Rough Approximations in Algebras of a Non-associative Generalization of the Łukasiewicz Infinite Valued Logic

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Keywords: Commutative Basic Algebra, MV-algebra, Non-associative Logic, Łukasiewicz Logic, Classical Rough Set, Approximation Space, Ideal, Congruence.

Abstract: Commutative basic algebras are non-associative generalizations of MV-algebras. They are an algebraic counterpart of a non-associative propositional logic which generalizes the Łukasiewicz infinite valued logic and which is related to reasoning under uncertainty. The paper investigates approximation spaces in commutative basic algebras based on their ideals.

1 INTRODUCTION

Rough sets were introduced by Pawlak (Pawlak, 1982) in 1982 to give a new mathematical approach to vagueness. The key idea is that our knowledge about the properties of the objects of a given universe of discourse may be inadequate or incomplete in the sense that the objects of this universe can be observed only within the accuracy of indiscernibility relations. Recall that in the classical rough set theory, subsets are approximated by means of pairs of ordinary sets, so-called lower and upper approximations, which are e.g. composed by some classes of given equivalences.

It is known that the basic (fuzzy) logic $\mathcal{BL}$ is the logic of continuous t-norms and their residua (Hájek, 1998). That means, if a continuous t-norm & is considered as the truth function of conjunction and its residuum $\to$ is the truth function of implication, then each evaluation of propositional variables by truth values from [0,1] extends to the evaluation of each formula. (See (Hájek, 1998), (Botur and Halaš, 2009).) In all these logics the conjunction & is associative, i.e., for arbitrary formulas ϕ, ψ, χ, the formula $\phi \mathbin{\&} (\psi \mathbin{\&} \chi) \longleftrightarrow (\phi \mathbin{\&} \psi) \mathbin{\&} \chi$ is provable.

But there are situations where the associativity of & need not be satisfied. Let we have expert systems where we need estimate for the degree of certainty of conjunction and disjunction of statements $S_1, \ldots, S_n$ of which they are not completely sure. This uncertainty is described by the probabilities p_i assigned to the statements S_i. The conclusion C of an expert system usually depends on several statements S_i. Then, e.g., the probability $p(S_1 \mathbin{\&} S_2)$ of $S_1 \mathbin{\&} S_2$ can take different values depending on whether S_1 and S_2 are independent or correlated. It is known that for given $p_1 = p(S_1)$ and $p_2 = p(S_2)$, possible values of $p(S_1 \mathbin{\&} S_2)$ form an interval $\mathbf{p} = [p^-, p^+] \subseteq [0, 1]$, where $p^- = \max(p_1 + p_2 - 1, 0)$ and $p^+ = \min(p_1, p_2)$. (See (Kreinovich, 2004) or (Botur and Halaš, 2009).)

Therefore we can use such interval estimates to get an interval $\mathbf{p}(C)$ of possible values of $p(C)$. But the interval $\mathbf{p}(C)$ can be too large. Then in such situations it is reasonable to select a point within this interval as an estimate for $p(S_1 \mathbin{\&} S_2)$, e.g., a midpoint of this interval. That means, we can evaluate $S_1 \mathbin{\&} S_2 := \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2)$. (See (Botur and Halaš, 2009).) It is obvious that operation & is not associative.

Hence we can see that in such situations we need to have a propositional logic which generalizes fuzzy logics, e.g. Łukasiewicz, Gödel or product logic, such that the conjunction is not necessarily associative.

In (Botur and Halaš, 2009), the authors proposed a logic foundation for fuzzy reasoning with non-associative conjunction in the form of a new formal deductive system $\mathcal{L}_{CBA}$. This logic is very close to the Łukasiewicz logic (differs just in this non-associativity of the conjunction). The authors have shown that $\mathcal{L}_{CBA}$ is algebraizable logic in the sense of (Blok and Pigozzi, 1989) and that its equivalent algebraic semantics is the variety of commutative basic algebras. Since MV-algebras are an algebraic coun-

Rachůnek J. and Šalounová D..
Rough Approximations in Algebras of a Non-associative Generalization of the Łukasiewicz Infinite Valued Logic.
DOI: 10.5220/0005129401580162
In *Proceedings of the International Conference on Fuzzy Computation Theory and Applications* (FCTA-2014), pages 158-162
ISBN: 978-989-758-053-6

terpart of the Łukasiewicz logic, commutative basic algebras are appropriate non-associative generalizations of MV-algebras.

MV-algebras are an algebraic semantics of a logic with truth values from the real interval [0, 1] and thus it is natural that rough sets in MV-algebras were introduced and investigated. (See (Rasouli and Davvaz, 2010).) The corresponding approximate spaces are based on congruences or, equivalently, on ideals of MV-algebras.

In the paper we introduce and study approximate spaces in commutative basic algebras. Analogously as in MV-algebras, congruences correspond to ideals and so we deal with approximate spaces based on ideals of these algebras.

2 PRELIMINARIES

An algebra $A = (A; \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ is called a *basic algebra* (Chajda et al., 2009) if for any $x, y, z \in A$:

(1) $x \oplus 0 = x$;
(2) $\neg\neg x = x$;
(3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;
(4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

If the groupoid $(A; \oplus)$ is commutative then $(A; \oplus, \neg, 0)$ is a *commutative basic algebra*.

Put $1 := \neg 0$. Let $\leq$ be the binary relation on A such that

$$x \leq y :\Longleftrightarrow \neg x \oplus y = 1.$$

Then $\leq$ is an order and the ordered set $(A; \leq)$ is a bounded lattice, where 0 is the least and 1 the greatest element, and for the lattice operations we have

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

The class of basic algebras contains certain classes of algebras of many-valued and quantum logics. For example, MV-algebras, orthomodular lattices and lattice effect algebras can be viewed as particular cases of basic algebras (see (Chajda et al., 2009)).

In what follows, we will deal with commutative basic algebras. Recall that in such a case the lattice $(A; \vee, \wedge)$ is distributive (Chajda et al., 2009). Moreover, every finite commutative basic algebra is an MV-algebra (Botur and Halaš, 2008), but there are commutative basic algebras which are not MV-algebras. (Recall that MV-algebras are just associative commutative basic algebras.)

Define, for any $x, y \in A$,

$$x \ominus y := \neg(\neg x \oplus y).$$

For the fundamental properties of commutative basic algebras see (Botur and Halaš, 2008), (Botur and Halaš, 2009) or (Botur et al., 2012).

Let A be a commutative basic algebra and $\emptyset \neq I \subseteq A$. Then I is called

(a) a *preideal* of A if
  (i) $x, y \in I \Longrightarrow x \oplus y \in I$;
  (ii) $x \in I,\ y \in A,\ y \leq x \Longrightarrow y \in I$;
(b) an *ideal* of A if I is the 0-class of some congruence on A.

(See (Krňávek and Kühr, 2011) or (Botur et al., 2012).)

Every ideal of A is a preideal of A but not conversely (Krňávek and Kühr, 2011). Ideals of A are exactly kernels of congruences and since the variety of commutative basic algebras is congruence regular, any ideal I is the 0-class of a unique congruence θ_I on A. Then $(x,y) \in \theta_I$ iff $x \ominus y,\ y \ominus x \in I$. Hence we will denote the quotient algebra A/θ_I also in the form A/I.

Let $\mathcal{P}(A)$ and $\mathcal{I}(A)$ be the set of preideals and ideals of A, respectively. Then by (Krňávek and Kühr, 2011), $(\mathcal{P}(A), \subseteq)$ is a distributive complete lattice and $(\mathcal{I}(A), \subseteq)$ is its complete sublattice.

An *additive term* is a commutative basic algebra term in which the symbol $\neg$ does not occure. If A is a commutative basic algebra and $\emptyset \neq B \subseteq A$, then the preideal $\langle B\rangle$ generated by B contains exactly those elements $a \in A$ such that $a \leq \tau(b_1, \dots, b_n)$ for some n-ary additive term τ and $b_1, \dots, b_n \in B$.

Now we recall some basic notions of the theory of classical approximation spaces. An *approximation space* is a pair (S, θ) where S is a set and θ an equivalence on S. For any approximation space (S, θ), by the *upper rough approximation* in (S, θ) we will mean the mapping $\overline{Apr} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ such that $\overline{Apr}(X) := \{x \in S : x/\theta \cap X \neq \emptyset\}$ and by the *lower rough approximation* in (S, θ) the mapping $\underline{Apr} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ such that $\underline{Apr}(X) := \{x \in S : x/\theta \subseteq X\}$, for any $X \subseteq S$. (x/θ is the class of S/θ containing x.)

If $\overline{Apr}(X) = \underline{Apr}(X)$ then X is called a *definable* set, otherwise X is called a *rough* set.

3 APPROXIMATIONS INDUCED BY IDEALS

In this section we introduce and investigate special approximation spaces (A, θ) such that A is the universe of a commutative basic algebra and θ is a congruence on this basic algebra.

If $A = (A; \oplus, \neg, 0)$ is a commutative basic algebra, θ a congruence on A and $I = I_\theta$ the corresponding ideal, then $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ will denote the lower and upper rough approximation of any $X \subseteq A$ in the approximation space (A, θ).

Proposition 3.1. *Let A be a commutative basic algebra, I an ideal of A and $a, b \in A$. Then $a/I = b/I$ if and only if there are $x, y \in I$ such that $a = (b \oplus y) \ominus x$.*

Proof. Let $a = (b \oplus y) \ominus x$, where $x, y \in I$. Then $b \ominus a = b \ominus ((b \oplus y) \ominus x) \leq b \ominus (b \ominus x) = b \wedge x \leq x$, thus $b \ominus a \in I$.

Further, $a \ominus b = ((b \oplus y) \ominus x) \ominus b \leq (b \oplus y) \ominus b = y \wedge \neg b \leq y \in I$, hence $a \ominus b \in I$.

Therefore $a/I = b/I$.

Conversely, let $a/I = b/I$, i.e. $x = b \ominus a$, $y = a \ominus b \in I$. We have $y \oplus b = (a \ominus b) \oplus b = a \vee b = (b \ominus a) \oplus a = x \oplus a$, thus $(y \oplus b) \ominus x = (x \oplus a) \ominus x = a \wedge \neg x$. At the same time $x = b \ominus a \leq 1 \ominus a = \neg a$, that means $x \leq \neg a$, hence $a \leq \neg x$, thus $a \wedge \neg x = a$.

Therefore we get $a = (b \oplus y) \ominus x$. □

If A is a commutative basic algebra and $\emptyset \neq X, Y \subseteq A$, denote by $\langle X, Y \rangle$ the preideal of A generated by $X \cup Y$.

If $\tau = \tau(z_1, \ldots, z_n)$ is an additive term, then by $l(\tau)$ we will mean the number of occurencies of the variables $z_1, \ldots, z_n$ in τ.

Lemma 3.2. *(Botur et al., 2012) Let A be a commutative basic algebra and I an ideal of A. Then, for all $a, b \in A$, $a \oplus (b \oplus I) = (a \oplus b) \oplus I$.*

Theorem 3.3. *Let I be an ideal of a commutative basic algebra A and $\emptyset \neq X, Y \subseteq A$. Then*

$$\overline{Apr}_I(\langle X, Y \rangle) \subseteq \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle.$$

If A is linearly ordered then

$$\overline{Apr}_I(\langle X, Y \rangle) = \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle.$$

Proof. If $a \in \overline{Apr}_I(\langle X, Y \rangle)$ then $a/I \cap \langle X, Y \rangle \neq \emptyset$. Let $b \in a/I \cap \langle X, Y \rangle$ and $b \leq \tau(z_1, \ldots, z_n)$ where τ is an n-ary additive term and $z_i \in X \cup Y$, $i = 1, \ldots, n$. Suppose that τ_1 and τ_2 are n-ary additive terms such that $l(\tau_1), l(\tau_2) < l(\tau)$ and $\tau(z_1, \ldots, z_n) = \tau_1(z_1, \ldots, z_n) \oplus \tau_2(z_1, \ldots, z_n)$. Since $a/I = b/I$, there are $x, y \in I$ such that $a = (b \oplus y) \ominus x$. Hence $a = (b \oplus y) \ominus x \leq b \oplus y \leq (\tau_1(z_1, \ldots, z_n) \oplus \tau_2(z_1, \ldots, z_n)) \oplus y = \tau_1(z_1, \ldots, z_n) \oplus (\tau_2(z_1, \ldots, z_n) \oplus u)$ where $u \in I$. Since $(\tau_2(z_1, \ldots, z_n) \oplus u)/I = \tau_2(z_1, \ldots, z_n)/I \oplus u/I = \tau_2(z_1, \ldots, z_n)/I$ and $z_i \in \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$, $i = 1, \ldots, n$, we obtain $a \in \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle$.

Suppose A is linearly ordered. Let $a \in \langle \overline{Apr}_I(X), \overline{Apr}_I(Y) \rangle$, $a \leq \tau(v_1, \ldots, v_n)$, where τ is an n-ary additive term, $v_i \in \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$, $i = 1, \ldots, n$. Let $w_i \in v_i/I \cap X$, provided $v_i \in X$, and $w_i \in v_i/I \cap Y$, provided $v_i \in Y$, and let $z \in a/I$. Suppose $a/I \neq \tau(w_1, \ldots w_n)/I$. Since A is linearly ordered, $z < \tau(w_1, \ldots w_n)$, hence $z \in \langle X, Y \rangle$. Therefore $a \in \overline{Apr}_I(\langle X, Y \rangle)$. □

Theorem 3.4. *Let I be an ideal of a commutative basic algebra A and $\emptyset \neq X, Y \subseteq A$. Then*

$$\langle \underline{Apr}_I(X), \underline{Apr}_I(Y) \rangle \subseteq \underline{Apr}_I(\langle X, Y \rangle).$$

Proof. Let $a \in \langle \underline{Apr}_I(X), \underline{Apr}_I(Y) \rangle$. Suppose $a \leq \tau(z_1, \ldots, z_n)$, where τ is an n-ary additive term and $z_i \in \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$, $i = 1, \ldots, n$. Let $b \in a/I$. Then there are $x, y \in I$ with $b = (a \oplus x) \ominus y$. If τ_1 and τ_2 are n-ary additive terms such that $l(\tau_1), l(\tau_2) < l(\tau)$ and $\tau(z_1, \ldots, z_n) = \tau_1(z_1, \ldots, z_n) \oplus \tau_2(z_1, \ldots, z_n)$, then $b = (a \oplus x) \ominus y \leq a \oplus x \leq \tau(z_1, \ldots, z_n) \oplus x = (\tau_1(z_1, \ldots, z_n) \oplus \tau_2(z_1, \ldots, z_n)) \oplus x = \tau_1(z_1, \ldots, z_n) \oplus (\tau_2(z_1, \ldots, z_n) \oplus u)$, where $u \in I$.

We have $(\tau_2(z_1, \ldots, z_n) \oplus u)/I = \tau_2(z_1, \ldots, z_n)/I = \tau_2(z_1/I, \ldots, z_n/I) \subseteq \langle X, Y \rangle$, because $z_i/I \subseteq X \cup Y$, $i = 1, \ldots, n$.

Analogously $\tau_1(z_1/I, \ldots, z_n/I) \subseteq \langle X, Y \rangle$, thus also $\tau(z_1/I, \ldots, z_n/I) \subseteq \langle X, Y \rangle$, i.e. $b \in \langle X, Y \rangle$. Therefore $a \in \underline{Apr}_I(\langle X, Y \rangle)$. □

Theorem 3.5. *Let A be a linearly ordered commutative basic algebra, I an ideal of A and $X \neq \emptyset$ a convex subset of A. Then also $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ are convex.*

Proof. Let $x, y \in \underline{Apr}_I(X)$, $z \in A$, $x \leq z \leq y$ and $x/I \neq z/I \neq y/I$. Suppose $a \in z/I$. The congruence θ_I has convex classes, hence for any elements $x_1 \in x/I$, $y_1 \in y/I$ and $z_1 \in z/I$ we have $x_1 < z_1 < y_1$, thus $z_1 \in \underline{Apr}_I(X)$, and therefore $z \in \underline{Apr}_I(X)$. That means $\underline{Apr}_I(X)$ is convex

Let now $x, y \in \overline{Apr}_I(X)$ and $z \in A$ such that $x \leq z \leq y$. Let $x_1 \in x/I \cap X$, $y_1 \in y/I \cap X$ and $x/I \neq z/I \neq y/I$. If $z_1 \in z/I$, then $x_1 < z_1 < y_1$. Since $x_1, y_1 \in X$, we get $z_1 \in z/I \cap X$, therefore $z \in \overline{Apr}_I(X)$. That means $\overline{Apr}_I(X)$ is convex. □

Let A be a commutative basic algebra. If $B \subseteq A$, put $\neg B := \{\neg b : b \in B\}$.

Theorem 3.6. *Let A be a commutative basic algebra, I an ideal of A and $\emptyset \neq X \subseteq A$. Then*

a) $\neg \overline{Apr}_I(X) = \overline{Apr}_I(\neg X)$;

b) $\neg \underline{Apr}_I(X) = \underline{Apr}_I(\neg X)$.

Proof. a) Let $x \in \neg \overline{Apr}_I(X)$. Then $\neg x \in \overline{Apr}_I(X)$, thus $\neg x/I \cap X \neq \emptyset$. Let $y \in \neg x/I \cap X$. Then $\neg x \ominus y$, $y \ominus \neg x \in I$, hence also $x \ominus \neg y$, $\neg y \ominus x \in I$ and $\neg y \in \neg X$. Therefore $x \in \overline{Apr}_I(\neg X)$, and so $\neg \overline{Apr}_I(X) \subseteq \overline{Apr}_I(\neg X)$.

Let $x \in \overline{Apr}_I(\neg X)$ and $y \in x/I \cap \neg X$. Then $x \ominus y$, $y \ominus x \in I$, hence also $\neg x \ominus \neg y$, $\neg y \ominus \neg x \in I$, and $\neg y \in X$. Thus $\neg x/I \cap X \neq \emptyset$, so $\neg x \in \overline{Apr}_I(X)$, and consequently $x \in \neg\overline{Apr}_I(X)$. That means $\overline{Apr}_I(\neg X) \subseteq \neg\overline{Apr}_I(X)$.

b) Let $x \in \neg\underline{Apr}_I(X)$. Then $\neg x \in \underline{Apr}_I(X)$, that means $\neg x/I \subseteq X$. Thus $x/I \subseteq \neg X$, hence $x \in \underline{Apr}_I(\neg X)$.

Let $x \in \underline{Apr}_I(\neg X)$, i.e. $x/I \subseteq \neg X$. Hence $\neg x/I \subseteq X$, therefore $x \in \neg\underline{Apr}_I(X)$. □

Lemma 3.7. *(Botur et al., 2012, Lemma 2.7) If A is a commutative basic algebra and I is a preideal of A, then the following are equivalent:*

(i) *I is an ideal of A;*

(ii) *$(a \oplus (b \oplus x)) \ominus (a \oplus b) \in I$ for all $a, b \in A$, $x \in I$.*

Theorem 3.8. *Let A be a linearly ordered commutative basic algebra and I and J ideals of A. Then $\overline{Apr}_I(J)$ is an ideal of A.*

Proof. Obviously $0 \in \overline{Apr}_I(J)$.

Let $x \in \overline{Apr}_I(J)$, $y \in A$, $y \leq x$. Let $x_1 \in x/I \cap J$. Suppose that $y_1 \in y/I$ and $y/I \neq x/I$. Then $y_1 < x_1$, hence $y_1 \in J$, and thus $y \in \overline{Apr}_I(J)$.

Now, let $x, y \in \overline{Apr}_I(J)$, $x_1 \in x/I \cap J$ and $y_1 \in y/I \cap J$. Then $x_1 \oplus y_1 \in J$ and $x_1 \oplus y_1 \in (x/I) \oplus (y/I) = (x \oplus y)/I$. Therefore $x \oplus y \in \overline{Apr}_I(J)$.

Let $x, y \in A$, $a \in \overline{Apr}_I(J)$ and $a_1 \in a/I \cap J$. Then $((x \oplus (y \oplus a)) \ominus (x \oplus y))/I = (x/I \oplus (y/I \oplus a/I)) \ominus (x/I \oplus y/I) = (x/I \oplus (y/I \oplus a_1/I)) \ominus (x/I \oplus y/I) = (x \oplus (y \oplus a_1)) \ominus (x \oplus y))/I$ and $(x \oplus (y \oplus a_1)) \ominus (x \oplus y) \in J$. Hence $(x \oplus (y \oplus a)) \ominus (x \oplus y) \in \overline{Apr}_I(J)$.

Therefore by Lemma 3.7, $\overline{Apr}_I(J)$ is an ideal of A. □

4 CONNECTIONS AMONG APPROXIMATION SPACES

In this section we investigate approximation spaces which are induced by different or special ideals.

Proposition 4.1. *If I and J are ideals of a commutative basic algebra and $\emptyset \neq X \subseteq A$, then*

$$\underline{Apr}_{\langle I,J \rangle}(X) \subseteq \langle \underline{Apr}_I(X), \underline{Apr}_J(X) \rangle.$$

Proof. If $a \in \underline{Apr}_{\langle I,J \rangle}(X)$, then $a/\langle I,J \rangle \subseteq X$, thus also a/I, $a/J \subseteq X$. Hence $a \leq a \oplus a \in \langle \underline{Apr}_I(X), \underline{Apr}_J(X) \rangle$. □

Lemma 4.2. *Let A_1 and A_2 be commutative basic algebras, I an ideal of A_2 and f a homomorphism of A_1 into A_2. Then $f^{-1}(I)$ is an ideal of A_1.*

Proof. Obviously $f^{-1}(I)$ is a preideal of A_1. Let $x, y \in A_1$ and $a \in f^{-1}(I)$. Then $f((x \oplus (y \oplus a)) \ominus (x \oplus y)) = (f(x) \oplus (f(y) \oplus f(a)) \ominus (f(x) \oplus f(y)) \in I$, therefore $f^{-1}(I)$ is an ideal of A_1. □

Theorem 4.3. *Let A_1 and A_2 be commutative basic algebras, f a homomorphism of A_1 into A_2, I an ideal of A_2 and $\emptyset \neq X \subseteq A_2$. Then*

$$f^{-1}(\overline{Apr}_I(X)) = \overline{Apr}_{f^{-1}(I)}(f^{-1}(X)).$$

Proof. Let $x \in A_1$. Then $x \in \overline{Apr}_{f^{-1}(I)}(f^{-1}(X))$ if and only if there exists $z \in x/f^{-1}(I) \cap f^{-1}(X)$ iff $z \ominus x$, $x \ominus z \in f^{-1}(I)$ iff $f(z \ominus x)$, $f(x \ominus x) \in I$ iff $f(z) \ominus f(x)$, $f(x) \ominus f(z) \in I$ iff $f(z)/I = f(x)/I$.

We have $f(z) \in f(x)/I$, $z \in f^{-1}(X)$, then $f(z) \in f(x)/I \cap X$, hence $f(x)/I \cap X \neq \emptyset$, and so $f(x) \in \overline{Apr}_I(X)$.

That means $x \in \overline{Apr}_{f^{-1}(I)}(f^{-1}(X))$ if and only if $x \in f^{-1}(\overline{Apr}_I(X))$. □

Theorem 4.4. *Let A_1 and A_2 be commutative basic algebras, $f : A_1 \longrightarrow A_2$ a homomorphism and $\emptyset \neq X \subseteq A_1$. Then*

$$f(\overline{Apr}_{\mathrm{Ker}(f)}(X)) = f(X).$$

Proof. Let $x \in f(\overline{Apr}_{\mathrm{Ker}(f)}(X))$ and $y \in \overline{Apr}_{\mathrm{Ker}(f)}(X)$ be such that $x = f(y)$. Let $z \in y/\mathrm{Ker}(f) \cap X$. Then $z \ominus y$, $y \ominus z \in \mathrm{Ker}(f)$ and $z \in X$. Hence $f(z \ominus y) = 0$, $f(y \ominus z) = 0$, so $f(z) \ominus f(y) = 0$, $f(y) \ominus f(z) = 0$. Therefore $f(z) = f(y) = x$, i.e. $f(z) \in f(X)$, and consequently $f(\overline{Apr}_{\mathrm{Ker}(f)}(X)) \subseteq f(X)$.

The converse inclusion is obvious. □

Proposition 4.5. *Let A be a commutative basic algebra, I and J ideals of A and $\emptyset \neq X \subseteq A$.*

a) If A is linearly ordered, then

$$\underline{Apr}_I(X) \cap \underline{Apr}_J(X) = \underline{Apr}_{I \cap J}(X).$$

b) If X is definable with respect to I or J, or if A is linearly ordered, then

$$\overline{Apr}_{I \cap J}(X) = \overline{Apr}_I(X) \cap \overline{Apr}_J(X).$$

Proof. a) Obvious.

b) Let X be definable, e.g., with respect to I. Then $\overline{Apr}_I(X) \cap \overline{Apr}_J(X) = X \cap \overline{Apr}_J(X) = X \subseteq \overline{Apr}_{I \cap J}(X)$.

The converse inclusion follows from the fact that $I \cap J \subseteq I$, J implies $\overline{Apr}_{I \cap J}(X) \subseteq \overline{Apr}_I(X)$, $\overline{Apr}_J(X)$.

For linearly ordered A it is obvious. □

5 CONCLUSIONS

It is known that there are situations concerning reasoning where the associativity of the logical connection conjunction need not be satisfied. Recently, a logic foundation for fuzzy reasoning with non-associative conjunction, as a generalization of the Lukasiewicz infinite valued logic, was proposed. Commutative basic algebras are an algebraic semantics of such logic. This paper introduces and investigates the concept of approximate spaces based on ideals of commutative basic algebras and shows that it is reasonable to study approximate spaces in non-associative structures.

ACKNOWLEDGEMENTS

The first author was supported by the ESF Project CZ.1.07/2.3.00/20.0051 and IGA PřF 2014016, the second author was supported by ESF Project CZ.1.07/2.3.00/20.0296.

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