Measure of Roughness for Rough Approximation of Fuzzy Sets and Its Topological Interpretation

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Keywords: *L*-fuzzy Set, *L*-relation, *L*-fuzzy Rough Set, cl-monoid, Measure of Inclusion, *L*-fuzzy Topology, *L*-fuzzy Co-topology.

Abstract: We define the measure of upper and the measure of lower rough approximation for *L*-fuzzy subsets of a set equipped with a reflexive transitive fuzzy relation *R*. In case when the relation *R* is also symmetric, these measures coincide and we call their value by the measure of roughness of rough approximation. Basic properties of such measures are studied. A realization of measures of rough approximation in terms of *L*-fuzzy topologies is presented.

1 INTRODUCTION

The concept of a rough subset of a set equipped with an equivalence relation was introduced by Z. Pawlak (Pawlak 1982). Rough sets found important applications in real-world problems, and also arouse interest among "pure" mathematicians as an interesting mathematical notion having deep relations with other fundamental mathematical concepts, in particular, with topology. Soon after Pawlak's work, the concept of roughness was extended to the context of fuzzy sets; D. Dubois' and H. Prade's paper (Dubois and Prade 1990) was the first work in this direction. At present there is a vast literature where fuzzy rough sets are investigated and applied. In particular, fuzzy rough sets are studied and used in (Kortelainen 1994, Ciuci 2009, Yao 1998, Qin and Pai 2005, Qin and Pai 2008, Hao and Li 2011, Radzikowska and Kerre 2002, Tiwari and Srivastava 2013, Mi and HU 2013, Yu and Zhou 2014) just to mention a few of numerous works dealing with (fuzzy) rough sets. However, as far as we know, there were no attempts undertaken to measure the degree of roughness of a fuzzy set. To state it in another way, to measure, "how much rough" is a given (fuzzy) subset of a set equipped with a (fuzzy) relation. We undertake such attempt in this paper. Namely, given an *L*-fuzzy subset *A* of a set equipped with a reflexive transitive *L*-relation we assign to *A* an element $\alpha \in L$ showing how much this set differs from its upper and lower rough approximations.

The structure of the paper is as follows. In the next section we recall two notions fundamental for our work, namely a cl-*monoid* and an *L-relation*. In the third section we introduce the measure of inclusion of one fuzzy set into another, and describe the behavior of this measure.

In Section 4 we define operators of upper and lower rough approximation for an *L*-fuzzy subset of a set endowed with an *L*-relation. Note that similarly defined operators under differemt assumptions appear also in the previous researches, see e.g. (Järvinen and Kortelainen, 2007, Qin and Pai 2005, Sostak 2010, Sostak 2012)

In Section 5 we define the measures of upper $\mathcal{K}(A)$ and lower $\mathcal{T}(A)$ rough approximation for an *L*-fuzzy subset *A* of a set endowed with an *L*-relation. Essentially, $\mathcal{K}(A)$ is the measure of inclusion of the upper approximation of an *L*-fuzzy set *A* into *A*, while $\mathcal{T}(A)$ is the measure of inclusion of *A* into its lower approximation. By showing $\mathcal{K}(A) = \mathcal{T}(A)$ whenever *R* is symmetric, we come to the measure of roughness $\mathcal{RA}(A)$ of an *L*-fuzzy set *A*.

In Section 6 we interpret the operator of measuring roughness of rough approximation as an *L*-fuzzy ditopology (that is a pair of a an *L*-fuzzy topology $\mathcal{T}$ and an *L*-fuzzy $\mathcal{K}$ co-topology) on a set *X* and discuss some issues of this interpretation.

In the last, Conclusion, section we discuss some directions for the prospective work.

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Measure of Roughness for Rough Approximation of Fuzzy Sets and Its Topological Interpretation.
DOI: 10.5220/0005080400610067
In *Proceedings of the International Conference on Fuzzy Computation Theory and Applications* (FCTA-2014), pages 61-67
ISBN: 978-989-758-053-6

2 PRELIMINARIES

2.1 cl-monoids

Let $(L,\leq,\wedge,\vee)$ denote a complete lattice, that is a lattice in which arbitrary suprema (joins) and infima (meets) exist. In particular, the top 1_L and the bottom 0_L elements in L exist and $0_L \neq 1_L$.

Definition 2.1. (Birkhoff 1995) *A tuple $(L,\leq,\wedge,\vee,*)$ is called a cl-monoid if $(L,\leq,\wedge,\vee)$ is a complete lattice and the binary operation $*: L\times L\to L$ satisfies conditions:*

(0$*$) *$*$ is monotone: $\alpha\leq\beta\Longrightarrow\alpha*\gamma\leq\beta*\gamma$ for all $\alpha,\beta,\gamma\in L$;*

(1$*$) *$*$ is commutative: $\alpha*\beta=\beta*\alpha$ for all $\alpha,\beta\in L$;*

(2$*$) *$*$ is associative: $(\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$ for all $\alpha,\beta,\gamma\in L$;*

(3$*$) *$*$ distributes over arbitrary joins: $\alpha*(\bigvee_{i\in I}\beta_i)=\bigvee_{i\in I}(\alpha*\beta_i)$ for all $\alpha\in L$, for all $\{\beta_i\mid i\in I\}\subseteq L$,*

(4$*$) *$\alpha*1_L=\alpha, \quad \alpha*0_L=0_L$ for all $\alpha\in L$.*

(Note, that a cl-monoid can be defined also as an integral commutative quantale in the sense of (Rosenthal 1990).)

Remark 2.2. In case $L=[0,1]$ the operation $*: [0,1]\times[0,1]\to[0,1]$ satisfying properties (0$*$), (1$*$), (2$*$) and (4$*$) (defined in a slightly different form) for the first time appeared in K. Menger's papers, see e.g. (Menger 1979)under the name *a triangular norm*, or a *t-norm*. Later t-norms were thoroughly studied by different authors, see e.g. (Schweitzer and Sclar 1983, Klement, Messiar and Pap 2000) A t-norm satisfying property (3$*$) is called lower semicontinuous.

In a cl-monoid a further binary operation $\mapsto$, residium, is defined:

$$\alpha\mapsto\beta=\bigvee\{\lambda\in L\mid\lambda*\alpha\leq\beta\}.$$

Residuation is connected with operation $*$ by the Galois connection:

$$\alpha*\beta\leq\gamma\Longleftrightarrow\alpha\leq(\beta\mapsto\gamma),$$

see e.g. (Höhle 1992). In the following proposition we collect well-known properties of the residium:

Proposition 2.3. (see e.g. Höhle 1992.)

(1$\mapsto$) $(\bigvee_i\alpha_i)\mapsto\beta=\bigwedge_i(\alpha_i\mapsto\beta)$ *for all $\{\alpha_i\mid i\in I\}\subseteq L$, for all $\beta\in L$;*

(2$\mapsto$) $\alpha\mapsto(\bigwedge_i\beta_i)=\bigwedge_i(\alpha\mapsto\beta_i)$ *for all $\alpha\in L$, for all $\{\beta_i\mid i\in I\}\subseteq L$,;*

(3$\mapsto$) $1_L\mapsto\alpha=\alpha$ *for all $\alpha\in L$;*

(4$\mapsto$) $\alpha\mapsto\beta=1_L$ *whenever $\alpha\leq\beta$;*

(5$\mapsto$) $\alpha*(\alpha\mapsto\beta)\leq\beta$ *for all $\alpha,\beta\in L$;*

(6$\mapsto$) $(\alpha\mapsto\beta)*(\beta\mapsto\gamma)\leq\alpha\mapsto\gamma$ *for all $\alpha,\beta,\gamma\in L$;*

(7$\mapsto$) $\alpha\mapsto\beta\leq(\alpha*\gamma\mapsto\beta*\gamma)$ *for all $\alpha,\beta,\gamma\in L$.*

In the sequel we will need the following two lemmas:

Lemma 2.4. *Let $(L,\leq,\wedge,\vee,*)$ be a cl-monoid. Then for every $\{\alpha_i\mid i\in I\}\subseteq L$ and every $\{\beta_i\mid i\in I\}\subseteq L$ it holds:*

$$\left(\bigwedge_i\alpha_i\right)\mapsto\left(\bigwedge_i\beta_i\right)\geq\bigwedge_i(\alpha_i\mapsto\beta_i).$$

Indeed, applying Proposition 2.3 we have:

$$\bigwedge_i\alpha_i\mapsto\bigwedge_j\beta_j=\bigwedge_j(\bigwedge_i\alpha_i\mapsto\beta_j)\geq\bigwedge_j(\alpha_j\mapsto\beta_j).$$

Lemma 2.5. *Let $(L,\leq,\wedge,\vee,*)$ be a cl-monoid. Then for every $\{\alpha_i\mid i\in I\}\subseteq L$, and every $\{\beta_i\mid i\in I\}\subseteq L$ it holds:*

$$\left(\bigvee_i\alpha_i\right)\mapsto\left(\bigvee_i\beta_i\right)\geq\bigwedge_i(\alpha_i\mapsto\beta_i).$$

Proof. Applying Proposition 2.3 we have

$$(\alpha_i\mapsto\beta_i)*\alpha_i\leq\beta_i$$

for each $i\in I$. Let $c=\bigwedge_i(\alpha_i\mapsto\beta_i)$. Then $c*\alpha_i\leq\beta_i$ for each $i\in I$. Taking suprema on the both sides of the above inequality over $i\in I$ we get $c*\bigvee_i\alpha_i\leq\bigvee_i\beta_i$ and hence, by the Galois connection,

$$\bigwedge_i(\alpha_i\mapsto\beta_i).\leq\bigvee_i\alpha_i\mapsto\bigvee_i\beta_i.$$

2.2 L-relations

The concept of a fuzzy relation (or an $[0,1]$-relation in our terminology) was first introduced by Zadeh and then redefined and studied by different authors.

Definition 2.6. (Zadeh 1971, Valverde 1985) *An L-relation on a set X is a mapping $R: X\times X\to L$.*

(r) *L-relation R is called reflexive if $R(x,x)=1$ for each $x\in X$;*

(s) *L-relation R is called symmetric if $R(x,y)=R(y,x)$ for all $x,y\in X$;*

(t) *L-relation R is called transitive if $R(x,y)*R(y,z)\leq R(x,z)$ for all $x,y,z\in X$.*

A reflexive symmetric transitive L-relation is called an L-equivalence, or a similarity L-relation.

Let a lattice L be fixed and let $\mathbf{REL}(L)$ be the category whose objects are pairs (X,R), where X is a set and $R: X\times X\to L$ is a transitive reflexive L-relation on it. Morphisms in $\mathbf{REL}(L)$ are mappings $f:(X,R_X)\to(Y,R_Y)$ such that

$$R_X(x,x')\leq R_Y(f(x),f(x')) \text{ for all } x,x'\in X.$$

3 THE MEASURE OF INCLUSION OF L-fuzzy SETS

If $(L, \leq, \wedge, \vee, *)$ is a cl-monoid, and X is a set, then the lattice and the monoidal structures of L can be pointwise lifted to the L-powerset L^X of X. Namely, given $A, B \in L^X$ we set $A \leq B$ iff $A(x) \leq B(x)$ for all $x \in X$, and define operations on L^X by setting

$$(A \wedge B)(x) = A(x) \wedge B(x), (A \vee B)(x) = A(x) \vee B(x),$$
$$(A * B)(x) = A(x) * B(x)$$

for all $x \in X$. One can easily notice that in this way $(L^X, \leq, \wedge, \vee, *)$ becomes a cl-monoid.

Definition 3.1. *By setting*

$$A \hookrightarrow B = \inf_{x \in X} (A(x) \mapsto B(x))$$

for all $A, B \in L^X$ we obtain a mapping

$$\hookrightarrow: L^X \times L^X \to L.$$

Equivalently, $\hookrightarrow$ can be defined by

$$A \hookrightarrow B = \inf(A \mapsto B),$$

where the infimum of the L-fuzzy set $A \mapsto B$ is taken in the lattice L^X. We call $A \hookrightarrow B$ by the measure of inclusion of the L-fuzzy set A in the L-fuzzy set B.

As the next proposition shows, the measure of inclusion

$$\hookrightarrow: L^X \times L^X \to L$$

has properties in a certain sense resembling the properties of the residuation:

Proposition 3.2. *Mapping $\hookrightarrow: L^X \times L^X \to L$ defined above satisfies the following properties:*

(1$\hookrightarrow$) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$ *for all* $\{A_i \mid i \in I\} \subseteq L^X$, *for all* $B \in L^X$;

(2$\hookrightarrow$) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i (A \hookrightarrow B_i)$ *for all* $A \in L^X$, *for all* $\{B_i \mid i \in I\} \subseteq L^X$;

(3$\hookrightarrow$) $A \hookrightarrow B = 1_L$ *whenever* $A \leq B$;

(4$\hookrightarrow$) $1_X \hookrightarrow A = \inf_x A(x)$ *for all* $A \in L^X$;

(5$\hookrightarrow$) $A * (A \hookrightarrow B) \leq B$ *for all* $A, B \in L^X$;

(6$\hookrightarrow$) $(A \hookrightarrow B) * (B \hookrightarrow C) \leq A \hookrightarrow C$ *for all* $A, B, C \in L^X$;

(7$\hookrightarrow$) $A \hookrightarrow B \leq (A * C \hookrightarrow B * C)$ *for all* $A, B, C \in L^X$;

(8$\hookrightarrow$) $(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i, B_i \in I\} \subseteq L^X$;

(9$\hookrightarrow$) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i, B_i \in I\} \subseteq L^X$.

Proof. The proof can be done straightforward from the definition of operation $\hookrightarrow$ and applying properties of the residium $\mapsto: L \times L \to L$ collected in Proposition 2.3, Lemma 1 and Lemma 2.

4 ROUGH APPROXIMATION OF A FUZZY SET

Let $R : X \times X \to L$ be a reflexive transitive L-relation on a set X and $A \in L^X$. By the rough approximation of the L-fuzzy set A we call the pair $(u_R(A), l_R(A))$ where $u_R : L^X \to L^X$ and $l_R : L^X \to L^X$ are respectively operators of upper and lower rough approximations of A defined below.

4.1 Upper Rough Approximation

Given a reflexive transitive L-relation $R : X \times X \to L$, we define the upper rough approximation operator $u_R : L^X \to L^X$ by

$$u_R(A)(x) = \sup_{x'} \left(R(x, x') * A(x')\right) \ \forall A \in L^X, \ \forall x \in X.$$

Theorem 4.1. *The upper rough approximation operator satisfies the following properties:*[1]

(1u) $u_R(0_X) = 0_X$;

(2u) $A \leq u_R(A) \ \forall A \in L^X$;

(3u) $u_R(\bigvee_i A_i) = \bigvee_i u_R(A_i) \ \forall \{A_i \mid i \in I\} \subseteq L^X$;

(4u) $u_R(u_R(A)) = u_R(A) \ \forall A \in L^X$;

Proof. Statement (1u) is obvious. Statement (2u) follows easily taking into account reflexivity of the L-relation R. We prove property (3u) as follows:

$$u_R(\bigvee_i A_i)(x) = \sup_{x'} (R(x, x') * (\bigvee_i A_i(x'))) =$$
$$\sup_{x'} (\bigvee_i R(x, x') * A_i(x')) = \bigvee_i (\sup_{x'} (R(x, x') * A_i(x')))$$
$$= \bigvee_i (u_R(A_i)(x)) = \left(\bigvee_i (u_R(A_i)\right)(x).$$

Finally, taking into account transitivity of the L-relation we have:

$$u_R(u_R(A))(x) = \sup_{x'} (u_R(A)(x') * R(x, x')) =$$
$$\sup_{x''} \sup_{x'} (A(x'') * R(x, x') * R(x', x'')) \leq$$
$$\sup_{x''} A(x'') * R(x, x'') = u_R(A)(x).$$

Since the converse inequality follows from (2u), we get property (4u).

4.2 Lower Rough Approximation Induced by a Reflexive Transitive L-relation

Given a reflexive transitive L-relation $R : X \times X \to L$, we define a lower rough approximation operator $l_R : L^X \to L^X$ by

$$l_R(A)(x) = \inf_{x'} \left(R(x, x') \mapsto A(x')\right) \ \forall A \in L^X \ \forall x \in X.$$

[1]Similar results can be found e.g. in (Järvinen and Kortelainen , Qin and Pei 2005, Sostak 2010.)

Theorem 4.2. *The lower rough approximation operator satisfies the following properties:*[2]

(1l) $l_R(1_X) = 1_X$;

(2l) $A \geq l_R(A)\ \forall A \in L^X$;

(3l) $l_R(\bigwedge_i A_i) = \bigwedge_i l_R(A_i)\ \forall \{A_i \mid i \in I\} \subseteq L^X$;

(4l) $l_R(l_R(A)) = l_R(A)\ \forall A \in L^X$;

Proof. Statement (1l) is obvious. Statement (2l) follows easily taking into account reflexivity of the L-relation R. We prove property (3l) as follows:

$$l_R(\bigwedge_i A_i)(x) = \inf_{x'} \left(R(x,x') \mapsto \bigwedge_i A_i(x')\right) =$$
$$\inf_{x'} \bigwedge_i \left(R(x,x') \mapsto A_i(x')\right) =$$
$$\bigwedge_i \inf_{x'} \left(R(x,x') \mapsto A_i(x')\right) = \bigwedge_i l_R(A_i).$$

Finally, taking into account transitivity of the L-relation we have:

$$l_R(l_R(A))(x) = \inf_{x'}(R(x,x') \mapsto l_R(A)(x')) =$$
$$\inf_{x'}(R(x,x') \mapsto \inf_{x'}(R(x',x'') \mapsto A(x''))) =$$
$$\inf_{x'}(\inf_{x''}(R(x,x') * R(x',x'') \mapsto A(x''))) \geq$$
$$\inf_{x''}(R(x,x'') \mapsto A(x'')) = l_R(A)(x).$$

Since the converse inequality follows from (2l), we get property (4l).

5 THE MEASURE OF ROUGHNESS OF AN L-FUZZY SET

Let $R : X \times X \to L$ be a reflexive transitive L-relation on a set X. Given an L-fuzzy set $A \in L^X$ we define the measure $\mathcal{K}(A)$ of its upper rough approximation by

$$\mathcal{K}(A) = u_R(A) \hookrightarrow A$$

and the measure $\mathcal{T}(A)$ of its lower rough approximation by

$$\mathcal{T}(A) = A \hookrightarrow l_R(A).$$

Theorem 5.1. *If R is also symmetric, that is an equivalence L-relation, then $\mathcal{K}(A) = \mathcal{T}(A)$ for every L-fuzzy set A.*

Proof. For the measure of the upper rough approximation we have

$$\mathcal{K}(A) = u_R(A) \hookrightarrow A = \inf_x(u_R(A)(x) \mapsto A(x)) =$$
$$\inf_x \sup_{x'}(A(x) * R(x,x') \mapsto A(x')) =$$
$$\inf_x \inf_{x'} \left(A(x) * R(x,x') \mapsto A(x')\right).$$

[2]Similar results can be found e.g. in (Järvinen and Kortelainen , Qin and Pei 2005, Sostak 2010.)

On the other hand, for the lower rough approximations we have

$$A \hookrightarrow l_R(A) = \inf_x(A(x) \mapsto l_R(A)(x)) =$$
$$\inf_x(A(x) \mapsto \inf_{x'} \left(R(x,x') \mapsto A(x')\right)) =$$
$$\inf_x \inf_{x'}(A(x) \mapsto (R(x,x') \mapsto A(x')) =$$
$$\inf_x \inf_{x'}(A(x) * R(x,x') \mapsto A(x').$$

Since $R(x,x') = R(x',x)$ in case R is symmetric, to complete the proof it is sufficient to notice that

$$(\alpha * \beta) \mapsto \gamma = \alpha \mapsto (\beta \mapsto \gamma)$$

for any $\alpha, \beta, \gamma \in L$. Indeed

$$(\alpha * \beta) \mapsto \gamma = \bigvee\{\lambda \mid \lambda * (\alpha * \beta) \leq \gamma\},$$

and

$$\alpha \mapsto (\beta \mapsto \gamma) = \bigvee\{\lambda \mid \lambda * \alpha \leq \beta \mapsto \gamma\} =$$
$$\bigvee\{\lambda \mid (\lambda * \alpha) * \beta \mapsto \gamma\};$$

the last equality is justified by Galois connection between $\mapsto$ and $*$.
$\square$

The previous theorem allows us to introduce the following definition:

Definition 5.2. *Let R be an equivalence L-relation on a set X and $A \in L^X$. The measure of rough approximation of A is defined by*

$$\mathcal{RA}(A) = u_R(A) \hookrightarrow A = A \hookrightarrow l_R(A).$$

In the next theorem we collect the main properties of the operators $\mathcal{K} : L^X \to L$ and $\mathcal{T} : L^X \to L$, and hence also of the operator $\mathcal{RA} : L^X \to L$ in case the relation R is symmetric.

Theorem 5.3. *Measures of roughness of upper and lower rough approximation $\mathcal{K}, \mathcal{T} : L^X \to L$ has the following properties:*

1. $\mathcal{K}(0_X) = 1_L$
   ($0_X : X \to L$ is the constant function $0_X(x) = 0_L\ \forall x \in X$);
2. $\mathcal{T}(1_X) = 1_L$
   ($1_X : X \to L$ is the constant function $1_X(x) = 1_L\ \forall x \in X$);
3. $\mathcal{K}(u(A)) = 1_L$
   for every $A \in L^X$;
4. $\mathcal{T}(l(A)) = 1_L$
   for every $A \in L^X$;
5. $\mathcal{K}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{K}(A_i)$
   for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$;
6. $\mathcal{T}(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$
   for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$.

Proof. (1) Referring to Theorem 4.1 and applying Proposition 3.2 ($3 \hookrightarrow$) we have

$$\mathcal{K}(A) = u_R(0_X) \hookrightarrow 0_X = 0_X \hookrightarrow 0_X = 1_L.$$

(2) Referring to Theorem 4.2 and applying Proposition 3.2 ($3 \hookrightarrow$), we have

$$\mathcal{T}(A) = 1_X \hookrightarrow l_R(1_X) = 1_X \hookrightarrow 1_X = 1_L.$$

(3) Referring to Theorem 4.1 and applying Proposition 3.2 ($3 \hookrightarrow$), we have

$$\mathcal{K}(u(A))) = u_R(u_R(A) \hookrightarrow u_R(A) = u_R(A) \hookrightarrow u_R(A) = 1.$$

(4) Referring to Theorem 4.2 and applying Proposition 3.2 ($3 \hookrightarrow$), we have

$$\mathcal{T}(l(A))) = l_R(A) \hookrightarrow l_R(l_R(A)) = l_R(A) \hookrightarrow l_L(A) = 1'$$

(5) Referring to Theorem 4.1 and applying Proposition 3.2 ($8 \hookrightarrow$), we have

$$\mathcal{K}(\bigvee_i A_i) = u_R(\bigvee_i A_i) \hookrightarrow \bigvee_i A_i =$$

$$\bigvee_i u_R(A_i) \hookrightarrow \bigvee_i A_i \geq \bigwedge_i (u(A_i) \hookrightarrow A_i).$$

(6) Referring to Theorem 4.2 and applying Proposition 3.2 ($9 \hookrightarrow$), we have

$$\mathcal{T}(\bigwedge_i A_i) = (\bigwedge_i A_i) \hookrightarrow l(\bigwedge_i A_i) =$$

$$\bigwedge_i A_i \hookrightarrow \bigwedge_i l(A_i) \geq \bigwedge_i A_i \hookrightarrow l(A_i).$$

□

Theorem 5.4. *Let $R_X : X \times X \to L$ and $R_Y : Y \times Y \to L$ be reflexive transitive L-relations on sets X and Y respectively. Further, let $f : X \to Y$ be a mapping such that*

$$R_X(x,x') \leq R_Y(f(x), f(x'))$$

for every $x,x' \in X$. Then

$$\mathcal{K}_X(f^{-1}(B)) \geq \mathcal{K}_Y(B) \text{ and } \mathcal{T}_X(f^{-1}(B)) \geq \mathcal{T}_Y(B)$$

for every $B \in L^Y$.

Proof. follows from the next sequences of (in)equalities:

$$\mathcal{K}_X(f^{-1}(B)) = u_{R_X}(f^{-1}(B)) \hookrightarrow f^{-1}(B) =$$

$$\inf_x (u_{R_X}(f^{-1}(B)(x) \mapsto f^{-1}(B)(x)) =$$

$$\inf_x (\sup_{x'} B(f(x)) * R(x,x') \mapsto B(f(x))) \geq$$

$$\inf_x (\sup_{x'} B(f(x') * R(f(x), f(x')) \mapsto B(f(x)) \geq$$

$$\inf_y (\sup'_y B(y') * R(y,y') \mapsto B(y)) = \mathcal{K}_Y(B)$$

and

$$\mathcal{T}_X(f^{-1}(B)) = f^{-1}(B) \hookrightarrow l_{R_X}(f^{-1}(B)) =$$

$$\inf_x (f^{-1}(B)(x) \mapsto l_{R_X}(f^{-1}(B))(x)) =$$

$$\inf_x (B(f(x)) \mapsto \inf_{x'} (R(x,x') \mapsto f^{-1}(x')) =$$

$$\inf_x inf_{x'} (B(f(x)) \mapsto (R(x,x') \mapsto B(f(x'))) \geq$$

$$\inf_x \inf_{x'} (B(f(x)) \mapsto (R(f(x), f(x')) \mapsto B(f(x)) \geq$$

$$\inf_y \inf_{y'} (B(y) \mapsto (R(y,y') \mapsto B(y)) =$$

$$B \hookrightarrow l_{R_Y}(B) = \mathcal{T}_Y(B).$$

□

Example 5.5. *Let $*_L$ be the Łukasiewicz t-norm on the unit interval $L = [0,1]$, that is*

$$\alpha * \beta = \min(\alpha + \beta - 1, 1)$$

and $\mapsto_L : L \times L \to L$ be the corresponding residium, that is

$$\alpha \mapsto_L \beta = \max\{1 - \alpha + \beta, 0\}.$$

Then, given an equivalence L-relation R on a set X and $A \in L^X$ we have:

$$\mathcal{K}(A) = \inf_x \inf_{x'} (2 - A(x) + A(x') - R(x,x'));$$

$$\mathcal{T}(A) = \inf_x \inf_{x'} (2 - A(x) + A(x') - R(x',x)).$$

In particular, if $R : X \times X \to [0,1]$ is the discrete relation, that is

$$R(x,x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\mathcal{RA}(A) = 1 \text{ for every } A \in L^X.$$

On the other hand for the indiscrete relation (that is $R(x,x') = 1$ for all $x,x' \in X$)

$$\mathcal{RA}(A) = 1 - \inf_{x,x'} \mid A(x) - A(x') \mid \text{ for all } A \in L^X.$$

Example 5.6. *Let $* = \wedge$ be the minimum t-norm on the unit interval $L = [0,1]$, and $\mapsto : L \times L$ be the corresponding residium, that is*

$$\alpha \mapsto \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}.$$

Then given a reflexive transitive L-relation R on a set X and $A \in L^X$ we have:

$$\mathcal{K}(A) = \inf_x \inf_{x'} (A(x') \wedge R(x,x') \mapsto A(x));$$

$$\mathcal{T}(A) = \inf_x \inf_{x'} (A(x') \wedge R(x',x) \mapsto A(x)).$$

In particular, $\mathcal{RA}(A) = 1$ for every $A \in L^X$ in case the relation R is symmetric.

Example 5.7. *Let $* = \cdot$ be the product t-norm on the unit interval [0,1], and $\mapsto: L \times L$ be the corresponding residium, that is*

$$\alpha \mapsto \beta = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \frac{\beta}{\alpha} & \text{if } \alpha > \beta \end{cases}.$$

Then, for a reflexive transitive L-relation we have:

$$\mathcal{K}(A) = \inf_x \inf_{x'} (A(x') \cdot R(x,x') \mapsto A(x));$$

$$\mathcal{T}(A) = \inf_x \inf_{x'} (A(x') \cdot R(x',x) \mapsto A(x)).$$

In particular

$$\mathcal{R}\mathcal{A}(A) = \inf_x \inf_{x'} (A(x') \cdot R(x,x') \mapsto A(x))$$

in case R is symmetric.

6 MEASURE OF ROUGHNESS OF A FUZZY SET: DITOPOLOGICAL INTERPRETATION[3]

Notice that conditions (2), (4), and (6) of Theorem 5.3 actually mean that the mapping $\mathcal{T} : L^X \to L$ is an L-fuzzy topology on the set X, (see e.g. Sostak 1989, Sostak 1996), while conditions (1), (3), and (5) of this Theorem mean that the mapping $\mathcal{K} : L^X \to L$ is an L-co-topology on this set (see e.g. (Sostak 1985, Kubiak 1985, Sostak 1989, Sostak 1996). Since the mappings $\mathcal{T}$ and $\mathcal{K}$ are not mutually related via complementation on the lattice L (which even need not exist on the lattice) we may interpret the pair $(\mathcal{T}, \mathcal{K})$ as an L-fuzzy ditopology on the set X (Brown, Ertürk and Dost 2000).

Let $\alpha \in L$ be fixed and let

$$\mathcal{K}_\alpha = \{A \mid \mathcal{K}(A) \geq \alpha\} \text{ and } \mathcal{T}_\alpha = \{A \mid \mathcal{T}(A) \geq \alpha\}.$$

Then, applying again Theorem 5.3, we easily conclude that $\mathcal{T}_A$ satisfies the axioms of a Chang-Goguen L-topology, see (Chang 1968, Goguen 1973) and $\mathcal{K}_A$ satisfies the axioms of a Chang-Goguen L-co-topology. Hence for each $\alpha \in L$ the pair $(\mathcal{T}_\alpha, \mathcal{K}_\alpha)$ can be realized as a a Chang-Goguen L-topology on X (Brown, Ertürk and Dost 2000).

From Theorem 5.4 we conclude that if $f : (X, R_X) \to (Y, R_Y)$ is a morphism in the category $\mathbf{REL}(L)$ of sets endowed with reflexive transitive L-relations, then

$$f : (X, \mathcal{T}_X, \mathcal{K}_X) \to (Y, \mathcal{T}_Y, \mathcal{K}_Y)$$

is continuous mapping of the corresponding L-fuzzy ditopological spaces. Thus we come to the following

Theorem 6.1. *By assigning to every object (X, R_X) from the category $\mathbf{REL}(L)$ (see subsection 2.2) an L-fuzzy ditopological space $(X, \mathcal{T}_X, \mathcal{K}_X)$, and interpreting a morphism $f : (X, R_X) \to (Y, R_Y)$ of $\mathbf{REL}(L)$ as a mapping $f : (X, \mathcal{T}_X, \mathcal{K}_X) \to (Y, \mathcal{T}_Y, \mathcal{K}_Y)$ we obtain a functor*

$$\Phi : \mathbf{REL}(L) \to L\text{-}\mathbf{DiTop}(L),$$

where L-$\mathbf{DiTop}(L)$, is the category of L-fuzzy ditopological spaces and their continuous mappings.

Corollary 6.2. *Let $\alpha \in L$ be fixed. By assigning to every object (X, R_X) from the category $\mathbf{REL(}L\mathbf{)}$ a Chang-Goguen L-ditopological space $(X, \mathcal{T}_{X\alpha}, \mathcal{K}_{X\alpha})$, and realizing a morphism $f : (X, R_X) \to (Y, R_Y)$ from $\mathbf{REL(}L\mathbf{)}$ as a mapping $f : (X, \mathcal{T}_{X\alpha}, \mathcal{K}_{X\alpha}) \to (Y, \mathcal{T}_{Y\alpha}, \mathcal{K}_{Y\alpha})$ we obtain a functor*

$$\Phi_\alpha : \mathbf{REL}(L) \to L\text{-}\mathbf{DiTop},$$

where L-$\mathbf{DiTop}$ is the category of Chang-Goguen L-ditopological spaces and their continuous mappings.

7 CONCLUSION

In this paper we proposed an approach allowing to measure the roughness of lower and upper rough approximation for fuzzy subsets of a set endowed with a reflexive transitive L-relation. The basics of the theory of roughness measure were developed here. Besides, a natural interpretation of the operator of measure of rough approximation as a fuzzy ditopology was sketched here. However, several crucial issues concerning this theory remain untouched in this work. As one of the first goals for the further work we see the development of a consistent categorical viewpoint on the measure of rough approximation. In particular, it is important to study the behavior of the measure of approximation under operations of products, direct sums, quotients, etc, and to research the behavior of the measure of roughness under images and preimages of special mappings between sets endowed with reflexive transitive fuzzy relations..

Another interesting, in our opinion, direction of the research is to develop the topological model of this theory sketched in Section 6. The restricted volume of this work does not allow us to linger on this subject. However, in our opinion the topological interpretation of the theory could be helpful for further studies.

Besides we hope that the concept of measure of rough approximation will be helpful also in some problems of practical nature, since since it allows in a certain sense to measure the quality of the rough approximation.

[3]In this section we give an alternative view on the concepts studied in the work. A reader not interested in the topological aspects of approximation, may omit this section.

ACKNOWLEDGEMENTS

The support of the ESF project 2013/0024/1DP/ 1.1.1.2.0/13/APIA/VIAA/045 is kindly announced.

REFERENCES

Birkhoff, G . Lattice Theory, AMS Providence, RI, (1995)

Brown, L.M., Ertürk, R. Dost Ş. *Ditopological texture spaces and fuzzy topology, I. Basic concepts,* Fuzzy Sets and Syst. 110, 227–236 (2000).

Ciucci, D. *Approximation algebra and framework,* Fund. Inform. 94(2) (2009) 147–161.

Chang, C.L. *Fuzzy topological spaces*, J. Math. Anal. Appl. 24, 182–190 (1968)

Chen, P., Zhang, D. *Alexandroff L-cotopological spaces*, Fuzzy Sets and Systems 161 (2010) 2505 – 2514.

Ciucci, D. *Approximation algebra and framework,* Fund. Inform. 94(2) (2009) 147–161.

Gierz, G., Hoffman, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S. Continuous Lattices and Domains, Cambridge Univ. Press, Cambridge, 2003

Goguen, J.A. *The fuzzy Tychonoff theorem,* J. Math. Anal. Appl. 43 (1973) 734–742.

Dubois, D., Prade, H. *Rough fuzzy sets and fuzzy rough sets*, International J. General Systems 17 (2–3) (1990) 191–209.

Eļkins, A., Šostak, A. *On some categories of approximate systems generated by L-relations.* In: 3^{rd} Rough Sets Theory Workshop, pp. 14–19 Milan, Italy (2011)

Hao J., Li Q. *The relation between L-fuzzy rough sets and L-topology*, Fuzzy Sets and Systems, 178 (2011)

Höhle U. *Commutative residuated l-monoids*, in: U. Höhle abd E.P. Klement eds., Nonclassical Logics and their Appl. to Fuzzy Subsets, 53–106, Kluwer Acad. Publ., Docrecht, Boston, 1995.

Järvinen J. *On the structure of rough approximations,* Fund. Inform. 53 (2002) 135–153.

Järvinen J., Kortelainen J. *A unified study between modal-like operators, topologies and Fuzzy Sets*, Fuzzy Sets and Systems 158 (2007) 1217–1225.

Kaufman A. Theory of Fuzzy Subsets, Academic Press, New York, 1975.

Klement, E.P., Mesiar, R., Pap, E. Triangular Norms. Kluwer Acad. Publ. (2000)

Kortelainen J. *On relationship between modified sets, topological spaces and rough sets*, Fuzzy Sets and Systems 61 (1994) 91-95.

Kubiak T. *On fuzzy topologies*, PhD Thesis, Adam Mickiewicz University Poznan, Poland (1985)

Menger K. Geometry and positivism - a probabilistic microgeometry, Selected Papers in Logic, Foundations, Didactics, Economics, Reidel, Dodrecht, 1979.

Mi J.S., Hu B.Q. *Topological and lattice structure of L-fuzzy rough stes determined by upper and lower sets,* Information Sciences 218 (2013) 194–204.

Pawlak, Z. *Rough sets*, Intern. J. of Computer and Inform. Sci. 11, 341–356 (1982)

Qin K., Pei Z. *On the topological properties of fuzzy rough sets*, Fuzzy Sets and Systems 151 (2005) 601–613.

Qin K., Pei Z. *Generalized rough sets based on reflexive and transitive relations*, Information Sciences 178 (2008) 4138–4141.

Radzikowska, A.M., Kerre, *E.E. A comparative study of fuzzy rough sets.* Fuzzy Sets and Syst. 126, 137–155 (2002)

Rosenthal, K.I. Quantales and Their Applications. Pirman Research Notes in Mathematics 234. Longman Scientific & Technical (1990)

Schweitzer, B., Sklar, A. Probabilistic Metric Spaces. North Holland, New York (1983).

Šostak, A. *On a fuzzy topological structure,* Suppl. Rend. Circ. Matem. Palermo, Ser II 11, 125–186 (1980). Russian Math. Surveys 44 (1989) 125–186

Šostak, A. *Two decades of fuzzy topology: Basic ideas, notions and results,* Russian Math. Surveys 44 (1989) 125–186

Šostak, A. *Basic structures of fuzzy topology,* J. Math. Sci. 78 (1996) 662–701.

Šostak A. *Towards the theory of M-approximate systems: Fundamentals and examples.* Fuzzy Sets and Syst. 161, 2440–2461 (2010)

Šostak, A. *Towards the theory of approximate systems: variable range categories.* In: Proceedings of ICTA2011, Islamabad, Pakistan. pp. 265–284. Cambridge University Publ. (2012)

Tiwari, S.P., Srivastava, A.K. *Fuzzy rough sets. fuzzy preoders and fuzzy topoloiges*, Fuzzy Sets and Syst., 210 (2013), 63–68.

Valverde, L. *On the structure of F-indistinguishability operators*, Fuzzy Sets and Syst. 17, 313–328 (1985)

Yao, Y.Y. *A comparative study of fuzzy sets and rough sets*. Inf. Sci. 109, 227–242 (1998)

Yao, Y.Y. *On generalizing Pawlak approximation operators.* In: Proc. First Intern. Conf. Rough Sets and Current Trends in Computing. pp. 298–307 (1998)

Yu H., Zhan W.R. *On the topological properties of generalized rough sets*, Information Sciences 263 (2014), 141–152.

Zadeh L. *Similarity relations and fuzzy orderings*, Inf. Sci. 3 (1971) 177–200.