Oscillatory Model of Neuromorphic Processors by Embedding Orthogonal Filters

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Abstract: The purpose of this article is to present a model of the computational intelligence system based on a network of coupled phase oscillators. The structure of such a model consists of a net of phase-locked loops (PLL) and orthogonal filters based on a Hamiltonian neural network embedded in this net.

1 INTRODUCTION

It is well known that true artificial intelligence cannot be implemented with traditional hardware. It should be clear as well that that in order to be able to build machines that learn, reason and recognize, one needs power efficient processors with computational efficiency unattainable even by supercomputers. Two such processors are theoretically known: quantum and neuromorphic structures. Up to date, several neuromorphic devices using different technologies (e.g. spiking, oscillatory and static artificial neurons and structures based on them) have been proposed (Mcdonnell et al., 2014). Neverthe-less, we claim that a biological brain is an almost lossless dynamic structure and, hence, the neuromorphic system should be sought in a class of lossless systems, especially Hamiltonian systems, i.e. Hamiltonian neural networks. Therefore, the main goal of this paper is to prove the following statement: The structure of oscillatory neuromorphic processors can be obtained by embedding orthogonal filters based on the Hamiltonian neural network into a network based on phase-locked loops. Using this method, one obtains an oscillatory model of self-sustaining memory, which can memorize an input information and simultaneously perform a different analysis, e.g. pattern recognition.

2 HAMILTONIAN NEURAL NETWORKS - BASED ORTHOGONAL FILTERS

It is well known that a general description of the Hamiltonian network is given by the following state–space equation:

$$\dot{\mathbf{x}} = \mathbf{J}\nabla \mathrm{H}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \qquad (1)$$

where: $\mathbf{x}$ – state vector, $\mathbf{x} \in \mathrm{R}^{2n}$
$\mathbf{v}(\mathbf{x})$ – a nonlinear vector field
$\mathbf{J}$ – skew-symmetric, orthogonal matrix e.g. Poisson matrix.

Function $\mathrm{H}(\mathbf{x})$ is energy absorbed in the network. Since Hamiltonian networks are lossless (dissipationless), their trajectories in the state space can be very complex for $t \in (-\infty, \infty)$. But Eq.(1) gives rise to the model of Hamiltonian Neural Networks (HNN), as follows (Sienko and Citko, 2009):

$$\dot{\mathbf{x}} = \mathbf{W}\boldsymbol{\Theta}(\mathbf{x}) + \mathbf{d} \qquad (2)$$

where: $\mathbf{W}$ – (2n×2n) skew-symmetric, orthogonal weight matrix ($\mathbf{W}^2 = \mathbf{-1}$)
$\boldsymbol{\Theta}(\mathbf{x})$ – vector of activation functions (output vector $\mathbf{y} = \boldsymbol{\Theta}(\mathbf{x})$) $\mathbf{d}$ – input data
and $\boldsymbol{\Theta}(\mathbf{x}) = \nabla \mathrm{H}(\mathbf{x})$

One assumes here that activation functions are passive i.e. :

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$$\mu_1 \le \frac{\Theta(x)}{x} \le \mu_2 \ ; \ \mu_1, \mu_2 \in (0, \infty) \tag{3}$$

The HNN described by Eq.(1) cannot be realized as a macroscopic scale physical object. Nevertheless, introducing the negative-feedback loops, Eq.(2) can be reformulated as follows:

$$\dot{\mathbf{x}} = \left(\mathbf{W} - w_0 \mathbf{1}\right)\mathbf{\Theta}(\mathbf{x}) + \mathbf{d} \tag{4}$$

where: $w_0 > 0$

and Eq.(4) sets up an orthogonal transformation (HNN-based orthogonal filter):

$$\mathbf{y} = \frac{1}{1 + w_0^2}(\mathbf{W} + w_0 \mathbf{1})\mathbf{d} \tag{5}$$

where: $\mathbf{W}^2 = \mathbf{-1}$

Thus, a 8-dim. orthogonal filter, referred to as <u>octonionic module</u>, can be synthesized by the formula:

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{bmatrix} = \frac{1}{\sum_{i=1}^{8} y_i^2} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ -y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & y_8 & -y_7 \\ -y_3 & y_4 & y_1 & -y_2 & -y_7 & -y_8 & y_5 & y_6 \\ -y_4 & -y_3 & y_2 & y_1 & -y_8 & y_7 & -y_6 & y_5 \\ -y_5 & y_6 & y_7 & y_8 & y_1 & -y_2 & -y_3 & -y_4 \\ -y_6 & -y_5 & y_8 & -y_7 & y_2 & y_1 & y_4 & -y_3 \\ -y_7 & -y_8 & -y_5 & y_6 & y_3 & -y_4 & y_1 & y_2 \\ -y_8 & y_7 & -y_6 & -y_5 & y_4 & y_3 & -y_2 & y_1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \tag{6}$$

i.e. $\mathbf{w} = \mathbf{Y}\ \mathbf{d}$

It can be seen that Eq.(6) is a solution of the following design problem: for a given input vector $\mathbf{d} = [d_1, \dots, d_8]^T$ and a given output vector $\mathbf{y} = [y_1, \dots, y_8]^T$ find the weight matrix $\mathbf{W}$ of the HNN based orthogonal filter (octonionic module). Thus:

$$\mathbf{W}_8 = \begin{bmatrix} 0 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \\ -w_1 & 0 & w_3 & -w_2 & w_5 & -w_4 & -w_7 & w_6 \\ -w_2 & -w_3 & 0 & w_1 & w_6 & w_7 & -w_4 & -w_5 \\ -w_3 & w_2 & -w_1 & 0 & w_7 & -w_6 & w_5 & -w_4 \\ -w_4 & -w_5 & -w_6 & -w_7 & 0 & w_1 & w_2 & w_3 \\ -w_5 & w_4 & -w_7 & w_6 & -w_1 & 0 & -w_3 & w_2 \\ -w_6 & w_7 & w_4 & -w_5 & -w_2 & w_3 & 0 & -w_1 \\ -w_7 & -w_6 & w_5 & w_4 & -w_3 & -w_2 & w_1 & 0 \end{bmatrix} \tag{7}$$

$\mathbf{W}_8$- matrix belongs to the family of matrices obtained by superposition of Hurwitz-Radon matrices.

The octonionic module can be seen as a basic building block for the construction of AI processors. Moreover, the output $\mathbf{y}$ of the filter in Eq.(4) is a Haar spectrum of the input vector $\mathbf{d}$. It is worth noting that the octonionic module sets up an elementary memory module as well. For example, designing an orthogonal filter, using Eq.(4) and Eq.(5), which performs the following transformation:

$$\mathbf{y}_{[1]} = \frac{1}{1+w_0^2}(\mathbf{W} + w_0\mathbf{1})\mathbf{m} \tag{8}$$

where: $\mathbf{y}_{[1]} = [1, 1, \dots, 1]^T$ i.e. synthesizing by Eq.(5) a flat Haar spectrum for a given input vector $\mathbf{m}$, so that

$$\sum_{i=1}^{8} m_i > 0 \tag{9}$$

one gets implementation of a linear perceptron, as shown in Fig.1.

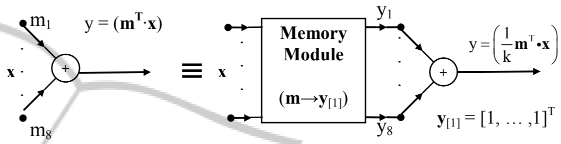


Figure 1: Implementation of an elementary memory module by the octonionic module.

Moreover, according to Eq. (5) and (7) the matrix $\mathbf{Y}$ with $y_1 = y_2 = \dots = y_8 = 1$ generates the structures of all memory modules. It is also worth noting that the transformation in Eq. (5) can be also realized by the octonionic modules, as shown in Fig.2.

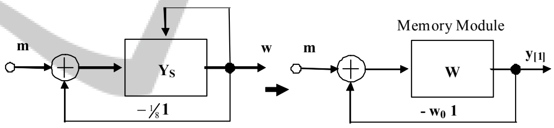


Figure 2: Self-creation of the memory module.

where: $\mathbf{Y}_s$–skew-symmetric part of the matrix $\mathbf{Y}$ (Eq.(5))

$\mathbf{W}$ - weight matrix of memory modules (Eq.(6) and Eq.(7)).

Such a transformation can be seen as a process of self creation of memory modules. To summarize the discussion above, one can state that the octonionic module is a universal building block realizing very large scale orthogonal filters in particular memory blocks. Multidimensional, octonionic modules based orthogonal filters can be realized by using the family of Hurwitz-Radon matrices. Thus, 16-dim orthogonal filter can be, for example, determined by the following matrix:

$$\mathbf{W}_{16} = \left[\begin{array}{c:c} \mathbf{W}_8 & \begin{matrix} w_8 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & w_8 \end{matrix} \\ \hdashline \begin{matrix} -w_8 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & -w_8 \end{matrix} & \mathbf{W}_8^T \end{array}\right] = \begin{bmatrix} \mathbf{W}_8 & \mathbf{0} \\ \mathbf{0} & -\mathbf{W}_8 \end{bmatrix} + w_8 \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix} \tag{10}$$

where: $w_8 \in R < \infty$

Similarly, for the dimension $N = 2^k$, k = 5, 6, 7, … all Hurwitz-Radon matrices can be found, as:

$$\mathbf{W}_{2^k} = \begin{bmatrix} \mathbf{W}_{2^{k-1}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{W}_{2^{k-1}} \end{bmatrix} + w_K \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix} \quad (11)$$

where: $w_K \in R < \infty$.

To conclude, we formulate the following statements:

1. N-dimensional HNN can be created by compatible connections of the octonionic modules.
2. The basic function of orthogonal filters is a Haar spectrum analysis of the input data **d**. Particularly, an orthogonal filter performs the function of memory, as given by Eq. (8).

3 ON MODELLING OSCILLATORY NEURAL NETWORKS

To our knowledge, the fundamental research in the field of oscillatory implementation of neural networks has been done by Hoppenstead and Izhikevich (Hoppenstead and Izhikevich, 1997, 2000; Izhikevich, 1999, 2006; Strogatz 2006). To review briefly, an oscillator can by described be the following state equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in R^m, \quad (12)$$

and it is a nonlinear dynamical system with a limit cycle. Hence, a net of weakly coupled oscillators is given by:

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \varepsilon \mathbf{g}_i(\mathbf{x}_1, \ldots, \mathbf{x}_n, \varepsilon),\ \varepsilon << 1,\ i = 1, \ldots, n \quad (13)$$

A synchronization phenomenon in such a network is one of the most challenging mathematical and engineering problems. According to (Izhikevich, 2006), the sufficient conditions for synchronization in the net Eq.(13) can be formulated as follows:

Transforming the state space Eq.(12) onto phase equations:

$$\dot{\varphi}_i = \Omega_i + \varepsilon h_i(\varphi_1, \ldots, \varphi_n, \varepsilon), \quad \varphi_i \in S^1 \quad (14)$$

where: Ω_i – natural frequency of i-th oscillator (i.e. for $\varepsilon = 0$).

Assuming a weak coupling of oscillators, the state equation and phase equation can be simplified, as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \varepsilon \sum_{j=1}^{n} \mathbf{g}_{ij}(\mathbf{x}_i, \mathbf{x}_j),\ \mathbf{x}_i \in R^m \quad (15)$$

and

$$\dot{\varphi}_i = \Omega_i + \varepsilon \sum_{j=1}^{n} h_{ij}(\varphi_i, \varphi_j),\ i=1,...,n \quad (16)$$

Introducing a phase deviation Ψ_i of i-th oscillator i.e.:

$$\varphi_i = \Omega_i t + \Psi_i \quad (17)$$

and averaging over a period $T = 2\pi/\Omega$, the phase equation (16) can be formulated as:

$$\dot{\Psi}_i = \varepsilon \sum_{j=1}^{n} H_{ij}(\Psi_i - \Psi_j),\ i=1,...,n \quad (18)$$

where nonlinear functions H_{ij}; i, j = 1, … n determine time evolution of momentary frequency of coupled oscillators in the net. It is clear that the state of synchronization is given by equilibria of differential Eq.(18), i.e. :

$$\varepsilon \sum_{j=1}^{n} H_{ij}(\Psi_i - \Psi_j) = 0,\ i=1,...,n \quad (19)$$

or

$$\Delta\omega_i + \sum_{j \neq i}^{n} H_{ij}(\Psi_i - \Psi_j) = 0; \quad \forall i \quad (20)$$

where: $\Delta\omega_i = H_{ii}(0)$ is a deviation of natural frequency Ω_i.

For the steady state of synchronization the equilibria have to be asymptotically stable. Unfortunately, the general solution of Eq.(20) is a nontrivial task, for n >> 1. In a special case, under the assumption that $H_{ij}(\bullet)$ has the form:

$$H_{ij}(\Psi_i - \Psi_j) = H(\Psi_i - \Psi_j) = -\sin(\Psi_i - \Psi_j) \quad (21)$$

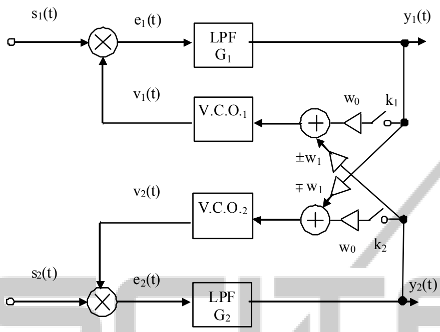
the solution of Eg.(20) can be analytically found. The above case is known and celebrated as the Kuramoto model (Strogatz, 2000). For example, for n =2, the Kuramoto model is given by:

$$\begin{aligned} \frac{d\Psi_1}{d\tau} &= \Delta\omega_1 - \sin(\Psi_1 - \Psi_2) \\ \frac{d\Psi_2}{d\tau} &= \Delta\omega_2 + \sin(\Psi_1 - \Psi_2) \end{aligned} \quad (22)$$

where: $\tau = \varepsilon\, t$.

It is worth noting that assuming Eq.(12) as a model of an oscillatory neuron, the state Eq.(15) describes an oscillatory neural network, which can be synchronized, as shown above. But, it seems that synchronization alone insufficiently determines a neural network as the information processor.

We claim that neural networks, to be treated as information processors, have to function as orthogonal filters. The authors of this publication have proposed a model of the oscillating net based on the structure of appropriately connected phase locked-loops (PLL) (Sienko, 1999; Citko and Sienko, 2008). Two connected PLLs create the neuron structure as shown in Fig. 3.



$s_i(t) = A_{Ci} \sin(\Omega_i t + \Psi_{si})$,
$v_i(t) = A_{Vi} \cos(\Omega_i t + \Psi_i)$; i = 1, 2.

Figure 3: The oscillating (PLL) neuron model.

It is easy to see that the model in Fig. 3. (PLL model) consists of two antisymmetrically coupled sinusoidal phase oscillators. The input signals $s_i(t)$, i =1, 2 are sinusoidal carriers.
Thus:

$$s_i(t) = A_{Ci} \sin(\Omega_i t + \Psi_{si}), \tag{23}$$

$$v_i(t) = A_{Vi} \cos(\Omega_i t + \Psi_i);\ i = 1, 2. \tag{24}$$

Assuming ideal transmittances of loop filters, i.e., $G_1 = G_2 \equiv 1$, the mean phase equation (Adler equation) of this model is as follows (keys k_1, k_2 open):

$$\frac{d}{dt}\begin{bmatrix}\Psi_{s1}-\Psi_1\\ \Psi_{s2}-\Psi_2\end{bmatrix} = 2\pi\begin{bmatrix}0 & \pm w_1 k_{V1} k_{m2} A_{C2} A_{V2}\\ \mp w_1 k_{V2} k_{m1} A_{C1} A_{V1} & 0\end{bmatrix}\begin{bmatrix}\sin(\Psi_{s1}-\Psi_1)\\ \sin(\Psi_{s2}-\Psi_2)\end{bmatrix} + \begin{bmatrix}\Delta\omega_1\\ \Delta\omega_2\end{bmatrix} \tag{25}$$

where: $\Delta\omega_i$ - frequency deviations of the input $s_i(t)$ signal
k_{Vi}, k_{mi} – sensitivity of VCO and phase-detector, respectively

The similarity between Eq. (25) and the Kuramoto model is worth noting. Closing k_1, k_2- keys in the model from Fig. 3. one obtains an elementary PLL orthogonal filter described by:

$$\frac{d}{dt}\begin{bmatrix}\Psi_{s1}-\Psi_1\\ \Psi_{s2}-\Psi_2\end{bmatrix} = 2\pi\begin{bmatrix}-w_0 k_{V1} k_{m1} A_{C1} A_{V1} & \pm w_1 k_{V1} k_{m2} A_{C2} A_{V2}\\ \mp w_1 k_{V2} k_{m1} A_{C1} A_{V1} & -w_0 k_{V2} k_{m2} A_{C2} A_{V2}\end{bmatrix}\begin{bmatrix}\sin(\Psi_{s1}-\Psi_1)\\ \sin(\Psi_{s2}-\Psi_2)\end{bmatrix} + \begin{bmatrix}\Delta\omega_1\\ \Delta\omega_2\end{bmatrix} \tag{25}$$

where it is assumed that the connection matrix has a form:

$$\mathbf{W}_c = \mathbf{W} - w_0\mathbf{1} \tag{27}$$

with

$$\mathbf{W}^2 = \mathbf{-1},\ \ \mathbf{W}^T = \mathbf{W}^{-1} = \mathbf{-W} \tag{28}$$

and $w_0 > 0$ (**W** –skew-symmetric, orthogonal)

Let us note that PLL implementation of the elementary orthogonal filter from Fig.3. can be easily scaled up to n-dimensional space. Such a generalization is shown in Fig.4. (Citko and Sienko, 2008).
The Adler equation of this model is given by:

$$\frac{d}{dt}\begin{bmatrix}\Psi_{s1}-\Psi_1\\ \Psi_{s2}-\Psi_2\\ \vdots\\ \Psi_{sk}-\Psi_k\end{bmatrix} = 2\pi\begin{bmatrix}-w_0 k_{v1} k_{m1} A_{C1} A_{v1} & \pm w_1 k_{v1} k_{m2} A_{C2} A_{v2} & \cdots & \pm w_{k-1} k_{v1} k_{mk} A_{Ck} A_{vk}\\ \mp w_1 k_{v2} k_{m1} A_{C1} A_{v1} & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \mp w_{2k-1} k_{vk} k_{m1} A_{C1} A_{v1} & \cdots & \cdots & -w_0 k_{vk} k_{mk} A_{Ck} A_{vk}\end{bmatrix}\begin{bmatrix}\sin(\Psi_{s1}-\Psi_1)\\ \sin(\Psi_{s2}-\Psi_2)\\ \vdots\\ \sin(\Psi_{sk}-\Psi_k)\end{bmatrix} + \begin{bmatrix}\Delta\omega_1\\ \Delta\omega_2\\ \vdots\\ \Delta\omega_k\end{bmatrix} \tag{29}$$

where : $s_i(t) = A_{Ci} \sin(\Omega_i t + \Psi_{si})$
$v_i(t) = A_{Vi} \cos(\Omega_i t + \Psi_i)$
$\Delta\omega_i$ – frequency deviation

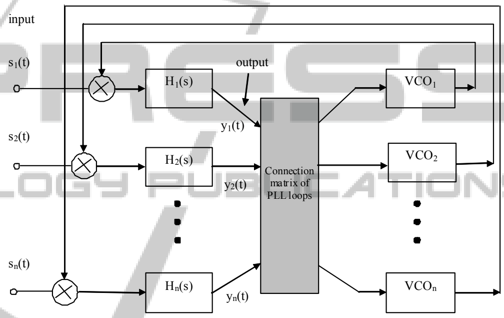


Figure 4: A PLL model of the n-dim neural network.

Equation (29) can be rewritten as:

$$\dot{\mathbf{z}} = \mathbf{W_c} \sin \mathbf{z} + \mathbf{\Delta\omega} \tag{30}$$

where: $\mathbf{z} = [z_1, \dots, z_n]^T = [\Psi_{s1} - \Psi_1, \dots, \Psi_{sn} - \Psi_n]^T$
$\mathbf{W_c}$ – matrix of connections.

It is worth noting that:

1. The hold range of a PLL network is determined by the stable equilibrium of Eq.(30). It means that, for a given $\mathbf{\Delta\omega}$, one can find loop gains ($k_v k_m A_c A_v$) such that the PLL network attains synchronization in the point: $|\sin z_i| < 1$, i = 1, … , n.
2. Under synchronization, the steady-state output of the PLL network is given by:

   $$\mathbf{y} = \mathbf{\sin z} = \mathbf{W_c}^{-1}(-\mathbf{\Delta\omega}). \tag{31}$$

   Taking the connection matrix $\mathbf{W}_c$ as the weight matrix in the orthogonal filter, the output **y** gives the Haar spectrum of the input vector. Moreover, the PLL network from Fig. 4. can be treated as a n-dimensional FM signal demodulator.

3. The PLL network from Fig. 4. can be seen as a model of the neural network with dynamical connections. The weight of connections can be changed by the parameter k_v (i.e. sensitivity of VCO).

4 OSCILLATORY MODEL OF NEUROMOPHIC PROCESSORS BY EMBEDDING ORTHOGONAL FILTERS

By embedding the HNN-based orthogonal filters into the net of PLL, one obtains a novel model of the neuromorphic processor. Such a model is presented in Fig. 5., where the structure from Fig.4. was accordingly utilized.

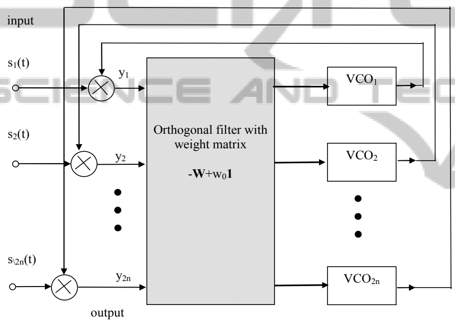
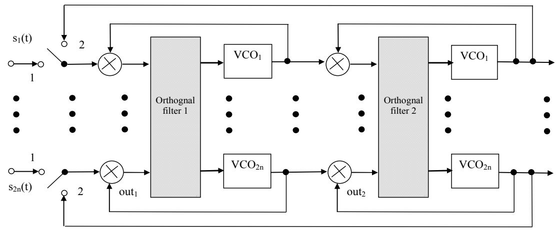


Figure 5: The oscillatory model of the neural network as the embedded system.

It is worth noting that this model consists of the network of "synaptic connections" hidden in the structure of the orthogonal filter (Eq.4). Hence, it could be a justification to name this structure as neuromorphic. Moreover, the dynamic of the model from Fig. 5 is given by Adler equations (29) and it can be seen as a basic bulinding block to create the oscillatory nets. The key contribution of this paper can be formulated by the following statement: by the chain connection of an even number of blocks from Fig. 5. one obtains a ring structure performing functions of self-sustaining memory with parallel analysis of the input information by embedded orthogonal filters.

A number of simulations were performed by using Matlab-Simulink macro-models of phase locked-loops. This analysis showed that oscillatory memory proposed above exactly performed algebraic functions of embedded orthogonal filters.



Switch: 1 - input information
2 - memory

Figure 6: The self-sustaining memory ring with two embedded orthogonal filters.

5 CONCLUSIONS

The main goal of this paper was to prove the following statements:

An AI compatible processor should be formulated in the form of a top-down structure via the following hierarchy: the Hamiltonian neural network (composed of lossless neurons) – the octonionic module (a basic building block). Furthermore, it has been confirmed that by using the octonionic module based structures, one obtains regularized and stable networks for learning. Thus, typical for AI tasks, such as realization of classifiers, pattern recognizers and memories, could be physically implemented for any number $N=2^k$ (dimension of input vectors). It is clear that the octonionic module cannot be ideally realized as an orthogonal filter (decoherence-like phenomena). Hence, the problem under consideration now is as follows: how exactly an octonionic module be realized by using cheap VLSI technology to preserve the main properties -orthogonality, power efficiency and scaleability. The possibility to directly transform the integrator structure in to the phase-locked loop (PLL)-based oscillatory structure is noteworthy. It is clear, however, that oscillatory neural network from Fig. 5. does not mimic the biological spiking tissue. Nevertheless, we claim that orthogonal filters-based data processing can be considered as inspired by biological solutions.

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