Inconsistency and Sequentiality in LTL

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Abstract: Inconsistency-tolerant temporal reasoning with sequential (ordered or hierarchical) information is of gaining increasing importance in the areas of computer science applications such as medical informatics. A logical system for representing such reasoning is required for obtaining a theoretical basis for such applications. In this paper, a new logic called a paraconsistent sequential linear-time temporal logic (PSLTL) is introduced extending the standard linear-time temporal logic (LTL). PSLTL can appropriately represent inconsistency-tolerant temporal reasoning with sequential information. The cut-elimination, complexity and completeness theorems for PSLTL are proved as the main results of this paper.

1 INTRODUCTION

Inconsistency-tolerant temporal reasoning with sequential (ordered or hierarchical) information is of growing importance in the areas of computer science applications such as medical informatics and agent communication. A logical system for representing such reasoning is required for obtaining a concrete theoretical basis for such applications. But, there was no logical system that can simultaneously represent inconsistency, sequentiality and temporality. Thus, the aim of this paper is to introduce a logical system for appropriately representing inconsistency-tolerant temporal reasoning with sequential information.

For this aim, a new logic called a *paraconsistent sequential linear-time temporal logic* (PSLTL) is introduced in this paper extending the standard *linear-time temporal logic* (LTL) (Pnueli, 1977). Inconsistency-tolerant reasoning in PSLTL is expressed by a paraconsistent negation connective, and sequential information in PSLTL is represented by some sequence modal operators. Temporal reasoning in PSLTL is, of course, expressed by some temporal operators used in LTL. As the main results of this paper, the cut-elimination, complexity and completeness theorems for PSLTL are proved using some theorems for semantically and syntactically embedding PSLTL into its fragments SLTL and LTL.

The proposed logic PSLTL is regarded as an extension of both LTL and *Nelson's paraconsistent four-valued logic with strong negation*, N4 (Almukdad and Nelson, 1984; Kamide and Wansing, 2012; Nelson, 1949; Wansing, 1993). On one hand, LTL is known to be one of the most useful temporal logics for verifying and specifying concurrent systems and temporal reasoning. On the other hand, N4 is known to be one of the most important base logics for inconsistency-tolerant reasoning. Combining the logics LTL and N4 was studied in (Kamide and Wansing, 2011), and such a combined logic is called a *paraconsistent LTL* (PLTL). PSLTL is obtained from PLTL by adding some sequence modal operators.

Combining LTL with some sequence modal operators was studied in (Kamide, 2010; Kaneiwa and Kamide, 2010; Kamide, 2013a), and such a combined logic was called a *sequence-indexed LTL* (SLTL). PSLTL is regarded as a modified paraconsistent extension of SLTL, and hence PSLTL is a modified extension of both PLTL (Kamide and Wansing, 2011) and SLTL (Kaneiwa and Kamide, 2010). In the following, we explain an important property of the paraconsistent negation connective and a plausible interpretation of sequence modal operators.

The paraconsistent negation connective $\sim$ used in PSLTL can suitably be expressed inconsistency-tolerant reasoning. One reason why $\sim$ is considered is that it can be added in such a way that the extended logics satisfy the property of *paraconsistency*. A semantic consequence relation $\models$ is called paraconsistent with respect to a negation connective $\sim$ if there

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are formulas α, β such that not $\{\alpha, \sim\alpha\} \models \beta$. In the case of LTL, this implies that there is a model M and a position i of a sequence $\sigma = t_0, t_1, t_2, \ldots$ of time-points in M with not $[(M,i) \models (\alpha \wedge \sim\alpha) \rightarrow \beta]$.

It is known that logical systems with paraconsistency can deal with inconsistency-tolerant and uncertainty reasoning more appropriately than systems that are non-paraconsistent. For example, we do not desire that $(s(x) \wedge \sim s(x)) \rightarrow d(x)$ is satisfied for any symptom s and disease d where $\sim s(x)$ means "person x does not have symptom s" and $d(x)$ means "person x suffers from disease d", because there may be situations that support the truth of both $s(a)$ and $\sim s(a)$ for some individual a but do not support the truth of $d(a)$.

If we cannot determine whether someone is healthy, then the vague concept *healthy* can be represented by asserting the inconsistent formula: $\mathit{healthy}(\mathit{john}) \wedge \sim\mathit{healthy}(\mathit{john})$. This is well-formalized in PSLTL because the formula: $\mathit{healthy}(\mathit{john}) \wedge \sim\mathit{healthy}(\mathit{john}) \rightarrow \mathit{hasCancer}(\mathit{john})$ where $\mathit{hasCancer}(\mathit{john})$ means John has cancer is not valid in PSLTL (i.e., PSLTL is inconsistency-tolerant). On the other hand, the formula $\mathit{healthy}(\mathit{john}) \wedge \neg\mathit{healthy}(\mathit{john}) \rightarrow \mathit{hasCancer}(\mathit{john})$ where $\neg$ is the classical negation connective is valid in classical logic (i.e., inconsistency has undesirable consequences). For more information on paraconsistency, see e.g., (Priest, 2002).

Some sequence modal operators (Kamide and Kaneiwa, 2009; Kamide, 2010; Kaneiwa and Kamide, 2010; Kaneiwa and Kamide, 2011; Kamide, 2013a; Kamide, 2013b) used in PSLTL can suitably be expressed sequential information. A sequence modal operator $[b]$ represents a sequence b of symbols. The notion of sequences is useful to represent the notions of "information," "trees," and "ontologies". Thus, "sequential (ordered or hierarchical) information" can be represented by sequences. This is plausible because a sequence structure gives a *monoid* $\langle M, ;, \emptyset \rangle$ with *informational interpretation* (Wansing, 1993): (1) M is a set of pieces of (ordered or prioritized) information (i.e., a set of sequences), (2) ; is a binary operator (on M) that combines two pieces of information (i.e., a concatenation operator on sequences), and (3) $\emptyset$ is the empty piece of information (i.e., the empty sequence).

A formula of the form $[b_1 ; b_2 ; \cdots ; b_n]\alpha$ in PSLTL intuitively means that "α is true based on a sequence b_1 ; b_2 ;$\cdots$; b_n of (ordered or prioritized) information pieces." Further, a formula of the form $[\emptyset]\alpha$ in PSLTL, which coincides with α, intuitively means that "α is true without any information (i.e., it is an eternal truth in the sense of classical logic)." Using a sequence modal operator, we can express the formula $[\mathit{john} ; \mathit{student} ; \mathit{human}]\mathrm{F}(\mathit{happy} \wedge \sim\mathit{happy})$ which means "a human student, John, will be both happy and unhappy sometime in the future." In this formula, the sequence modal operator $[\mathit{john} ; \mathit{student} ; \mathit{human}]$ represents the hierarchy $\mathit{John} \subseteq \mathit{student} \subseteq \mathit{human}$.

The structure of this paper is then presented as follows. In Section 2, PSLTL is introduced as a semantics by extending (a semantics of) LTL with a paraconsistent negation connective and some sequence modal operators. Firstly in this section, LTL is presented as the standard semantics, and next, SLTL is presented as the semantics with some sequence modal operators. Finally, PSLTL is obtained from SLTL by adding a paraconsistent negation connective similar to that of N4. In Section 3, a Genten-type sequent calculus PSLT_ω for PSLTL is introduced extending a Gentzen-type sequent calculus LT_ω for LTL. Firstly in this section, a Gentzen-type sequent calculus LT_ω, which was introduced by Kawai (Kawai, 1987), is presented, and next, a Gentzen-type sequent calculus SLT_ω for SLTL is presented based on (Kamide, 2010; Kaneiwa and Kamide, 2010). Finally, PSLT_ω is obtained from SLT_ω by adding some inference rules concerning the paraconsistent negation connective. In Section 4, the cut-elimination, complexity and completeness theorems for PSLTL (and PSLT_ω) are proved using two theorems for semantically and syntactically embedding PSLTL (and PSLT_ω) into SLTL (SLT_ω) and LTL (LT_ω). In Section 5, this paper is concluded.

2 SEMANTICS

Formulas of LTL are constructed from countably many propositional variables, $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), X (next), G (globally) and F (eventually). Lower-case letters $p, q, \ldots$ are used to denote propositional variables, and Greek lower-case letters $\alpha, \beta, \ldots$ are used to denote formulas. An expression $\alpha \leftrightarrow \beta$ is used to denote $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. We write $A \equiv B$ to indicate the syntactical identity between A and B. The symbol ω is used to represent the set of natural numbers. Lower-case letters i, j and k are used to denote any natural numbers. The symbol $\geq$ or $\leq$ is used to represent a linear order on ω.

Definition 2.1. *Formulas of* LTL *are defined by the following grammar, assuming p represents propositional variables:* $\alpha ::= p \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \neg\alpha \mid \mathrm{X}\alpha \mid \mathrm{G}\alpha \mid \mathrm{F}\alpha$

Definition 2.2 (LTL). *Let S be a non-empty set of states. A structure $M := (\sigma, I)$ is a* model *if*

1. *σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,*
2. *I is a mapping from the set Φ of propositional variables to the power set of S.*

A satisfaction relation $(M,i) \models \alpha$ for any formula α, where M is a model (σ, I) and i ($\in \omega$) represents some position within σ, is defined inductively by

1. *for any $p \in \Phi$, $(M,i) \models p$ iff $s_i \in I(p)$,*
2. *$(M,i) \models \alpha \wedge \beta$ iff $(M,i) \models \alpha$ and $(M,i) \models \beta$,*
3. *$(M,i) \models \alpha \vee \beta$ iff $(M,i) \models \alpha$ or $(M,i) \models \beta$,*
4. *$(M,i) \models \alpha \rightarrow \beta$ iff $(M,i) \models \alpha$ implies $(M,i) \models \beta$,*
5. *$(M,i) \models \neg\alpha$ iff not-$[(M,i) \models \alpha]$,*
6. *$(M,i) \models \mathrm{X}\alpha$ iff $(M,i+1) \models \alpha$,*
7. *$(M,i) \models \mathrm{G}\alpha$ iff $\forall j \geq i[(M,j) \models \alpha]$,*
8. *$(M,i) \models \mathrm{F}\alpha$ iff $\exists j \geq i[(M,j) \models \alpha]$.*

A formula α is valid *in* LTL *if $(M,0) \models \alpha$ for any model $M := (\sigma, I)$.*

Formulas of SLTL are obtained from that of LTL by adding $[b]$ (sequence modal operator) where b is a sequence. *Sequences* are constructed from countable atomic sequences, $\emptyset$ (empty sequence) and ; (composition). Lower-case letters $b, c, \ldots$ are used for sequences. An expression $[\emptyset]\alpha$ means α, and expressions $[\emptyset\ ;\ b]\alpha$ and $[b\ ;\ \emptyset]\alpha$ mean $[b]\alpha$. The set of sequences (including $\emptyset$) is denoted as SE. An expression $[\hat{d}]$ is used to represent $[d_0][d_1]\cdots[d_i]$ with $i \in \omega$ and $d_0 \equiv \emptyset$. Note that $[\hat{d}]$ can be the empty sequence. Also, an expression $\hat{d}$ is used to represent $d_0\ ;\ d_1\ ;\ \cdots\ ;\ d_i$ with $i \in \omega$.

Definition 2.3. *Formulas and sequences of* SLTL *are defined by the following grammar, assuming p and e represent propositional variables and atomic sequences, respectively:* $\alpha ::= p \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \neg\alpha \mid \mathrm{X}\alpha \mid \mathrm{G}\alpha \mid \mathrm{F}\alpha \mid [b]\alpha$. $b ::= e \mid \emptyset \mid b\ ;\ b$.

Definition 2.4 (SLTL)**.** *Let S be a non-empty set of states. A structure $M := (\sigma, \{I^{\hat{d}}\}_{\hat{d} \in \mathrm{SE}})$ is a* sequential model *if*

1. *σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,*
2. *$I^{\hat{d}}$ ($\hat{d} \in \mathrm{SE}$) are mappings from the set Φ of propositional variables to the power set of S.*

Satisfaction relations $(M,i) \models^{\hat{d}} \alpha$ ($\hat{d} \in \mathrm{SE}$) for any formula α, where M is a sequential model $(\sigma, \{I^{\hat{d}}\}_{\hat{d} \in \mathrm{SE}})$ and i ($\in \omega$) represents some position within σ, is defined inductively by

1. *for any $p \in \Phi$, $(M,i) \models^{\hat{d}} p$ iff $s_i \in I^{\hat{d}}(p)$,*
2. *$(M,i) \models^{\hat{d}} \alpha \wedge \beta$ iff $(M,i) \models^{\hat{d}} \alpha$ and $(M,i) \models^{\hat{d}} \beta$,*
3. *$(M,i) \models^{\hat{d}} \alpha \vee \beta$ iff $(M,i) \models^{\hat{d}} \alpha$ or $(M,i) \models^{\hat{d}} \beta$,*
4. *$(M,i) \models^{\hat{d}} \alpha \rightarrow \beta$ iff $(M,i) \models^{\hat{d}} \alpha$ implies $(M,i) \models^{\hat{d}} \beta$,*
5. *$(M,i) \models^{\hat{d}} \neg\alpha$ iff not-$[(M,i) \models^{\hat{d}} \alpha]$,*
6. *$(M,i) \models^{\hat{d}} \mathrm{X}\alpha$ iff $(M,i+1) \models^{\hat{d}} \alpha$,*
7. *$(M,i) \models^{\hat{d}} \mathrm{G}\alpha$ iff $\forall j \geq i[(M,j) \models^{\hat{d}} \alpha]$,*
8. *$(M,i) \models^{\hat{d}} \mathrm{F}\alpha$ iff $\exists j \geq i[(M,j) \models^{\hat{d}} \alpha]$.*
9. *for any atomic sequence e, $(M,i) \models^{\hat{d}} [e]\alpha$ iff $(M,i) \models^{\hat{d}\ ;\ e} \alpha$,*
10. *$(M,i) \models^{\hat{d}} [b\ ;\ c]\alpha$ iff $(M,i) \models^{\hat{d}} [b][c]\alpha$.*

A formula α is valid *in* SLTL *if $(M,0) \models^{\emptyset} \alpha$ for any sequential model $M := (\sigma, \{I^{\hat{d}}\}_{\hat{d} \in \mathrm{SE}})$.*

Some remarks on SLTL are given below.

1. SLTL is an extension of LTL since $\models^{\hat{d}}$ of SLTL includes $\models$ of LTL.
2. The following clauses hold for SLTL: For any formula α and any sequences c and $\hat{d}$,
   (a) $(M,i) \models^{\hat{d}} [c]\alpha$ iff $(M,i) \models^{\hat{d}\ ;\ c} \alpha$,
   (b) $(M,i) \models^{\emptyset} [\hat{d}]\alpha$ iff $(M,i) \models^{\hat{d}} \alpha$.
3. The following formulas are valid in SLTL: for any formulas α and β and any $b, c \in \mathrm{SE}$,
   (a) $[b](\alpha \circ \beta) \leftrightarrow ([b]\alpha) \circ ([b]\beta)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$,
   (b) $[b](\sharp\alpha) \leftrightarrow \sharp([b]\alpha)$ where $\sharp \in \{\neg, \mathrm{X}, \mathrm{G}, \mathrm{F}\}$,
   (c) $[b\ ;\ c]\alpha \leftrightarrow [b][c]\alpha$.

Formulas of PSLTL are obtained from that of SLTL by adding $\sim$ (paraconsistent negation).

Definition 2.5. *Formulas and sequences of* PSLTL *are defined by the following grammar, assuming p and e represent propositional variables and atomic sequences, respectively:* $\alpha ::= p \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid \neg\alpha \mid {\sim}\alpha \mid \mathrm{X}\alpha \mid \mathrm{G}\alpha \mid \mathrm{F}\alpha \mid [b]\alpha$. $b ::= e \mid \emptyset \mid b\ ;\ b$.

Definition 2.6 (PSLTL)**.** *Let S be a non-empty set of states. A structure $M := (\sigma, \{I^{+\hat{d}}\}_{\hat{d} \in \mathrm{SE}}, \{I^{-\hat{d}}\}_{\hat{d} \in \mathrm{SE}})$ is a* paraconsistent sequential model *if*

1. *σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S,*
2. *$I^{*\hat{d}}$ ($* \in \{+,-\}$, $\hat{d} \in \mathrm{SE}$) are mappings from the set Φ of propositional variables to the power set of S.*

Satisfaction relations $(M,i) \models^{*\hat{d}} \alpha$ ($* \in \{+,-\}$, $\hat{d} \in \mathrm{SE}$) for any formula α, where M is a paraconsistent sequential model $(\sigma, \{I^{+\hat{d}}\}_{\hat{d} \in \mathrm{SE}}, \{I^{-\hat{d}}\}_{\hat{d} \in \mathrm{SE}})$ and i ($\in \omega$) represents some position within σ, are defined by

1. *for any $p \in \Phi$, $(M,i) \models^{+\hat{d}} p$ iff $s_i \in I^{+\hat{d}}(p)$,*
2. *$(M,i) \models^{+\hat{d}} \alpha \wedge \beta$ iff $(M,i) \models^{+\hat{d}} \alpha$ and $(M,i) \models^{+\hat{d}} \beta$,*

3. $(M,i) \models^{+\hat{d}} \alpha \vee \beta$ *iff* $(M,i) \models^{+\hat{d}} \alpha$ *or* $(M,i) \models^{+\hat{d}} \beta$,
4. $(M,i) \models^{+\hat{d}} \alpha \to \beta$ *iff* $(M,i) \models^{+\hat{d}} \alpha$ *implies* $(M,i) \models^{+\hat{d}} \beta$,
5. $(M,i) \models^{+\hat{d}} \neg\alpha$ *iff not*-$[(M,i) \models^{+\hat{d}} \alpha]$,
6. $(M,i) \models^{+\hat{d}} {\sim}\alpha$ *iff* $(M,i) \models^{-\hat{d}} \alpha$,
7. $(M,i) \models^{+\hat{d}} \mathrm{X}\alpha$ *iff* $(M,i+1) \models^{+\hat{d}} \alpha$,
8. $(M,i) \models^{+\hat{d}} \mathrm{G}\alpha$ *iff* $\forall j \geq i[(M,j) \models^{+\hat{d}} \alpha]$,
9. $(M,i) \models^{+\hat{d}} \mathrm{F}\alpha$ *iff* $\exists j \geq i[(M,j) \models^{+\hat{d}} \alpha]$,
10. *for any* $p \in \Phi$, $(M,i) \models^{-\hat{d}} p$ *iff* $s_i \in I^{-\hat{d}}(p)$,
11. $(M,i) \models^{-\hat{d}} \alpha \wedge \beta$ *iff* $(M,i) \models^{-\hat{d}} \alpha$ *or* $(M,i) \models^{-\hat{d}} \beta$,
12. $(M,i) \models^{-\hat{d}} \alpha \vee \beta$ *iff* $(M,i) \models^{-\hat{d}} \alpha$ *and* $(M,i) \models^{-\hat{d}} \beta$,
13. $(M,i) \models^{-\hat{d}} \alpha \to \beta$ *iff* $(M,i) \models^{+\hat{d}} \alpha$ *and* $(M,i) \models^{-\hat{d}} \beta$,
14. $(M,i) \models^{-\hat{d}} \neg\alpha$ *iff not*-$[(M,i) \models^{-\hat{d}} \alpha]$,
15. $(M,i) \models^{-\hat{d}} {\sim}\alpha$ *iff* $(M,i) \models^{+\hat{d}} \alpha$,
16. $(M,i) \models^{-\hat{d}} \mathrm{X}\alpha$ *iff* $(M,i+1) \models^{-\hat{d}} \alpha$,
17. $(M,i) \models^{-\hat{d}} \mathrm{G}\alpha$ *iff* $\exists j \geq i[(M,j) \models^{-\hat{d}} \alpha]$,
18. $(M,i) \models^{-\hat{d}} \mathrm{F}\alpha$ *iff* $\forall j \geq i[(M,j) \models^{-\hat{d}} \alpha]$,
19. *for any atomic sequence* e *and any* $* \in \{+,-\}$, $(M,i) \models^{*\hat{d}} [e]\alpha$ *iff* $(M,i) \models^{*\hat{d}\,;\,e} \alpha$,
20. *for any* $* \in \{+,-\}$, $(M,i) \models^{*\hat{d}} [b\,;c]\alpha$ *iff* $(M,i) \models^{*\hat{d}} [b][c]\alpha$.

A formula α *is* valid *in* PSLTL *iff* $(M,0) \models^{+\emptyset} \alpha$ *for any paraconsistent sequential model* $M := (\sigma, \{I^{+\hat{d}}\}_{\hat{d}\in \mathrm{SE}}, \{I^{+\hat{d}}\}_{\hat{d}\in \mathrm{SE}})$.

Some remarks on PSLTL are given below.

1. The intuitive meanings of $\models^{+\hat{d}}$ and $\models^{-\hat{d}}$ are "verification (or justification) with sequential information" and "refutation (or falsification) with sequential information," respectively.
2. F and G are duals of each other not only with respect to $\neg$ but also with respect to $\sim$. X is a self dual not only with respect to $\neg$ but also with respect to $\sim$. $[b]$ is a self dual not only with respect to $\neg$ but also with respect to $\sim$. $\neg$ and $\sim$ are self-duals with respect to $\sim$ and $\neg$, respectively.
3. The falsification conditions for $\neg$ may be felt to be in need of some justification. Suppose that a is a person who is neither rich nor poor and that, as a matter of fact, no one is both rich and poor. Let p stand for the claim that a is poor and r for the claim that a is rich. Intuitively, a state definitely verifies p iff it falsifies r, and vice versa. Suppose now that $\neg p$ is indeed falsified at a state i in model M: $(M,i) \models^{-\hat{d}} \neg p$. This should mean that it is verified at i that p is poor or neither poor or rich. But this is the case iff r is not verified at i, which means that p is not falsified at i.
4. PSLTL is paraconsistent with respect to $\sim$. The reason is presented as follows. Assume a paraconsistent sequential model $M := (\sigma, \{I^{+\hat{d}}\}_{\hat{d}\in \mathrm{SE}}, \{I^{+\hat{d}}\}_{\hat{d}\in \mathrm{SE}})$ such that $s_i \in I^{+\hat{d}}(p)$, $s_i \in I^{-\hat{d}}(p)$ and $s_i \notin I^{+\hat{d}}(q)$ for a pair of distinct propositional variables p and q. Then, $(M,i) \models^{+\hat{d}} (p \wedge {\sim}p) \to q$ does not hold.
5. The following clauses hold for PSLTL: For any formula α, any sequences c, $\hat{d}$ and any $* \in \{+,-\}$,
   (a) $(M,i) \models^{*\hat{d}} [c]\alpha$ iff $(M,i) \models^{*\hat{d}\,;\,c} \alpha$,
   (b) $(M,i) \models^{*\emptyset} [\hat{d}]\alpha$ iff $(M,i) \models^{*\hat{d}} \alpha$.

3 SEQUENT CALCULUS

Greek capital letters $\Gamma, \Delta, \ldots$ are used to represent finite (possibly empty) sets of formulas. An expression $\mathrm{X}^i\alpha$ for any $i \in \omega$ is defined inductively by $\mathrm{X}^0\alpha \equiv \alpha$ and $\mathrm{X}^{n+1}\alpha \equiv \mathrm{X}^n\mathrm{X}\alpha$. An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*. An expression $L \vdash S$ is used to denote the fact that a sequent S is provable in a sequent calculus L. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance $\frac{S_1 \cdots S_n}{S}$ of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

Kawai's sequent calculus LT_ω (Kawai, 1987) for LTL is presented below.

Definition 3.1 (LT_ω). *The initial sequents of* LT_ω *are of the form: for any propositional variable* p,

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p.$$

The structural rules of LT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The logical inference rules of LT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \Sigma, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \Pi}{\mathrm{X}^i(\alpha \to \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\to\text{left})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \to \beta)}\ (\to\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left1}) \quad \frac{\mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \wedge \beta)}\ (\wedge\text{right})$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right1}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\beta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\alpha \vee \beta)}\ (\vee\text{right2})$$

$$\frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha}{\mathrm{X}^i\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}) \quad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\neg\alpha}\ (\neg\text{right})$$

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \quad \frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright})$$

$$\frac{\{\ \mathrm{X}^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\mathrm{X}^i\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \quad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^{i+k}\alpha}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{F}\alpha}\ (\text{Fright}).$$

Some remarks on LT_ω are given below.

1. The rules (Gright) and (Fleft) have infinite premises.
2. The sequents of the form: $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free LT_ω. This fact can be proved by induction on the complexity of α.
3. The cut-elimination and completeness theorems for LT_ω were proved by Kawai (Kawai, 1987).

Prior to introduce a sequent calculus for SLTL, we have to introduce some notations. The symbol K is used to represent the set $\{\mathrm{X}\} \cup \{[b] \mid b \in \mathrm{SE}\}$, and the symbol K^* is used to represent the set of all words of finite length of the alphabet K. For example, $\mathrm{X}^i[\hat{b}]\mathrm{X}^j[\hat{c}]$ is in K^*. Remark that K^* includes $\emptyset$, and hence $\{\dagger\alpha \mid \dagger \in K^*\}$ includes α. An expression $\sharp$ is used to represent an arbitrary member of K^*.

A sequent calculus SLT_ω for SLTL is then introduced below.

Definition 3.2 (SLT_ω)**.** *The initial sequents of* SLT_ω *are of the form: for any propositional variable p,*

$$\sharp p \Rightarrow \sharp p.$$

The structural rules of SLT_ω *are* (cut), (we-left) *and* (we-right) *in Definition 3.1.*

The logical inference rules of SLT_ω *are of the form:*

$$\frac{\Gamma \Rightarrow \Sigma, \sharp\alpha \quad \sharp\beta, \Delta \Rightarrow \Pi}{\sharp(\alpha\rightarrow\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\rightarrow\text{left}^s)$$

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta, \sharp\beta}{\Gamma \Rightarrow \Delta, \sharp(\alpha\rightarrow\beta)}\ (\rightarrow\text{right}^s)$$

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta}{\sharp(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left1}^s) \quad \frac{\sharp\beta, \Gamma \Rightarrow \Delta}{\sharp(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left2}^s)$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp\alpha \quad \Gamma \Rightarrow \Delta, \sharp\beta}{\Gamma \Rightarrow \Delta, \sharp(\alpha\wedge\beta)}\ (\wedge\text{right}^s)$$

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta \quad \sharp\beta, \Gamma \Rightarrow \Delta}{\sharp(\alpha\vee\beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}^s)$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp\alpha}{\Gamma \Rightarrow \Delta, \sharp(\alpha\vee\beta)}\ (\vee\text{right1}^s) \quad \frac{\Gamma \Rightarrow \Delta, \sharp\beta}{\Gamma \Rightarrow \Delta, \sharp(\alpha\vee\beta)}\ (\vee\text{right2}^s)$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp\alpha}{\sharp\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}^s) \quad \frac{\sharp\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sharp\neg\alpha}\ (\neg\text{right}^s)$$

$$\frac{\sharp\mathrm{X}^k\alpha, \Gamma \Rightarrow \Delta}{\sharp\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}^s) \quad \frac{\{\ \Gamma \Rightarrow \Delta, \sharp\mathrm{X}^j\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \sharp\mathrm{G}\alpha}\ (\text{Gright}^s)$$

$$\frac{\{\ \sharp\mathrm{X}^j\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\sharp\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}^s) \quad \frac{\Gamma \Rightarrow \Delta, \sharp\mathrm{X}^k\alpha}{\Gamma \Rightarrow \Delta, \sharp\mathrm{F}\alpha}\ (\text{Fright}^s)$$

$$\frac{\sharp[b]\mathrm{X}\alpha, \Gamma \Rightarrow \Delta}{\sharp\mathrm{X}[b]\alpha, \Gamma \Rightarrow \Delta}\ (\text{Xleft}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp[b]\mathrm{X}\alpha}{\Gamma \Rightarrow \Delta, \sharp\mathrm{X}[b]\alpha}\ (\text{Xright}).$$

The sequence inference rules of SLT_ω *are of the form:*

$$\frac{\sharp[b][c]\alpha, \Gamma \Rightarrow \Delta}{\sharp[b\,;c]\alpha, \Gamma \Rightarrow \Delta}\ (;\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp[b][c]\alpha}{\Gamma \Rightarrow \Delta, \sharp[b\,;c]\alpha}\ (;\text{right}).$$

Some remarks on SLT_ω are given below.

1. The sequents of the form $\sharp\alpha \Rightarrow \sharp\alpha$ for any formula α are provable in cut-free SLT_ω. This fact can be proved by induction on the complexity of α.
2. The following rules are admissible in cut-free SLT_ω:

$$\frac{\sharp\mathrm{X}[b]\alpha, \Gamma \Rightarrow \Delta}{\sharp[b]\mathrm{X}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Xleft}^{-1}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp\mathrm{X}[b]\alpha}{\Gamma \Rightarrow \Delta, \sharp[b]\mathrm{X}\alpha}\ (\text{Xright}^{-1}).$$

A sequent calculus PSLT_ω for PSLTL is introduced below.

Definition 3.3 (PSLT_ω)**.** PSLT_ω *is obtained from* SLT_ω *by adding the initial sequents of the form: for any propositional variable p,*

$$\sharp{\sim}p \Rightarrow \sharp{\sim}p,$$

and adding the logical and sequence inference rules of the form:

$$\frac{\sharp{\sim}\alpha, \Gamma \Rightarrow \Delta}{{\sim}\sharp\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\sharp\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp{\sim}\alpha}{\Gamma \Rightarrow \Delta, {\sim}\sharp\alpha}\ ({\sim}\sharp\text{right})$$

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp\alpha}{\Gamma \Rightarrow \Delta, \sharp{\sim}{\sim}\alpha}\ ({\sim}{\sim}\text{right})$$

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left1})$$

$$\frac{\sharp{\sim}\beta, \Gamma \Rightarrow \Delta}{\sharp{\sim}(\alpha\rightarrow\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\rightarrow}\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp\alpha \quad \Gamma \Rightarrow \Delta, \sharp{\sim}\beta}{\Gamma \Rightarrow \Delta, \sharp{\sim}(\alpha\rightarrow\beta)}\ ({\sim}{\rightarrow}\text{right})$$

$$\frac{\sharp{\sim}\alpha, \Gamma \Rightarrow \Delta \quad \sharp{\sim}\beta, \Gamma \Rightarrow \Delta}{\sharp{\sim}(\alpha\wedge\beta), \Gamma \Rightarrow \Delta}\ ({\sim}{\wedge}\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp{\sim}\alpha}{\Gamma \Rightarrow \Delta, \sharp{\sim}(\alpha\wedge\beta)}\ ({\sim}{\wedge}\text{right1})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp{\sim}\beta}{\Gamma \Rightarrow \Delta, \sharp{\sim}(\alpha\wedge\beta)}\ ({\sim}{\wedge}\text{right2})$$

$$\frac{\sharp{\sim}\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left1})$$

$$\frac{\sharp{\sim}\beta, \Gamma \Rightarrow \Delta}{\sharp{\sim}(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ ({\sim}\vee\text{left2})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp{\sim}\alpha \quad \Gamma \Rightarrow \Delta, \sharp{\sim}\beta}{\Gamma \Rightarrow \Delta, \sharp{\sim}(\alpha \vee \beta)}\ ({\sim}\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp{\sim}\alpha}{\sharp{\sim}\neg\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\neg\text{left}) \quad \frac{\sharp{\sim}\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sharp{\sim}\neg\alpha}\ ({\sim}\neg\text{right})$$

$$\frac{\{\ \sharp\mathrm{X}^j{\sim}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{\sharp{\sim}\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{Gleft})$$

$$\frac{\Gamma \Rightarrow \Delta, \sharp\mathrm{X}^k{\sim}\alpha}{\Gamma \Rightarrow \Delta, \sharp{\sim}\mathrm{G}\alpha}\ ({\sim}\text{Gright})$$

$$\frac{\sharp\mathrm{X}^k{\sim}\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}\mathrm{F}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\text{Fleft})$$

$$\frac{\{\ \Gamma \Rightarrow \Delta, \sharp\mathrm{X}^j{\sim}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \sharp{\sim}\mathrm{F}\alpha}\ ({\sim}\text{Fright})$$

$$\frac{\sharp{\sim}[b][c]\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}[b\,;c]\alpha, \Gamma \Rightarrow \Delta}\ ({\sim};\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp{\sim}[b][c]\alpha}{\Gamma \Rightarrow \Delta, \sharp{\sim}[b\,;c]\alpha}\ ({\sim};\text{right}).$$

Some remarks on PSLT_ω are given below.

1. The sequents of the form $\sharp\alpha \Rightarrow \sharp\alpha$ for any formula α are provable in cut-free PSLT_ω. This fact can be proved by induction on the complexity of α.
2. The following rules are admissible in cut-free PSLT_ω:

$$\frac{\sharp\mathrm{X}[b]\alpha, \Gamma \Rightarrow \Delta}{\sharp[b]\mathrm{X}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Xleft}^{-1}) \quad \frac{\Gamma \Rightarrow \Delta, \sharp\mathrm{X}[b]\alpha}{\Gamma \Rightarrow \Delta, \sharp[b]\mathrm{X}\alpha}\ (\text{Xright}^{-1})$$

$$\frac{{\sim}\sharp\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}\sharp\text{left}^{-1}) \quad \frac{\Gamma \Rightarrow \Delta, {\sim}\sharp\alpha}{\Gamma \Rightarrow \Delta, \sharp{\sim}\alpha}\ ({\sim}\sharp\text{right}^{-1}).$$

4 MAIN RESULTS

In this section, we introduce a translation function f from SLTL into LTL, and a translation function g from PSLTL into SLTL. Using these functions, we obtain a translation function gf from PSLTL into LTL. Using these translation functions, we will show a theorem for semantically and syntactically embedding PSLTL into SLTL and LTL. Using these embedding theorems, we will show the cut-elimination, complexity and completeness theorems for PSLTL.

Definition 4.1 (Translation from SLTL into LTL)**.** *Let Φ be a non-empty set of propositional variables and $\Phi^{\hat{d}}$ be the set $\{p^{\hat{d}} \mid p \in \Phi\}$ ($\hat{d} \in$ SE) of propositional variables where $p^{\emptyset} := p$ (i.e., $\Phi^{\emptyset} := \Phi$). The language $\mathcal{L}^s$ (the set of formulas) of* SLTL *is defined using Φ, $[b]$, $\wedge, \vee, \rightarrow, \neg$, X, F and G. The language $\mathcal{L}$ of* LTL *is obtained from $\mathcal{L}^s$ by adding $\Phi^{\hat{d}}$ and deleting $[b]$.*

A mapping f from $\mathcal{L}^s$ to $\mathcal{L}$ is defined by:

1. *for any $p \in \Phi$, $f([\hat{d}]p) := p^{\hat{d}} \in \Phi^{\hat{d}}$, esp., $f(p) = p \in \Phi^{\emptyset}$,*
2. *$f(\sharp(\alpha \circ \beta)) := f(\sharp\alpha) \circ f(\sharp\beta)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$,*
3. *$f(\sharp\dagger\alpha) := \dagger f(\sharp\alpha)$ where $\dagger \in \{\neg, \mathrm{X}, \mathrm{G}, \mathrm{F}\}$,*
4. *$f(\sharp[b\,;c]\alpha) := f(\sharp[b][c]\alpha)$.*

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

Proposition 4.2 ((Kamide, 2010; Kaneiwa and Kamide, 2010))**.** *Let f be the mapping defined in Definition 4.1.*

1. *(Semantical embedding): For any formula α, α is valid in* SLTL *iff $f(\alpha)$ is valid in* LTL.
2. *(Syntactical embedding): For any sets Γ and Δ of formulas in $\mathcal{L}^s$,*
   (a) $\mathrm{SLT}_\omega \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{LT}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$,
   (b) SLT_ω − (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* LT_ω − (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$.
3. *(Cut-elimination): The rule* (cut) *is admissible in cut-free* SLT_ω.
4. *(Completeness): For any formula α,* $\mathrm{SLT}_\omega \vdash\ \Rightarrow \alpha$ *iff α is valid in* SLTL.

We now introduce a translation of PSLTL into SLTL, and by using this translation, we show some theorems for embedding PSLTL into SLTL. A similar translation has been used by Vorob'ev (Vorob'ev, 1952), Gurevich (Gurevich, 1977), and Rautenberg (Rautenberg, 1979) to embed Nelson's three-valued constructive logic (Almukdad and Nelson, 1984; Nelson, 1949) into intuitionistic logic.

Definition 4.3 (Translation from PSLTL into SLTL)**.** *Let Φ be a non-empty set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. The language $\mathcal{L}^{ps}$ (the set of formulas) of* PSLTL *is defined using Φ, ${\sim}$, $\rightarrow, \wedge, \vee, \neg$, X, F, G and $[b]$. The language $\mathcal{L}^s$ of* SLTL *is obtained from $\mathcal{L}^{ps}$ by adding Φ' and deleting ${\sim}$.*

A mapping g from $\mathcal{L}^{ps}$ to $\mathcal{L}^s$ is defined by

1. *for any $p \in \Phi$, $g(p) := p$ and $g({\sim}p) := p' \in \Phi'$,*
2. *$g(\alpha \circ \beta) := g(\alpha) \circ g(\beta)$ where $\circ \in \{\wedge, \vee, \rightarrow\}$,*
3. *$g(\dagger\alpha) := \dagger g(\alpha)$ where $\dagger \in \{\neg, \mathrm{X}, \mathrm{F}, \mathrm{G}, [b]\}$,*
4. *$g({\sim}{\sim}\alpha) := g(\alpha)$,*
5. *$g({\sim}\dagger\alpha) := \dagger g({\sim}\alpha)$ where $\dagger \in \{\neg, \mathrm{X}, [b]\}$,*
6. *$g({\sim}(\alpha \wedge \beta)) := g({\sim}\alpha) \vee g({\sim}\beta)$,*
7. *$g({\sim}(\alpha \vee \beta)) := g({\sim}\alpha) \wedge g({\sim}\beta)$,*
8. *$g({\sim}(\alpha \rightarrow \beta)) := g(\alpha) \wedge g({\sim}\beta)$,*
9. *$g({\sim}\mathrm{F}\alpha) := \mathrm{G}g({\sim}\alpha)$,*
10. *$g({\sim}\mathrm{G}\alpha) := \mathrm{F}g({\sim}\alpha)$.*

We have: $g(\sharp\alpha) = \sharp g(\alpha)$ for any formula α and any $\sharp \in K^*$.

The following is a translation example from PSLTL into LTL, by using the translation functions f and g.

Example 4.4. *We consider a formula* $\mathrm{G}({\sim}([b]p \wedge {\sim}[c]q))$ *where* b, c *are atomic sequences, and* p, q *are propositional variables.*

Firstly, we translate this PSLTL-formula into a SLTL-formula by the translation function g as follows.

$$
\begin{aligned}
& g(\mathrm{G}({\sim}([b]p \wedge {\sim}[c]q))) \\
& = \mathrm{G}g({\sim}([b]p \wedge {\sim}[c]q)) \\
& = \mathrm{G}(g({\sim}[b]p) \vee g({\sim}{\sim}[c]q)) \\
& = \mathrm{G}([b]g({\sim}p) \vee g([c]q)) \\
& = \mathrm{G}([b]p' \vee [c]g(q)) \\
& = \mathrm{G}([b]p' \vee [c]q)
\end{aligned}
$$

where p' *is a propositional variable in* SLTL.

Next, we translate this SLTL-formula into a LTL-formula by the translation function f *as follows.*

$$
\begin{aligned}
& f(\mathrm{G}([b]p' \vee [c]q)) \\
& = \mathrm{G}f([b]p' \vee [c]q) \\
& = \mathrm{G}(f([b]p') \vee f([c]q)) \\
& = \mathrm{G}(p'^b \vee q^c)
\end{aligned}
$$

where p'^b, q^c *are propositional variables in* LTL.

Thus, the formula $\mathrm{G}({\sim}([b]p \wedge {\sim}[c]q))$ *of* PSLTL *is translated into the formula* $\mathrm{G}(p'^b \vee q^c)$ *of* LTL.

Next, we will show a theorem for semantically embedding PSLTL into SLT. To show this theorem, we need two lemmas which are presented below.

Lemma 4.5. *Let g be the mapping defined in Definition 4.3, and S be a non-empty set of states. For any paraconsistent sequential model* $M := (\sigma, \{I^{+\hat{d}}\}_{\hat{d}\in\mathrm{SE}}, \{I^{-\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ *of* PSLTL*, any satisfaction relations* $\models^{*\hat{d}}$ $(* \in \{+,-\}, \hat{d} \in \mathrm{SE})$ *on M, and any state* s_i *in* σ*, we can construct a sequential model* $N := (\sigma, \{I^{\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ *of* SLTL *and satisfaction relations* $\models^{\hat{d}}$ *on N such that for any formula* α *in* $\mathcal{L}^{ps}$,

1. $(M,i) \models^{+\hat{d}} \alpha$ *iff* $(N,i) \models^{\hat{d}} g(\alpha)$.
2. $(M,i) \models^{-\hat{d}} \alpha$ *iff* $(N,i) \models^{\hat{d}} g({\sim}\alpha)$.

Proof. Let Φ be a non-empty set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. Suppose that M is a paraconsistent sequential model $(\sigma, \{I^{+\hat{d}}\}_{\hat{d}\in\mathrm{SE}}, \{I^{-\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ where $I^{+\hat{d}}$ and $I^{-\hat{d}}$ are mappings from Φ to the power set of S. Suppose that N is a sequential model $(\sigma, \{I^{\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ where $I^{\hat{d}}$ are mappings from $\Phi \cup \Phi'$ to the power set of S. Suppose moreover that M and N satisfy the following conditions: for any s_i in σ and any $p \in \Phi$,

1. $s_i \in I^{+\hat{d}}(p)$ iff $s_i \in I^{\hat{d}}(p)$,
2. $s_i \in I^{-\hat{d}}(p)$ iff $s_i \in I^{\hat{d}}(p')$.

The lemma is then proved by (simultaneous) induction on the complexity of α.

• Base step: Case $\alpha \equiv p \in \Phi$: For (1), we obtain: $(M,i) \models^{+\hat{d}} p$ iff $s_i \in I^{+\hat{d}}(p)$ iff $s_i \in I^{\hat{d}}(p)$ iff $(N,i) \models^{\hat{d}} p$ iff $(N,i) \models^{\hat{d}} g(p)$ (by the def. of g). For (2), we obtain: $(M,i) \models^{-\hat{d}} p$ iff $s_i \in I^{-\hat{d}}(p)$ iff $s_i \in I^{\hat{d}}(p')$ iff $(N,i) \models^{\hat{d}} p'$ iff $(N,i) \models^{\hat{d}} g({\sim}p)$ (by the def. of g).

• Induction step: We show some cases.

Case $\alpha \equiv {\sim}\beta$: For (1), we obtain: $(M,i) \models^{+\hat{d}} {\sim}\beta$ iff $(M,i) \models^{-\hat{d}} \beta$ iff $(N,i) \models^{\hat{d}} g({\sim}\beta)$ (by ind. hypo. for 2). For (2), we obtain: $(M,i) \models^{-\hat{d}} {\sim}\beta$ iff $(M,i) \models^{+\hat{d}} \beta$ iff $(N,i) \models^{\hat{d}} g(\beta)$ (by ind. hypo. for 1) iff $(N,i) \models^{\hat{d}} g({\sim}{\sim}\beta)$ (by the def. of g).

Case $\alpha \equiv \mathrm{X}\beta$: For (1), we obtain: $(M,i) \models^{+\hat{d}} \mathrm{X}\beta$ iff $(M,i+1) \models^{+\hat{d}} \beta$ iff $(N,i+1) \models^{\hat{d}} g(\beta)$ (by induction hypothesis for 1) iff $(N,i) \models^{\hat{d}} \mathrm{X}g(\beta)$ iff $(N,i) \models^{\hat{d}} g(\mathrm{X}\beta)$ (by the definition of g). For (2), we obtain: $(M,i) \models^{-\hat{d}} \mathrm{X}\beta$ iff $(M,i+1) \models^{-\hat{d}} \beta$ iff $(N,i+1) \models^{\hat{d}} g({\sim}\beta)$ (by induction hypothesis for 2) iff $(N,i) \models^{\hat{d}} \mathrm{X}g({\sim}\beta)$ iff $(N,i) \models^{\hat{d}} g({\sim}\mathrm{X}\beta)$ (by the definition of g).

Case $\alpha \equiv [b]\beta$: For (1), we obtain: $(M,i) \models^{+\hat{d}} [b]\beta$ iff $(M,i) \models^{+\hat{d}\,;\,b} \beta$ iff $(N,i) \models^{\hat{d}\,;\,b} g(\beta)$ (by induction hypothesis for 1) iff $(N,i) \models^{\hat{d}} [b]g(\beta)$ iff $(N,i) \models^{\hat{d}} g([b]\beta)$ (by the definition of g). For (2), we obtain: $(M,i) \models^{-\hat{d}} [b]\beta$ iff $(M,i) \models^{-\hat{d}\,;\,b} \beta$ iff $(N,i) \models^{\hat{d}\,;\,b} g({\sim}\beta)$ (by induction hypothesis for 2) iff $(N,i) \models^{\hat{d}} [b]g({\sim}\beta)$ iff $(N,i) \models^{\hat{d}} g({\sim}[b]\beta)$ (by the definition of g). **Q.E.D.**

Lemma 4.6. *Let g be the mapping defined in Definition 4.3, and S be a non-empty set of states. For any sequential model* $N := (\sigma, \{I^{\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ *of* SLTL *and any satisfaction relations* $\models^{\hat{d}}$ $(\hat{d} \in \mathrm{SE})$ *on N, and any state* s_i *in* σ*, we can construct a paraconsistent sequential model* $M := (\sigma, \{I^{+\hat{d}}\}_{\hat{d}\in\mathrm{SE}}, \{I^{-\hat{d}}\}_{\hat{d}\in\mathrm{SE}})$ *of* PSLTL *and satisfaction relations* $\models^{*\hat{d}}$ $(* \in \{+,-\}, \hat{d} \in \mathrm{SE})$ *on M such that*

1. $(M,i) \models^{+\hat{d}} \alpha$ *iff* $(N,i) \models^{\hat{d}} g(\alpha)$.
2. $(M,i) \models^{-\hat{d}} \alpha$ *iff* $(N,i) \models^{\hat{d}} g({\sim}\alpha)$.

Proof. Similar to the proof of Lemma 4.5. **Q.E.D.**

Theorem 4.7 (Semantical embedding from PSLTL into SLTL)**.** *Let g be the mapping defined in Definition 4.3. For any formula* α*,* α *is valid in* PSLTL *iff* $g(\alpha)$ *is valid in* SLTL.

Proof. By Lemmas 4.5 and 4.6. **Q.E.D.**

Theorem 4.8 (Semantical embedding from PSLTL into LTL)**.** *Let f and g be the mappings defined in Definitions 4.1 and 4.3, respectively. For any formula α, α is valid in* PSLTL *iff $fg(\alpha)$ is valid in* LTL.

Proof. By Proposition 4.2 (1) and Theorem 4.7. **Q.E.D.**

Theorem 4.9 (Complexity)**.** PSLTL *is PSPACE-complete.*

Proof. By decidability of LTL, for each α, it is possible to decide if $fg(\alpha)$ is valid in LTL. Then, by Theorem 4.8, PSLTL is also decidable. Moreover the mapping fg is a polynomial time translation, and LTL is know to be PSPACE-complete (Sistla and Clarke, 1985). Thus, PSLTL is also PSPACE-complete. **Q.E.D.**

Theorem 4.10 (Weak syntactical embedding from PSLT$_\omega$ into SLT$_\omega$)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}^{ps}$, and g be the mapping defined in Definition 4.3. Then:*

1. *If* PSLT$_\omega \vdash \Gamma \Rightarrow \Delta$*, then* SLT$_\omega \vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. *If* SLT$_\omega$ $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$*, then* PSLT$_\omega$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$.

Proof. • (1) : By induction on the proofs P of $\Gamma \Rightarrow \Delta$ in PSLT$_\omega$. We distinguish the cases according to the last inference of P, and show some cases.

Case ($\sharp{\sim}p \Rightarrow \sharp{\sim}p$): The last inference of P is of the form: $\sharp{\sim}p \Rightarrow \sharp{\sim}p$. In this case, we obtain the required fact LT$_\omega \vdash g(\sharp{\sim}p) \Rightarrow g(\sharp{\sim}p)$, since $g(\sharp{\sim}p)$ coincides with $\sharp p'$ by the definition of g.

Case (${\sim}{\sim}$left): The last inference of P is of the form:

$$\frac{\sharp\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}{\sim}\alpha, \Gamma \Rightarrow \Delta}\ ({\sim}{\sim}\text{left}).$$

By induction hypothesis, we have the required fact: SLT$_\omega \vdash g(\sharp\alpha), g(\Gamma) \Rightarrow g(\Delta)$ where $g(\sharp\alpha)$ coincides with $g(\sharp{\sim}{\sim}\alpha)$ by the definition of g.

Case (${\sim}$;left): The last inference of P is of the form:

$$\frac{\sharp{\sim}[b][c]\alpha, \Gamma \Rightarrow \Delta}{\sharp{\sim}[b\,;\,c]\alpha, \Gamma \Rightarrow \Delta}\ ({\sim};\text{left}).$$

By induction hypothesis, we have: SLT$_\omega \vdash g(\sharp{\sim}[b][c]\alpha), g(\Gamma) \Rightarrow g(\Delta)$ where $g(\sharp{\sim}[b][c]\alpha)$ coincides with $\sharp{\sim}[b][c]g(\alpha)$ by the definition of g. Then, we obtain:

$$\frac{\begin{array}{c}\vdots\\ \sharp{\sim}[b][c]g(\alpha), g(\Gamma) \Rightarrow g(\Delta)\end{array}}{\sharp{\sim}[b\,;\,c]g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ ({\sim};\text{left})$$

where $\sharp{\sim}[b\,;\,c]g(\alpha)$ coincides with $g(\sharp{\sim}[b\,;\,c]\alpha)$ by the definition of g.

• (2) : By induction on the proofs Q of $g(\Gamma) \Rightarrow g(\Delta)$ in SLT$_\omega$. We distinguish the cases according to the last inference of Q, and show some cases.

Case (;left): The last inference of Q is (;left).

Subcase (1): The last inference of Q is of the form:

$$\frac{\sharp[b][c]g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}{\sharp[b\,;\,c]g(\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (;\text{left})$$

where $\sharp[b][c]g(\alpha)$ and $\sharp[b\,;\,c]g(\alpha)$ respectively coincide with $g(\sharp[b][c]\alpha)$ and $g(\sharp[b\,;\,c]\alpha)$ by the definition of g. By induction hypothesis, we have: PSLT$_\omega \vdash \sharp[b][c]\alpha, \Gamma \Rightarrow \Delta$, and hence obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sharp[b][c]\alpha, \Gamma \Rightarrow \Delta\end{array}}{\sharp[b\,;\,c]\alpha, \Gamma \Rightarrow \Delta}\ (;\text{left}).$$

Subcase (2): The last inference of Q is of the form:

$$\frac{\sharp[b][c]g({\sim}\alpha), g(\Gamma) \Rightarrow g(\Delta)}{\sharp[b\,;\,c]g({\sim}\alpha), g(\Gamma) \Rightarrow g(\Delta)}\ (;\text{left})$$

where $\sharp[b][c]g({\sim}\alpha)$ and $\sharp[b\,;\,c]g({\sim}\alpha)$ respectively coincide with $g(\sharp{\sim}[b][c]\alpha)$ and $g(\sharp{\sim}[b\,;\,c]\alpha)$ by the definition of g. By induction hypothesis, we have: PSLT$_\omega \vdash \sharp{\sim}[b][c]\alpha, \Gamma \Rightarrow \Delta$, and hence obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ \sharp{\sim}[b][c]\alpha, \Gamma \Rightarrow \Delta\end{array}}{\sharp{\sim}[b\,;\,c]\alpha, \Gamma \Rightarrow \Delta}\ ({\sim};\text{left}).$$

Q.E.D.

Theorem 4.11 (Cut-elimination)**.** *The rule* (cut) *is admissible in cut-free* PSLT$_\omega$.

Proof. Suppose PSLT$_\omega \vdash \Gamma \Rightarrow \Delta$. Then, we have SLT$_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$ by Theorem 4.10 (1), and hence SLT$_\omega$ $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\Delta)$ by Proposition 4.2 (3). By Theorem 4.10 (2), we obtain PSLT$_\omega$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. **Q.E.D.**

Theorem 4.12 (Syntactical embedding from PSLT$_\omega$ into SLT$_\omega$)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}^{ps}$, and g be the mapping defined in Definition 4.3. Then:*

1. PSLT$_\omega \vdash \Gamma \Rightarrow \Delta$ *iff* SLT$_\omega \vdash g(\Gamma) \Rightarrow g(\Delta)$.
2. PSLT$_\omega$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ *iff* SLT$_\omega$ $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$.

Proof. • (1). ($\Longrightarrow$): By Theorem 4.10 (1). ($\Longleftarrow$): Suppose SLT$_\omega \vdash g(\Gamma) \Rightarrow g(\Delta)$. We then have SLT$_\omega$ $-$ (cut) $\vdash g(\Gamma) \Rightarrow g(\Delta)$ by Proposition 4.2 (3). Thus, we obtain PSLT$_\omega$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$ by Theorem 4.10 (2). Therefore we have PSLT$_\omega \vdash \Gamma \Rightarrow \Delta$.

• (2). ($\Longrightarrow$): Suppose PSLT$_\omega$ $-$ (cut) $\vdash \Gamma \Rightarrow \Delta$. Then we have PSLT$_\omega \vdash \Gamma \Rightarrow \Delta$. We then obtain SLT$_\omega$

$\vdash g(\Gamma) \Rightarrow g(\Delta)$ by Theorem 4.10 (1). Therefore we obtain $\mathrm{SLT}_\omega - (\mathrm{cut}) \vdash g(\Gamma) \Rightarrow g(\Delta)$ by Proposition 4.2 (3). ($\Longleftarrow$): By Theorem 4.10 (2). **Q.E.D.**

Theorem 4.13 (Syntactical embedding from PSLT_ω into LT_ω)**.** *Let Γ and Δ be sets of formulas in $\mathcal{L}^{ps}$. Let f and g be the mappings defined in Definitions 4.1 and 4.3, respectively. Then:*

1. $\mathrm{PSLT}_\omega \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{SLT}_\omega \vdash fg(\Gamma) \Rightarrow fg(\Delta)$.
2. $\mathrm{PSLT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathrm{SLT}_\omega - (\mathrm{cut}) \vdash fg(\Gamma) \Rightarrow fg(\Delta)$.

Proof. By Proposition 4.2 (2) and Theorem 4.12. **Q.E.D.**

Theorem 4.14 (Completeness)**.** *For any formula α,*

$\mathrm{PSLT}_\omega \vdash \Rightarrow \alpha$ *iff* α *is valid in* PSLTL.

Proof. $\mathrm{PSLT}_\omega \vdash \Rightarrow \alpha$ iff $\mathrm{SLT}_\omega \vdash \Rightarrow g(\alpha)$ (by Theorem 4.12) iff $g(\alpha)$ is valid in SLTL (by Proposition 4.2 (4)) iff α is valid in PSLTL (by Theorem 4.7). **Q.E.D.**

5 CONCLUSIONS

In this paper, the logic PSLTL (paraconsistent sequential linear-time temporal logic) was introduced as a semantics by extending the standard logic (a semantics of) LTL (linear-time temporal logic). PSLTL can appropriately represent inconsistency-tolerant reasoning by the paraconsistent negation connective, and sequential (hierarchical) information by some sequence modal operators. By using the semantical embedding theorem of PSLTL into LTL, it was shown that PSLTL is PSPACE-complete. The Gentzen-type sequent calculus PSLT_ω for PSLTL was introduced, and the cut-elimination theorem for this calculus was proved using the syntactical embedding theorem of PSLT_ω into its non-paraconsistent fragment SLT_ω. The completeness theorem for PSLTL (and PSLT_ω) was proved using both syntactical and semantical embedding theorems of PSLTL (and PSLT_ω) into SLTL (and SLT_ω). It was thus shown in this paper that PSLTL and PSLT_ω are a good theoretical basis for inconsistency-tolerant temporal reasoning with sequential information.

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