PWP}\right](\hat{\mathbf{x}}-\mathbf{x_d}) + \mathbf{2}\dot{\tilde{\mathbf{x}}}^T\mathbf{RA}_c\dot{\tilde{\mathbf{x}}}\end{aligned}$$

The property of a quadratic form allows one to expand $\dot{V}$ as

$$\begin{aligned}\dot{V} = &(\hat{\mathbf{x}}-\mathbf{x_d})^T \Big[\mathbf{A}_c^T(\mathbf{D}^T+\mathbf{W}^T\mathbf{Q_2}\mathbf{A^{-1}})^T\mathbf{P} \\ &+\mathbf{P}(\mathbf{D}^T+\mathbf{W}^T\mathbf{Q_2}\mathbf{A^{-1}})\mathbf{A}_c+\mathbf{P}(\mathbf{W}+\mathbf{W}^T)\mathbf{P}\Big](\hat{\mathbf{x}}-\mathbf{x_d}) \\ &+\dot{\tilde{\mathbf{x}}}^{\mathbf{T}}(\mathbf{R}\mathbf{A}_c+\mathbf{A}_c^T\mathbf{R})\dot{\tilde{\mathbf{x}}},\end{aligned} \tag{18}$$

so according to the definitions of **P** and **R** respectively in (13)and (9), $\dot{V}$ eventually becomes

$$\begin{aligned}\dot{V} &= -(\hat{\mathbf{x}}-\mathbf{x_d})^T\mathbf{Q}_3(\hat{\mathbf{x}}-\mathbf{x_d})-\dot{\tilde{\mathbf{x}}}^{\mathbf{T}}\mathbf{Q}_1\dot{\tilde{\mathbf{x}}} \\ &= -(\tilde{\mathbf{x}}-\mathbf{C}\tilde{\varphi})^T\mathbf{Q}_3(\tilde{\mathbf{x}}-\mathbf{C}\tilde{\varphi})-\dot{\tilde{\mathbf{x}}}^{\mathbf{T}}\mathbf{Q}_1\dot{\tilde{\mathbf{x}}}\end{aligned} \tag{19}$$

Since $\mathbf{Q}_1,\mathbf{Q}_3 > \mathbf{0}$, $\dot{V} \le 0$. Furthermore, $\dot{V} = 0$ if and only if $\tilde{\mathbf{x}} = \mathbf{C}\tilde{\varphi}$, and $\dot{\tilde{\mathbf{x}}} = \mathbf{0}$. From the closed-loop system equations, $\dot{\tilde{\mathbf{x}}} = \mathbf{0}$ gives $\mathbf{A}_c\tilde{\mathbf{x}}+\mathbf{BKC}\tilde{\varphi} = \mathbf{0}$, and with $\tilde{\mathbf{x}} = \mathbf{C}\tilde{\varphi}$, one can infer that $(\mathbf{A}_c+\mathbf{BK})\mathbf{C}\tilde{\varphi} = \mathbf{AC}\tilde{\varphi} = \mathbf{0}$. The invertibility of **A** and full rank condition of **C** lead to $\tilde{\varphi} = \mathbf{0}$, which also means $\tilde{\mathbf{x}} = \mathbf{0}$. Consequently, the necessary and sufficient condition for $\dot{V} = 0$ is equivalent to $\tilde{\mathbf{x}} = 0$, and $\tilde{\varphi} = \mathbf{0}$. This concludes that $\dot{V}$ is negative definite.