Fuzzy Function and the Generalized Extension Principle

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Abstract: The aim of this contribution is to develop a theory of such concepts as fuzzy point, fuzzy set and fuzzy function in a similar style as is common in classical mathematical analysis. We recall some known notions and propose new ones with the purpose to show that, similarly to the classical case, a (fuzzy) set is a collection of (fuzzy) points or singletons. We show a relationship between a fuzzy function and its ordinary "skeleton" that can be naturally associated with the original function. We show that any fuzzy function can be extended to the domain of fuzzy subsets and this extension is analogous to the Extension Principle of L. A. Zadeh.

1 INTRODUCTION

The notion of fuzzy function has at least two different meanings in fuzzy literature. On the one side (see e.g., (Hájek, 1998; Klawonn, 2000; Demirci, 1999; Demirci, 2002; Höhle et al., 2000; Šostak, 2001)), a fuzzy function is a special fuzzy relation with a generalized property of uniqueness. According to this approach, each element from the ordinary domain of thus defined fuzzy function is associated with a certain fuzzy set. Thus, a fuzzy function establishes a "point"-to-"fuzzy set" correspondence.

On the other hand (see (Novák, 1989; Perfilieva, 2004; Perfilieva, 2011; Perfilieva et al., 2012)), a fuzzy function is a mapping between two universes of fuzzy sets, i.e. establishes a "fuzzy set"-to-"fuzzy set" correspondence. This approach is implicitly used in many papers devoted to fuzzy IF-THEN rule models where the latter are actually partially given fuzzy functions.

In this contribution, we show that both viewpoints can be connected by a natural generalization of the Extension Principle of L. Zadeh (Zadeh, 1975). In details, a fuzzy function as a mapping is an extension of a fuzzy function as a relation to the domain of fuzzy sets. The similar approach has been used in (Šostak, 2001).

In order to establish the above mentioned extension, we introduce various spaces of fuzzy objects with fuzzy equivalence relations on them. We show that similar to the classical case, a (fuzzy) set is a collection of (fuzzy) points or (fuzzy) singletons.

Last, but not least, we analyze a relationship between a surjective fuzzy function and its ordinary core function. The similar study has been attempted in (Demirci, 1999) for a perfect fuzzy function and in (Klawonn, 2000) for one particular example of a fuzzy function. We propose a solution in the general case.

The present paper is organized as follows. In Section 2, we give preliminary information about extension principle, residuated lattices, fuzzy sets and fuzzy spaces. Fuzzy functions and two approaches to this notion are discussed in Section 3. Section 3 contains also main results of the paper.

2 PRELIMINARIES

2.1 Extension Principle and Its Relational Form

An extension principle has been proposed by L. Zadeh (Zadeh, 1975) in 1975 and since then it is widely used in the fuzzy set theory and its applications. Let us recall the principle and propose its relation form which will be later on used in a relationship to fuzzy function.

Assume that X,Y are universal sets and $f : X \to Y$ is a function with the domain X. Let moreover, $\mathscr{F}(X), \mathscr{F}(Y)$ be respective universes of fuzzy sets on

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X and Y identified with their membership functions, i.e. $\mathscr{F}(X) = \{A : X \to [0,1]\}$ and similarly, $\mathscr{F}(Y)$. By the extension principle, f induces a function $f^{\to} : \mathscr{F}(X) \to \mathscr{F}(Y)$ such that for all $A \in \mathscr{F}(X)$,

$$f^{\to}(A)(y) = \sup_{y=f(x)} A(x). \tag{1}$$

Let R_f be a binary relation on $X \times Y$ which corresponds to the function f, i.e.

$$R_f(x,y) = 1 \Leftrightarrow y = f(x).$$

Then it is easy to see that (1) can be equivalently represented by

$$f^{\to}(A)(y) = \bigvee_{y \in Y} (A(x) \cdot R_f(x,y)). \tag{2}$$

Expression (2) is the *relational form of the extension principle*. The meaning of expression (2) becomes more general when A is an L-fuzzy set (see Definition 3 below), binary relation R_f is a fuzzy relation, and multiplication $\cdot$ changes to a monoidal operation (see Section 2.2). In Section 3, we will discuss the proposed generalization and its relationship to fuzzy functions.

2.2 Residuated Lattice

Our basic algebra of operations is a residuated lattice.

Definition 1. *A residuated lattice is an algebra*

$$\mathscr{L} = \langle L, \vee, \wedge, *, \to, 0, 1\rangle.$$

with a support L and four binary operations and two constants such that

- $\langle L, \vee, \wedge, 0, 1\rangle$ *is a lattice where the ordering $\le$ defined using operations $\vee, \wedge$ as usual, and $0,1$ are the least and the greatest elements, respectively;*
- $\langle L, *, 1\rangle$ *is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * 1 = a$;*
- *the operation $\to$ is a residuation operation with respect to $*$, i.e.*

$$a * b \le c \iff a \le b \to c.$$

A residuated lattice is complete if it is complete as a lattice.

The following is a binary operation of biresiduation on $\mathscr{L}$:

$$x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

The well known examples of residuated lattices are: boolean algebra, Gödel, Łukasiewicz and product algebras. In the particular case $L = [0,1]$, multiplication $*$ is a left continuous t-norm.

From now on we fix a complete residuated lattice $\mathscr{L}$.

2.3 L-fuzzy Sets, Fuzzy Relations and Fuzzy Spaces

Below, we recall definitions of principal notions in the fuzzy set theory.

Fuzzy Sets with Crisp Equality. Let X be a non-empty universal set, $\mathscr{L}$ a complete residuated lattice. An (L-)fuzzy set A of X (fuzzy set, shortly) is a map $A : X \to L$ that establishes a relationship between elements of X and degrees of *membership* to A.

Fuzzy set A is *normal* if there exists $x_A \in X$ such that $A(x_A) = 1$. The (ordinary) set $\mathrm{Core}(A) = \{x \in X \mid A(x) = 1\}$ is a *core* of the normal fuzzy set A. The (ordinary) set $\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\}$ is a *support* set of fuzzy set A.

A class of L-fuzzy sets of X will be denoted L^X. The couple $(L^X, =)$ is called an *ordinary fuzzy space* on X. The elements of $(L^X, =)$ are fuzzy sets equipped with a crisp equality relation, i.e. for all $A, B \in L^X$,

$$A = B \textit{ if and only if } (\forall x \in X)\, A(x) = B(x).$$

In $(L^X, =)$, we strictly distinguish between fuzzy sets even if their membership functions differ in one point. On $(L^X, =)$, we can define the structure of residuated lattice using pointwise operations over fuzzy sets. Moreover, the underlying lattice $\langle L^X, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$ is complete, where the bottom $\mathbf{0}$ and the top $\mathbf{1}$ are constant fuzzy sets, respectively.

A class of normal L-fuzzy sets of X will be denoted $\mathscr{N}(X)$. The space $(\mathscr{N}(X), =)$ is a subspace of $(L^X, =)$.

By identifying a point $u \in X$ with a fuzzy subset $I_u : X \to L$ such that $I_u(u) = 1$ and $I_u(x) = 0$ whenever $x \neq u$ we may view X as a subspace of $(L^X, =)$ and as a subspace of $(\mathscr{N}(X), =)$

Space with Fuzzy Equivalence. Fuzzy Points. Let X, Y be universal sets. Similarly to L-valued fuzzy sets, we define *(binary) (L-)fuzzy relations* as fuzzy sets of $X \times Y$. If $X = Y$, then a fuzzy set of $X \times X$ is called a (binary) (L-)fuzzy relation on X.

A binary fuzzy relation E on X is called *fuzzy equivalence* on X (see (Klawonn and Castro, 1995; Höhle, 1998; De Baets and Mesiar, 1998))[1] if for all $x, y, z \in X$, the following holds:

1. $E(x,x) = 1$, *reflexivity,*
2. $E(x,y) = E(y,x)$, *symmetry,*
3. $E(x,y) * E(y,z) \le E(x,z)$, *transitivity.*

[1] Fuzzy equivalence appears in the literature under the names *similarity* or *indistinguishability* as well.

If fuzzy equivalence E fulfills

1. $E(x,y) = 1$ if and only if $x = y$,

then it is called *separated* or a *fuzzy equality* on X.

Let us remark that fuzzy equivalence E creates fuzzy sets on X, we will call them *E-fuzzy points*[2] of X or simply fuzzy points if E is clear from the context. Every E-fuzzy point is a class of fuzzy equivalence E of just one point of X. In more details, if $t \in X$, then E-fuzzy point E_t is a fuzzy set $E_t : X \to L$ such that for all $x \in X$, $E_t(x) = E(t,x)$. It is easy to see that E_t is a normal fuzzy set and $t \in \mathrm{Core}(E_t)$.

The set of all E-fuzzy points of X will be denoted by

$$\mathscr{P}_X^E = \{E_t \mid t \in X\}.$$

Obviously, $\mathscr{P}_X^E \subseteq L^X$ and $(\mathscr{P}_X^E, =)$ is a subspace of $(L^X, =)$. If E is a fuzzy equivalence on X, then it may happen that the same element, say E_t from $(\mathscr{P}_X^E, =)$ has different representations, i.e. there exists $u \in X$ such that $E_u = E_t$. It can be shown that this holds true if and only if $E(t,u) = 1$, or $u \in \mathrm{Core}(E_t)$.

On the other side, if E is a fuzzy equality on X, then the core of every E-fuzzy point consists of one element and thus, a representation of any E-fuzzy point in the form E_t is unique.

Space with Fuzzy Equivalence and Crisp Equality. Fuzzy Singletons and Sub-singletons. Let us equip the space X with both crisp $=$ and fuzzy E equalities and denote it by $(X, =, E)$. In this space, we are able to distinguish degrees of coincidence $E(t,u)$ between any two elements t,u from X. As we discussed above, crisp and fuzzy equalities put into the correspondence with each element t of X its characteristic function I_t and its E-fuzzy point E_t. Both are normal fuzzy sets in L^X with the same one-element core. Let us consider fuzzy sets $S_t \in L^X$, that are in between I_t and E_t, i.e. for all $x \in X$,

$$I_t(x) \le S_t(x) \le E_t(x). \qquad (3)$$

We will call them *fuzzy singletons*. In (Klawonn, 2000), fuzzy singletons were introduced as normal fuzzy sets $S_t \in L^X$ with $\{t\}$ as a one-element core, i.e. $S_t(t) = 1$, and such that for all $x,y \in X$,

$$S_t(x) * S_t(y) \le E(x,y), \qquad (4)$$

where $*$ is the monoidal operation from a chosen residuated lattice L. It is easy to show that this is equivalent to our definition. Indeed, if S_t fulfills (3), then it is normal, it has $\{t\}$ as a one-element core, and for all $x,y \in X$,

$$S_t(x) * S_t(y) \le E(t,x) * E(t,y) \le E(x,y).$$

[2] This notion was introduced in (Klawonn, 2000)

On the other side, if S_t has $\{t\}$ as a one-element core and fulfills (4), then for all $x \in X$, $I_t(x) \le S_t(x)$ and

$$S_t(x) = S_t(x) * S_t(t) \le E(t,x) = E_t(x).$$

From (4) and the discussion above it follows that E-fuzzy point E_t is the greatest fuzzy singleton with the one-element core $\{t\}$. The space of all fuzzy singletons, considered in $(X, =, E)$, will be denoted by $\mathscr{S}_X^E$. Obviously, $\mathscr{S}_X^E \subseteq L^X$ and $(\mathscr{S}_X^E, =)$ is a subspace of $(L^X, =)$.

Let us discard normality in the definition of fuzzy singleton and define *fuzzy sub-singleton* as a fuzzy set $U \in L^X$, such that there exists $t \in X$, so that

$$\mathbf{0} < U(x) \le E_t(x), x \in X. \qquad (5)$$

In order to stress that a fuzzy sub-singleton is connected with a certain fuzzy point E_t, we will denote it as U_t. Similarly to the above, we can prove that any fuzzy sub-singleton fulfills (4). The space of all fuzzy sub-singletons, considered in $(X, =, E)$, will be denoted by $\mathscr{U}_X^E$. Obviously, $\mathscr{S}_X^E \subseteq \mathscr{U}_X^E \subseteq L^X$ and $(\mathscr{U}_X^E, =)$ is a subspace of $(L^X, =)$.

Extensional Hulls. Let again our space be $(X, =, E)$ – a space with fuzzy equivalence and crisp equality. We remind (Klawonn, 2000) that fuzzy set A is *extensional* (with respect to E) if for all $x,y \in X$,

$$A(x) * E(x,y) \le A(y).$$

The smallest extensional fuzzy set A^E containing fuzzy set A is called the *extensional hull* of A. It is not difficult to prove the following representation of A^E.

Lemma 1. *The extensional hull A^E of every fuzzy set $A \in L^X$ can be represented as follows:*

$$A^E(y) = \sup_{x \in X} A(x) * E(x,y). \qquad (6)$$

Representation (6) has been obtained in many papers (see e.g.,(Höhle, 1998)), therefore will not prove this again.

Lemma 1 has two important corollaries.

Corollary 1. *Extensional hull of element $t \in X$ identified with I_t is equal to fuzzy point E_t.*

Corollary 2. *Extensional hull of fuzzy singleton $S_t \in L^X$, $t \in X$, is equal to the corresponding fuzzy point E_t.*

Decomposition of a Fuzzy Set into Fuzzy Sub-singletons

Theorem 1. *Let $A \in L^X$ be a non-zero fuzzy set. Then*

- *A can be represented as a supremum of fuzzy sub-singletons U_t^A, $t \in Supp(A)$, such that*

$$U_t^A(x) = A(x) \wedge E_t(x), x \in X, \tag{7}$$

- *A can be represented as a supremum of fuzzy sub-singletons W_t^A, $t \in Supp(A)$, such that*

$$W_t^A(x) = A(x) * E_t(x), x \in X, \tag{8}$$

In both cases, for all $x \in X$,

$$A(x) = \sup_{t \in Supp(A)} U_t^A(x) = \sup_{t \in Supp(A)} (A(x) \wedge E_t(x)), \tag{9}$$

and

$$A(x) = \sup_{t \in Supp(A)} W_t^A(x) = \sup_{t \in Supp(A)} (A(x) * E_t(x)). \tag{10}$$

Proof. At first, we will prove that for all $t \in \mathrm{Supp}(A)$, U_t^A and W_t^A are fuzzy sub-singletons, i.e. U_t^A and W_t^A are non-zero and less than E_t. The first assertion follows from the assumption $t \in \mathrm{Supp}(A)$, so that

$$U_t^A(t) = A(t) \wedge E_t(t) = A(t) > 0,$$
$$W_t^A(t) = A(t) * E_t(t) = A(t) > 0.$$

The second assertion easily follows from (7) and (8).

To prove (9) and (10), we first notice that both of them are trivially valid for $x \notin \mathrm{Supp}(A)$. Therefore, we assume that $x \in \mathrm{Supp}(A)$. Then (9) follows from the two inequalities below:

$$A(x) = \sup_{t \in \mathrm{Supp}(A)} U_t^A(x) =$$
$$= \sup_{t \in \mathrm{Supp}(A)} (A(x) \wedge E_t(x)) \geq A(x) \wedge E_x(x) = A(x),$$

and

$$A(x) = \sup_{t \in \mathrm{Supp}(A)} U_t^A(x) = \sup_{t \in \mathrm{Supp}(A)} (A(x) \wedge E_t(x)) \leq$$
$$\leq \sup_{t \in \mathrm{Supp}(A)} A(x) = A(x).$$

To prove (10), we recall that in every complete residuated lattice the following holds true:

$$\sup_{t \in \mathrm{Supp}(A)} (A(x) * E_t(x)) = A(x) * \sup_{t \in \mathrm{Supp}(A)} E_t(x).$$

Because for $x \in \mathrm{Supp}(A)$, $\sup_{t \in \mathrm{Supp}(A)} E_t(x) = 1$, we easily get

$$A(x) = \sup_{t \in \mathrm{Supp}(A)} W_t^A(x) = \sup_{t \in \mathrm{Supp}(A)} (A(x) * E_t(x)) =$$
$$= A(x) * \sup_{t \in \mathrm{Supp}(A)} E_t(x) = A(x).$$

□

3 FUZZY FUNCTIONS

The notion of fuzzy function has many definitions in the literature, see e.g. (Hájek, 1998; Klawonn, 2000; Demirci, 2002; Perfilieva, 2004). In (Hájek, 1998; Klawonn, 2000; Demirci, 2002), a fuzzy function is considered as a special fuzzy relation. Below, we remind the notion of fuzzy function as it appeared (independently) in (Klawonn, 2000), (Höhle et al., 2000) and (Demirci, 2002):

Definition 2. *Let E, F be fuzzy equivalences on X and Y, respectively. A fuzzy function is a binary fuzzy relation ρ on $X \times Y$ such that for all $x, x' \in X$, $y, y' \in Y$ the following axioms hold true:*

1. $\rho(x,y) * E(x,x') \leq \rho(x',y)$,
2. $\rho(x,y) * F(y,y') \leq \rho(x,y')$,
3. $\rho(x,y) * \rho(x,y') \leq F(y,y')$,

A fuzzy function is called perfect (Demirci, 1999), (cf also (Höhle et al., 2000, Section 3.2)) if it additionally fulfills

1. *for all $x \in X$, there exists $y \in Y$, such that $\rho(x,y) = 1$.*

A fuzzy function is called (strong) *surjective* (Demirci, 1999), cf also (Höhle et al., 2000, Section 4.2) if

1. for all $y \in Y$, there exists $x \in X$, such that $\rho(x,y) = 1$.

Actually, a fuzzy function ρ establishes a double extensional correspondence between the space $(X, =, E)$ and the space of $(Y, =, F)$ (axioms **FF.1, FF.2**) which is weakly functional (axioms **FF.3**). Moreover, it is a point-to-(fuzzy set) mapping between X and L^Y such that for all $x \in X$, $\rho(x, \cdot)$ is a fuzzy set on Y. If for all $x \in X$, $\rho(x, \cdot)$ is a normal fuzzy set then ρ is perfect, and there is an ordinary function $g : X \to Y$ such that for all $y \in Y$, $\rho(x,y) = F(g(x),y)$ (see (Demirci, 2002)). This means that every F-fuzzy point $F_{g(x)}$ of Y determined by $g(x)$ is a fuzzy value of ρ at $x \in X$.

In our study, we will consider the case where ρ is surjective and defined everywhere on X, i.e.

$$(\forall x \in X)(\exists y \in Y) \quad \rho(x,y) > 0. \tag{11}$$

In this case, we will propose an analytic representation of ρ and use ρ in the generalized extension principle. Moreover, we will discover a relationship between a fuzzy function, its ordinary core function and its extension to a mapping over the domain of fuzzy sets.

3.1 Fuzzy Function and Its Core

In this Section, we will show that each surjective fuzzy function ρ on $X \times Y$ determines a corresponding ordinary core function $g : X' \to Y$, where $X' \subseteq X$, such that at any $x' \in X'$, the value $\rho(x', \cdot)$ is equal to the F-fuzzy point $F_{g(x')(\cdot)}$. The proofs of the below given Theorems 2 and 3 are in (Perfilieva, 2011).

Theorem 2. *Let fuzzy relations E on X and F on Y be fuzzy equivalences and moreover, F be a fuzzy equality. Let fuzzy relation ρ on $X \times Y$ be a surjective fuzzy function. For every $y \in Y$, let us choose and fix $x_y \in Core(\rho(x,y))$. Denote $X' = \{x_y \mid x_y \in X, y \in Y\}$. Then the following fuzzy relation on X*

$$E'(x,x') = \bigwedge_{y \in Y} (\rho(x,y) \leftrightarrow \rho(x',y)), \qquad (12)$$

is a fuzzy equivalence E' such that

1. *$E \leq E'$ and ρ is a fuzzy function with respect to fuzzy equivalences E' and F,*
2. *for all $x \in X$, $y \in Y$,*
$$\rho(x,y) = E'(x,x_y), \qquad (13)$$
3. *for all $y, y' \in Y$,*
$$E'(x_y, x_{y'}) = F(y,y'), \qquad (14)$$
4. *the mapping $g : X' \to Y$ such that $g(x_y) = y$ is surjective and extensional with respect to E' and F, i.e. for all $x,t \in X'$,*
$$E'(x,t) \leq F(g(x), g(t)). \qquad (15)$$

Corollary 3. *Fuzzy equivalence E', given by (12), is the greatest one (in the sense of $\leq$) that fulfils Theorem 2.*

Corollary 4. *Fuzzy equivalence E', given by (12), covers X, i.e. for all $x \in X$ there exists $x_y \in X'$ such that $E'(x,x_y) > 0$.*

Proof. By (11), for arbitrary $x \in X$ there exists $y \in Y$, such that $\rho(x,y) > 0$. By (13), $\rho(x,y) = E'(x,x_y)$, and therefore, $E'(x,x_y) > 0$. □

The meaning of the assertions below is that a surjective fuzzy function ρ is indeed a fuzzified version of its core function $g : X' \to Y$, where $X' \subseteq X$. If $x \in X$, then the fuzzy value of $\rho(x,\cdot)$ is a "linear"-like combination of F-fuzzy points $F_{g(x')}(\cdot)$. In particular, if $x' \in X'$ - domain of g, then the fuzzy value of $\rho(x',\cdot)$ is equal to the corresponding F-fuzzy point $F_{g(x')}(\cdot)$.

Theorem 3. *Let fuzzy relations E, E', F, ρ and function $g : X' \to Y$ where $X' = \{x_y \mid y \in Y\}$ fulfil assumptions and conclusions of Theorem 2. Then*

1. *for all $x \in X$, $y \in Y$,*
$$\rho(x,y) = \bigvee_{x' \in X'} (E'_{x'}(x) * F_{g(x')}(y)), \qquad (16)$$
2. *for all $t \in X'$, $y \in Y$,*
$$\rho(t,y) = F_{g(t)}(y). \qquad (17)$$

3.2 Generalized Extension Principle

In this Section, we will show that every fuzzy function ρ that establishes a point-to-(fuzzy set) mapping between X and L^Y can be extended (via the Generalized extension principle) to a (fuzzy set)-to-(fuzzy set) mapping between L^X and L^Y. We will use expression (2), where we replace $\cdot$ by $*$ and use fuzzy relation ρ instead of ordinary R_f. Moreover, we will use our representation (10) of a non-zero fuzzy set and show that the extended mapping between $L^{X'}$ and L^Y is fully determined by its reduction to a certain set of fuzzy sub-singletons, and in particular, to sets of E'-fuzzy points of X.

Definition 3 (Generalized extension principle)**.** *Let $\mathscr{L}$ be a complete residuated lattice and $(L^X, =)$, $(L^Y, =)$ fuzzy spaces. Let E, F be fuzzy equivalences on X and Y, respectively, and fuzzy relation ρ on $X \times Y$ be a fuzzy function. Then ρ induces the map $f^{\rightarrow}_{\rho} : L^X \to L^Y$ such that for every $A \in L^X$,*

$$f^{\rightarrow}_{\rho}(A)(y) = \bigvee_{x \in X} (A(x) * \rho(x,y)). \qquad (18)$$

Theorem 4. *Let fuzzy relations E on X and F on Y be fuzzy equivalences and moreover, F be a fuzzy equality. Let fuzzy relation ρ on $X \times Y$ be a surjective fuzzy function and E' be fuzzy equivalence given by (12). Then for any $A \in L^X$,*

$$f^{\rightarrow}_{\rho}(A) = \bigvee_{t \in Supp(A)} f^{\rightarrow}_{\rho}(W^A_t), \qquad (19)$$

where W^A_t is a fuzzy sub-singleton (8) in the space $(X, =, E')$.

In particular, if A is represented as a supremum of fuzzy points E'_t, i.e. $A = \bigvee_{t \in Supp(A)} E'_t$, then

$$f^{\rightarrow}_{\rho}(A)(y) = \bigvee_{t \in Supp(A)} f^{\rightarrow}_{\rho}(E'_t)(y) = \bigvee_{t \in Supp(A)} \rho(t,y). \qquad (20)$$

Proof. By Theorem 1, A can be represented as a supremum of fuzzy sub-singletons W^A_t, $t \in \mathrm{Supp}(A)$, where $W^A_t(x) = A(x) * E_t(x)$, $x \in X$. Thence,

$$f_\rho^\rightarrow(A)(y) = \bigvee_{x\in X}(A(x) * \rho(x,y)) =$$
$$= \bigvee_{x\in X}\left(\bigvee_{t\in \mathrm{Supp}(A)} W_t^A(x)\right) * \rho(x,y) =$$
$$\bigvee_{t\in \mathrm{Supp}(A)}\bigvee_{x\in X} W_t^A(x) * \rho(x,y) = \bigvee_{t\in \mathrm{Supp}(A)} f_\rho^\rightarrow(W_t^A)(y).$$

To prove (20), we first decompose

$$f_\rho^\rightarrow(A)(y) = \bigvee_{t\in \mathrm{Supp}(A)} f_\rho^\rightarrow(E_t')(y),$$

and then continue with the following chain of equalities

$$f_\rho^\rightarrow(E_t')(y) = \bigvee_{x\in X}(E_t'(x) * \rho(x,y)) =$$
$$= \bigvee_{x\in X} E_t'(x) * \bigvee_{x'\in X'}(E_{x'}'(x) * F_{g(x')}(y)) =$$
$$\bigvee_{x'\in X'}\left(\bigvee_{x\in X} E_{x'}'(x) * E_t'(x)\right) * F_{g(x')}(y) =$$
$$= \bigvee_{x'\in X'} E_{x'}'(x) * F_{g(x')}(y) = \rho(t,y),$$

where we made use of representation ρ by (16). □

4 CONCLUSION

In this contribution we started a mathematical analysis of basic concepts in the fuzzy set theory such as fuzzy point, fuzzy set and fuzzy function. We introduced various spaces of elements equipped with crisp and fuzzy equivalences with the purpose to show that similar to the classical case, a (fuzzy) set is a collection of (fuzzy) points or singletons. We recalled the notion of a fuzzy function as a special fuzzy relation and showed that similarly to the classical case, any fuzzy function can be extended to the domain of fuzzy subsets and this extension is similar to the Extension Principle of L. Zadeh. We clarified a relationship between a fuzzy function and its ordinary core function.

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