Selection-based Approach to Cooperative Interval Games

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Abstract: Cooperative interval games are a generalized model of cooperative games in which worth of every coalition corresponds to a closed interval representing the possible outcomes of its cooperation. Selections are all possible outcomes of the interval game with no additional uncertainty. We introduce new selection-based classes of interval games and prove their characterizations and relations to existing classes based on the weakly better operator. We show new results regarding the core and imputations. Then we introduce the definition of strong imputation and strong core. We also examine a problem of equality of two different versions of core, which is the main stability solution of cooperative games.

1 INTRODUCTION

Uncertainty and inaccurate data are issues occurring very often in the real world situations. Therefore it is important to be able to make decisions even when the exact data are not available and only bounds on them are known.

In classical cooperative game theory, every group of players (*coalition*) knows precise reward for their cooperation; in cooperative interval games, only the worst and the best possible outcome are known. Such situations can be naturally modeled with intervals encapsulating these outcomes.

Cooperation under interval uncertainty was first considered by Branzei, Dimitrov and Tijs in 2003 to study bankruptcy situations (Branzei et al., 2004) and later further extensively studied by Gök in her PhD thesis (Alparslan-Gök, 2009) and other papers written together with Branzei et al. (see the references section of (Branzei et al., 2010) for more).

However, their approach is almost exclusively aimed on interval solutions, that is on payoff distributions consisting of intervals and thus containing another uncertainty. This is in contrast with selections – possible outcomes of an interval game with no additional uncertainty. Selection-based approach was never systematically studied and not very much is known. This paper is trying to fix this and summarizes our results regarding a selection-based approach to interval games.

This paper has the following structure. Section 2 is a preliminary section which concisely presents necessary definitions and facts on classical cooperative games, interval analysis and cooperative interval games. Section 3 is devoted to new selection-based classes of interval games. We consequently prove their characterizations and relations to existing classes. Section 4 focuses on the so called core incidence problem which asks under which conditions are the selection core and the set of payoffs generated by the interval core equal. In Section 5, definitions of strong core and strong imputation are introduced as new concepts. We show some remarks on strong core, one of them being a characterization of games with the strong imputation and strong core.

On Mathematical Notation

We will use $\leq$ relation on real vectors. For every $x, y \in \mathbb{R}^N$ we write $x \leq y$ if $x_i \leq y_i$ holds for every $1 \leq i \leq N$.

We do not use symbol $\subset$ in this paper. Instead, $\subseteq$ and $\subsetneq$ are used for subset and proper subset, respectively, to avoid ambiguity.

2 PRELIMINARIES

2.1 Classical Cooperative Game Theory

Comprehensive sources on classical cooperative game theory are for example (Branzei et al., 2000) (Driessen, 1988) (Gilles, 2010) (Peleg and Sudhölter,

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2007). For more info about on applications, see e.g. (Bilbao, 2000) (Curiel, 1997) (Lemaire, 1991). Here we present only necessary background theory for study of interval games. We examine games with transferable utility (TU), therefore by cooperative game we mean cooperative TU game.

Definition 2.1. (Cooperative Game) *Cooperative game is an ordered pair* (N,v)*, where* $N = \{1,2,\ldots,n\}$ *is a set of players and* $v : 2^N \to \mathbb{R}$ *is a characteristic function of the cooperative game. We further assume that* $v(\emptyset) = 0$*.*

The set of all cooperative games with player set N is denoted by G^N.

Subsets of N are called *coalitions* and N itself is called a *grand coalition*.

Note 2.2. *We often write* v *instead of* (N,v)*, because we can easily identify game with its characteristic function without loss of generality.*

To further analyze players' gains, we will need a *payoff vector* which can be interpreted as a proposed distribution of reward between players.

Definition 2.3. (Payoff vector) *Payoff vector for a cooperative game* (N,v) *is a vector* $x \in \mathbb{R}^N$ *with* x_i *denoting reward given to ith player.*

Definition 2.4. (Imputation) *An imputation of* $(N,v) \in G^N$ *is a vector* $x \in \mathbb{R}^N$ *such that* $\sum_{i\in N} x_i = v(N)$ *and* $x_i \geq v(\{i\})$ *for every* $i \in N$*.*

The set of all imputations of a given cooperative game (N,v) *is denoted by* $I(v)$*.*

Definition 2.5. (Core) *The core of* $(N,v) \in G^N$ *is the set*

$$C(v) = \left\{ x \in I(v),\ \sum_{i\in S} x_i \geq v(S), \forall S \subseteq N \right\}.$$

There are many important classes of cooperative games. Here we show the most important ones.

Definition 2.6. (Monotonic game) *A game* (N,v) *is monotonic if for every* $T \subseteq S \subseteq N$ *we have*

$$v(T) \leq v(S).$$

Informally, in monotonic games, bigger coalitions are stronger.

Definition 2.7. (Superadditive game) *A game* (N,v) *is superadditive if for every* $S,T \subseteq N$*,* $S \cap T = \emptyset$ *we have*

$$v(T) + v(S) \leq v(S \cup T).$$

In a superadditive game, coalition has no incentive to divide itself, since together, they will always achieve at least as much as separated.

Superadditive game is not necessarily monotonic. Conversely, monotonic game is not necessarily superadditive. However, these classes have a nonempty intersection. Check Caulier's paper (Caulier, 2009) for more details on relation of these two classes.

Definition 2.8. (Additive game) *A game* (N,v) *is additive if for every* $S,T \subseteq N$*,* $S \cap T = \emptyset$ *we have*

$$v(T) + v(S) = v(S \cup T).$$

Observe that additive games are superadditive as well.

Another important type of game is a *convex game*.

Definition 2.9. (Convex game) *A game* (N,v) *is convex if its characteristic function is supermodular. The characteristic function is supermodular if for every* $S \subseteq T \subseteq N$ *holds*

$$v(T) + v(S) \leq v(S \cup T) + v(S \cap T).$$

Clearly, supermodularity implies superadditivity.

Convex games have many nice properties. we show the most important one.

Theorem 2.10. *(Shapley 1971 (Shapley, 1971)) If a game* (N,v) *is convex, then its core is nonempty.*

2.2 Interval Analysis

Definition 2.11. (Interval) *The interval* X *is a set*

$$X := [\underline{X}, \overline{X}] = \{x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X}\}.$$

With $\underline{X}$ *being the lower bound and* $\overline{X}$ *being the upper bound of the interval.*

From now on, when we say an interval we mean a closed interval. The set of all real intervals is denoted by $\mathbb{IR}$.

The following definition shows how to do a basic arithmetics with intervals (Moore et al., 2009).

Definition 2.12. *For every* $X,Y,Z \in \mathbb{IR}$ *and* $0 \notin Z$ *define*

$$X + Y := [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}],$$

$$X - Y := [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}],$$

$$X \cdot Y := [\min S, \max S],\ S = \{\underline{X}\overline{Y}, \overline{X}\underline{Y}, \underline{X}\underline{Y}, \overline{X}\overline{Y}\},$$

$$X \,/\, Z := [\min S, \max S],\ S = \{\underline{X}/\overline{Z}, \overline{X}/\underline{Z}, \underline{X}/\underline{Z}, \overline{X}/\overline{Z}\}.$$

2.3 Cooperative Interval Games

Definition 2.13. (Cooperative Interval Game) *A cooperative game is an ordered pair* (N,w)*, where* $N = \{1,2,\ldots,n\}$ *is a set of players and* $w : 2^N \to \mathbb{IR}$ *is a characteristic function of the cooperative game. We further assume that* $w(\emptyset) = [0,0]$*.*

The set of all interval cooperative games on player set N *is denoted by* IG^N*.*

Note 2.14. *We often write $w(i)$ instead of $w(\{i\})$.*

Remark 2.15. *Each cooperative interval game in which the characteristic function maps to degenerate intervals only can be associated with some classical cooperative game. Converse holds as well.*

Definition 2.16. (Border games) *For every $(N,w) \in IG^N$, border games $(N,\underline{w}) \in G^N$ (lower border game) and $(N,\overline{w}) \in G^N$ (upper border game) are given by $\underline{w}(S) = \underline{w(S)}$ and $\overline{w}(S) = \overline{w(S)}$ for every $S \in 2^N$.*

Definition 2.17. (Length game) *The length game of $(N,w) \in IG^N$ is the game $(N,|w|) \in G^N$ with*

$$|w|(S) = \overline{w}(S) - \underline{w}(S),\ \forall S \in 2^N.$$

The basic notion of our approach will be a selection and consequently a selection imputation and a selection core.

Definition 2.18. (Selection) *A game $(N,v) \in G^N$ is a selection of $(N,w) \in IG^N$ if for every $S \in 2^N$ we have $v(S) \in w(S)$. Set of all selections of (N,w) is denoted by* Sel(w).

Note that border games are particular examples of selections.

Definition 2.19. (Interval selection imputation) *The set of interval selection imputations (or just selection imputations) of $(N,w) \in IG^N$ is defined as*

$$\mathcal{SI}(w) = \bigcup \big\{ I(v) \mid v \in \mathrm{Sel}(w) \big\}.$$

Definition 2.20. (Interval selection core) *Interval selection core (or just selection core) of $(N,w) \in IG^N$ is defined as*

$$\mathcal{SC}(w) = \bigcup \big\{ C(v) \mid v \in \mathrm{Sel}(w) \big\}.$$

Gök (Alparslan-Gök, 2009) choose an approach using a weakly better operator. That was inspired by (Puerto et al., 2008).

Definition 2.21. (Weakly better Operator $\succeq$) *Interval I is weakly better than interval J ($I \succeq J$) if $\underline{I} \geq \underline{J}$ and $\overline{I} \geq \overline{J}$. Furthermore, $I \preceq J$ if and only if $\underline{I} \leq \underline{J}$ and $\overline{I} \leq \overline{J}$. Interval I is better than J ($I \succ J$) if and only if $I \succeq J$ and $I \neq J$.*

Their definition of imputation and Core is as follows.

Definition 2.22. (Interval imputation) *The set of interval imputations of $(N,w) \in IG^N$ is defined as*

$$\mathcal{I}(w) := \Big\{ (I_1, I_2, \dots, I_N) \in \mathbb{IR}^N \mid \sum_{i \in N} I_i = w(N),\ I_i \succeq w(i),\ \forall i \in N \Big\}.$$

Definition 2.23. (Interval core) *An interval core of $(N,w) \in IG^N$ is defined as*

$$\mathcal{C}(w) := \Big\{ (I_1, I_2, \dots, I_N) \in \mathcal{I}(w) \mid \sum_{i \in S} I_i \succeq w(S),\ \forall S \in 2^N \setminus \{\emptyset\} \Big\}.$$

Important difference between definitions of interval and selection core and imputation is that selection concepts yield a payoff vectors from $\mathbb{R}^N$, while $\mathcal{I}$ and $\mathcal{C}$ yield vectors from $\mathbb{IR}^N$.

Note 2.24. (Notation) *Throughout the papers on cooperative interval games, notation, especially of core and imputations, is not unified. It is therefore possible to encounter different notation from ours.*

Also, in these papers, selection core is called core of interval game. We consider that confusing and that is why do we use term selection core instead. The term selection imputation is used because of its connection with selection core.

The following classes of interval games have been studied earlier (see e.g. (Alparslan-Gök et al., 2009b)).

Definition 2.25. (Size monotonicity) *A game $(N,w) \in IG^N$ is size monotonic if for every $T \subseteq S \subseteq N$ we have*

$$|w|(T) \leq |w|(S).$$

That is when its length game is monotonic.

The class of size monotonic games on player set N is denoted by SMIG^N.

As we can see, size monotonic games capture situations in which an interval uncertainty grows with the size of coalition.

Definition 2.26. (Superadditive interval game) *A game $(N,w) \in IG^N$ is superadditive interval game if for every $S,T \subseteq N$, $S \cap T = \emptyset$ holds*

$$w(T) + w(S) \preceq w(S \cup T),$$

and its length game is superadditive. We denote by SIG^N *class of superadditive interval games on player set N.*

We should be careful with the following analogy of convex game, since unlike for classical games, supermodularity is not the same as convexity.

Definition 2.27. (Supermodular interval game) *An interval game (N,v) is supermodular interval if for every $S \subseteq T \subseteq N$ holds*

$$v(T) + v(S) \preceq v(S \cup T) + v(S \cap T).$$

We get immediately that interval game is supermodular interval if and only if its border games are convex.

Definition 2.28. (Convex interval game) *An interval game (N,v) is convex interval if its border games and length game are convex.*

We write CIG^N *for a set of convex interval games on player set N.*

Convex interval game is supermodular as well but converse does not hold in general. See (Alparslan-Gök et al., 2009b) for characterizations of convex interval games and discussion on their properties.

3 SELECTION-BASED CLASSES OF INTERVAL GAMES

We will now introduce new classes of interval games based on the properties of their selections. We think that it is natural way to generalize special classes from classical cooperative game theory. Consequently, we show their characterizations and relation to classes from preceding subsection.

Definition 3.1. (Selection Monotonic Interval Game) *An interval game (N,v) is selection monotonic if all its selections are monotonic games. The class of such games on player set N is denoted by* SeMIG^N.

Definition 3.2. (Selection Superadditive Interval Game) *An interval game (N,v) is selection superadditive if all its selections are superadditive games. The class of such games on player set N is denoted by* SeSIG^N.

Definition 3.3. (Selection Convex Interval Game) *An interval game (N,v) is selection convex if all its selections are convex games. The class of such games on player set N is denoted by* SeCIG^N.

We see that many properties persist. For example, a selection convex game is a selection superadditive as well. Selection monotonic and selection superadditive are not subset of each other but their intersection is nonempty. Furthermore, selection core of selection convex game is nonempty, which is an easy observation.

We will now show characterizations of these three classes and consequently show their relations to existing classes presented in Subsection 2.3.

Proposition 3.4. *An interval game (N,w) is selection monotonic if and only if for every $S,T \in 2^N$, $S \subsetneq T$ holds*

$$\overline{w}(S) \leq \underline{w}(T).$$

Proof. For the "only if" part, suppose that (N,w) is a selection monotonic and $\overline{w}(S) > \underline{w}(T)$ for some $S,T \in 2^N$, $S \subsetneq T$. Then selection (N,v) with $v(S) = \overline{w}(S)$ and $v(T) = \underline{w}(T)$ clearly violates monotonicity and we arrive at a contradiction with assumptions.

Now for the "if" part. For any two subsets S,T of N, one of the situations $S \subsetneq T$, $T \subsetneq S$ or $S = T$ occur. For $S = T$, in every selection v, $v(S) \leq v(S)$ holds. As for the other two situations, it is obvious that monotonicity cannot be violated as well since $v(S) \leq \overline{w}(S) \leq \underline{w}(T) \leq v(T)$. □

Note 3.5. *Notice the importance of using $S \subsetneq T$ in the formulation of Proposition 3.4. It is because using of $S \subseteq T$ (thus allowing situation $S = T$) would imply $\overline{w}(S) \leq \underline{w}(S)$ for every S in selection monotonic game which is obviously not true in general. In characterizations of selection superadditive and selection convex games, similar situation arises.*

Proposition 3.6. *An interval game (N,w) is selection superadditive if and only if for every $S,T \in 2^N$ such that $S \cap T = \emptyset$, $S \neq \emptyset$, $T \neq \emptyset$ holds*

$$\overline{w}(S) + \overline{w}(T) \leq \underline{w}(S \cup T).$$

Proof. Similar to proof of Proposition 3.4. □

Proposition 3.7. *An interval game (N,w) is selection convex if and only if for every $S,T \in 2^N$ such that $S \not\subseteq T$, $T \not\subseteq S$, $S \neq \emptyset$, $T \neq \emptyset$ holds*

$$\overline{w}(S) + \overline{w}(T) \leq \underline{w}(S \cup T) + \underline{w}(S \cap T).$$

Proof. Similar to proof of Proposition 3.4. Note that we exclude cases $S \subseteq T$ and $S \subseteq T$ since $\overline{w}(S) + \overline{w}(T) \leq \underline{w}(S) + \underline{w}(T)$ is too restrictive. □

Now let us look on relation with existing classes of interval games.

For selection monotonic and size monotonic games, their relation is obvious. For nontrivial games (that is games with the player set size greater than one), a selection monotonic game is not necessarily size monotonic. Converse is the same.

Proposition 3.8. *For every player set N with $|N| > 1$, the following assertions hold.*

(i) $\mathrm{SeSIG}^N \not\subseteq \mathrm{SIG}^N$.
(ii) $\mathrm{SIG}^N \not\subseteq \mathrm{SeSIG}^N$.
(iii) $\mathrm{SeSIG}^N \cap \mathrm{SIG}^N \neq \emptyset$.

Proof. In *(i)*, we can construct the counterexample in the following way.

Let us construct game (N,w). For $w(\emptyset)$, interval is given. Now for any nonempty coalition, set $w(S) := [2|S|-2, 2|S|-1]$. For any $S,T \in 2^N$ with S and T being nonempty, the following holds with the fact that $|S| + |T| = |S \cup T|$ taken into account.

$$\begin{aligned}\overline{w}(S) + \overline{w}(T) &= (2|S|-1) + (2|T|-1)\\ &= 2|S \cup T| - 2\\ &= \underline{w}(S \cup T)\end{aligned}$$

So (N,w) is selection superadditive by Proposition 3.6. Its length game, however, is not superadditive, since for any two nonempty coalitions with the empty intersection $|w|(S)+|w|(T)=2\not\leq 1=|w|(S\cup T)$ holds.

In *(ii)*, we can construct the following counterexample (N,w'). Set $w'(S)=[0,|S|]$ for any nonempty S. The lower border game is trivially superadditive. For the upper game, $\overline{w'}(S)+\overline{w'}(T)=|S|+|T|=|S\cup T|=\overline{w'}(S\cup T)$ for any S,T with the empty intersection, so the upper game is superadditive. Observe that the length game is the same as the upper border game. This shows interval superadditivity.

However, (N,w') is clearly not selection superadditive because of nonzero upper bounds, zero lower bounds of nonempty coalitions and the characterization of SeSIG^N taken into account.

(iii) Nonempty intersection can be argumented easily by taking some superadditive game $(N,c)\in G^N$. Then we can define corresponding game $(N,d)\in IG^N$ with

$$d(S)=[c(S),c(S)],\ \forall S\in 2^N.$$

Game (N,d) is selection superadditive since its only selection is (N,c). And it is superadditive interval game since border games are supermodular and length game is $|w|(S)=0$ for every coalition, which trivially implies its superadditivity. $\square$

Proposition 3.9. *For every player set N with $|N|>1$, following assertions hold.*

(i) $\text{SeCIG}^N\not\subseteq\text{CIG}^N$.

(ii) $\text{CIG}^N\not\subseteq\text{SeCIG}^N$.

(iii) $\text{SeCIG}^N\cap\text{CIG}^N\neq\emptyset$.

Proof. For *(i)*, take a game (N,w) assigning to each nonempty coalition S interval $[2^{|S|}-2,2^{|S|}-1]$. From Proposition 3.7, we get that for inequalities which must hold in order to meet necessary conditions of game to be selection convex, $|S|<|S\cup T|$ and $|T|<|S\cup T|$ must hold. That gives the following inequality.

$$\begin{aligned}\overline{w}(S)+\overline{w}(T)&\leq(2^{|S\cup T|-1}-1)+(2^{|S\cup T|-1}-1)\\&=2^{|S\cup T|}-2\\&=\underline{w}(S\cup T)\\&\leq\underline{w}(S\cup T)+\underline{w}(S\cap T)\end{aligned}$$

This concludes that (N,w) is selection convex. We see that the border games and the length game are convex too. To have a game so that it is selection convex and not convex interval game, we can take (N,c) and set $c(S):=w(S)$ for $S\neq N$ and $v(N):=[\underline{w}(S),\underline{w}(S)]$. Now the game (N,c) is still selection convex, but its length game is not convex, so (N,v) is not convex interval game, which is what we wanted.

In *(ii)*, we can take a game (N,w') from the proof of Proposition 3.8*(ii)*. From the fact that $|S|+|T|=|S\cup T|+|S\cap T|$, it is clear that $\overline{w'}$ is convex. The lower border game is trivially convex and the length game is the same as upper. However, for nonempty $S,T\in 2^N$ such that $S\not\subseteq T$, $T\not\subseteq S$, $S\neq\emptyset$, $T\neq\emptyset$, convex selection games characterization is clearly violated.

As for *(iii)*, we can use the same steps as in *(iii)* of Proposition 3.8 or we can use a game (N,w) from *(i)* of this theorem. $\square$

4 CORE COINCIDENCE

In Gök's PhD thesis (Alparslan-Gök, 2009), the following topic is suggested: *"A difficult topic might be to analyze under which conditions the set of payoff vectors generated by the interval core of a cooperative interval game coincides with the core of the game in terms of selections of the interval game."*

We decided to examine this topic. We call it a *core coincidence problem.* This subsection shows our results.

Note 4.1. *We remind the reader that whenever we talk about relation and maximum, minimum, maximal, minimal vectors, we mean relation $\leq$ on real vectors unless we say otherwise.*

The main thing to notice is that while the interval core gives us a set of interval vectors, selection core gives us a set of real numbered vectors. To be able to compare them, we need to assign to a set of interval vectors a set of real vectors generated by these interval vectors. That is exactly what the following function gen does.

Definition 4.2. *The function* gen : $2^{\mathbb{IR}^N}\to 2^{\mathbb{R}^N}$ *maps to every set of interval vectors a set of real vectors. It is defined as*

$$\text{gen}(S)=\bigcup_{s\in S}\left\{(x_1,x_2,\ldots,x_n)\mid x_i\in s_i,\ s\in\mathbb{IR}^N\right\}.$$

Core coincidence problem can be formulated as this: What are the necessary and sufficient condition to satisfy $\text{gen}(\mathcal{C}(w))=\mathcal{SC}(w)$?

The main results of this subsection are the two following theorems which give an answer to the aforementioned question.

Note 4.3. *In the following text, by mixed system we mean a system of equalities and inequalities.*

Proposition 4.4. *For every interval game (M,w) we have* $\text{gen}(\mathcal{C}(w))\subseteq\mathcal{SC}(w)$.

Proof. For any $x'\in\text{gen}(\mathcal{C}(w))$, inequality $\underline{v}(N)\leq\sum_{i\in N}x'_i\leq\overline{v}(N)$ obviously holds. Furthermore, x' is in the core for any selection of the interval game (N,s) with s given by

$$s(S) = \begin{cases} \left[\sum_{i\in N} x_i', \sum_{i\in N} x_i'\right] & \text{if } S = N \\ \left[\underline{w}(S), \min(\sum_{i\in S} x_i', \overline{w}(S))\right] & \text{if otherwise.} \end{cases}$$

Clearly, $\mathrm{Sel}(s) \subseteq \mathrm{Sel}(w)$ and $\mathrm{Sel}(s) \neq \emptyset$. That concludes $\mathrm{gen}(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$. □

Theorem 4.5. (Core coincidence characterization) *For every interval game (N,w) we have $\mathrm{gen}(\mathcal{C}(w)) = \mathcal{SC}(w)$ if and only if for every $x \in \mathcal{SC}(w)$ there exist nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that*

$$\sum_{i\in N} (x_i - l_i^{(x)}) = \underline{w}(N), \quad (4.1)$$

$$\sum_{i\in N} (x_i + u_i^{(x)}) = \overline{w}(N), \quad (4.2)$$

$$\sum_{i\in S} (x_i - l_i^{(x)}) \geq \underline{w}(S),\ \forall S \in 2^N \setminus \{\emptyset\}, \quad (4.3)$$

$$\sum_{i\in S})x_i + u_i^{(x)}) \geq \overline{w}(S),\ \forall S \in 2^N \setminus \{\emptyset\}. \quad (4.4)$$

Proof. First, we observe that with Proposition 4.4 taken into account, we only need to take care of $\mathrm{gen}(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$ to obtain equality.

For $\mathrm{gen}(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$, suppose we have some $x \in \mathcal{SC}(w)$. For this vector, we need to find some interval $X \in \mathcal{C}(w)$ such that $x \in \mathrm{gen}(X)$. This is equivalent to the task of finding two nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that

$$([x_1 - l_1^{(x)}, x_1 + u_1^{(x)}]), [x_2 - l_2^{(x)}, x_2 + u_2^{(x)}], \ldots, [x_n - l_n^{(x)}, x_n + u_n^{(x)}]) \in \mathcal{C}(w).$$

From the definition of interval core, we can see that these two vectors have to satisfy exactly the mixed system $(4.1) - (4.4)$. That completes the proof. □

Example 4.6. *Consider an interval game with $N = \{1,2\}$ and $w(\{1\}) = w(\{2\}) = [1,3]$ and $w(N) = [1,4]$. Then vector $(2,2)$ lies in the core of the selection with $v(\{1\}) = v(\{2\}) = 2$ and $v(N) = 4$. However, to satisfy equation (4.1), we need to have $\sum_{i\in N} l_i = 3$ which means that either l_1 or l_2 has to be greater than 1. That means we cannot satisfy (4.3) and we conclude that $\mathrm{gen}(\mathcal{C}(w)) \neq \mathcal{SC}(w)$.*

The following theorem shows that it suffices to check only minimal and maximal vectors of $\mathcal{SC}(w)$.

Theorem 4.7. *For every interval game (N,w), if there exist vectors $q,r,x \in \mathbb{R}^N$ such that $q,r \in \mathrm{gen}(\mathcal{C}(w))$ and $q_i \leq x_i \leq r_i$ for every $i \in N$, then $x \in \mathrm{gen}(\mathcal{C}(w))$.*

Proof. Let $l^{(r)}, u^{(r)}, l^{(q)}, u^{(q)}$ be the corresponding vectors in sense of Theorem 4.5. We need to find vectors $l^{(x)}$ and $u^{(x)}$ satisfying $(4.1) - (4.4)$ of Theorem 4.5.

Let's define vectors $dq, dr \in \mathbb{R}^N$:

$$dq_i = x_i - q_i,$$

$$dr_i = r_i - x_i.$$

Finally, we define $l^{(x)}$ and $u^{(x)}$ in this way:

$$l_i^{(x)} = dq_i + l_i^{(q)},$$

$$u_i^{(x)} = dr_i + u_i^{(r)}.$$

We need to check that we satisfy $(4.1) - (4.4)$ for x, $l^{(x)}$ and $u^{(x)}$ We will show (4.2) only, since remaining ones can be done in a similar way.

$$\begin{aligned} \sum_{i\in N} (x_i - l_i^{(x)}) &= \sum_{i\in N} (x_i - dq_i - l_i^{(q)}) \\ &= \sum_{i\in N} (x_i - x_i + q_i - l_i^{(q)}) \\ &= \sum_{i\in N} (q_i - l_i^{(q)}) \\ &= \underline{w}(N). \end{aligned}$$

□

For games with additive border games (see Definition 2.8), we got a following result.

Theorem 4.8. *For an interval game (N,w) with additive border games, the payoff vector $(\underline{w}(1), \underline{w}(2), \ldots, \underline{w}(n)) \in \mathrm{gen}(\mathcal{C}(w))$.*

Proof. First, let us look on an arbitrary additive game (A, v_A). From additivity condition and the fact that we can write any subset of A as a union of one-player sets we conclude that $v_A(A) = \bigcup_{i\in A} v_A(\{i\})$ for every coalition A. This implies that vector a with $a_i = v_A(\{i\})$ is in the core.

This argument can be applied to border games of (N,w). The vector $q \in \mathbb{R}^N$ with $q_i = \underline{w}(i)$ is an element of the core of $(N, \underline{w})$ and an element of $\mathcal{SC}(w)$.

For the vector q we want to satisfy the mixed system (4.1)-(4.4) of Theorem 4.5.

Take the vector l containing zeros only and the vector u with $u_i = |w|(i)$. From the additivity, we get that $\sum_{i\in N} q_i - l_i = \underline{w}(N)$ and $\sum_{i\in N} q_i + u_i = \overline{w}(N)$.

Additivity further implies that inequalities (4.3) and (4.4) hold for q, l and u. Therefore, q is an element of $\mathrm{gen}(\mathcal{C}(w))$. □

Theorem 4.8 implies that for games with additive border games, we need to check the existence of vectors l and u from $(4.1) - (4.4)$ of Theorem 4.5 for maximal vectors of $\mathcal{SC}$ only. That follows from the fact that for any vector $y \in \mathcal{SC}(w)$ holds $(\underline{w}(1), \underline{w}(2), \ldots, \underline{w}(n)) \leq y$. In other words, $(\underline{w}(1), \underline{w}(2), \ldots, \underline{w}(n))$ is a minimum vector of $\mathcal{SC}(w)$.

5 STRONG IMPUTATION AND CORE

In this subsection, our focus will be on a new concept of *strong imputation* and *strong core*.

Definition 5.1. (Strong imputation) *For a game* $(N,w) \in IG^N$ *a strong imputation is a vector* $x \in \mathbb{R}^N$ *such that* x *is an imputation for every selection of* (N,w).

Definition 5.2. (Strong core) *For a game* $(N,w) \in IG^N$ *the strong core is the union of vectors* $x \in \mathbb{R}^N$ *such that* x *is an element of core of every selection of* (N,w).

Strong imputation and strong core can be considered as somewhat "universal" solutions. We show the following three simple facts about the strong core.

Proposition 5.3. *For every interval game with nonempty strong core,* $w(N)$ *is a degenerate interval.*

Proof. Easily by the fact that an element c of strong core must be efficient for every selection and therefore $\sum_{i \in N} c_i = \underline{w}(N) = \overline{w}(N)$. □

This leads us to characterizing games with nonempty strong core.

Proposition 5.4. (Strong core characterization) *An interval game* (N,w) *has a nonempty strong core if and only if* $w(N)$ *is a degenerate interval and the upper game* $\overline{w}$ *has a nonempty core.*

Proof. Combination of Proposition 5.3 and the fact that an element c of the strong core has to satisfy $\sum_{i \in S} c_i \geq v(S), \ \forall v \in \mathrm{Sel}(w), \ \forall S \in 2^N \setminus \emptyset$. We see that this fact is equivalent to condition $\sum_{i \in S} c_i \geq \overline{w}(S), \ \forall S \in 2^N \setminus \emptyset$. Proving an equivalence is then straightforward. □

Strong core also has the following important property.

Proposition 5.5. *For every element* c *of the strong core of* (N,w)*,* $c \in \mathrm{gen}(\mathcal{C}(w))$.

Proof. The vector c has to satisfy mixed system (4.1)-(4.4) of Theorem 4.5 for some $l,u \in \mathbb{IR}^N$. We show that $l_i = u_i = 0$ will do the thing.

Equations (4.1) and (4.2) are satisfied by taking Proposition 5.3 into account. Inequalities (4.3) and (4.4) are satisfied as the consequence of Proposition 5.4. □

Note 5.6. *The reason behind the using of name strong core and strong imputation comes form the interval linear algebra, where strong solutions of interval system are a solutions for any realization (selection) of interval matrices* A *and* b *in* $Ax = b$.

Note 5.7. *One could say why we did not introduce a strong game as game in which each of its selection does have an nonempty core. This is because such games are already defined as* strongly balanced games *(see e.g. (Alparslan-Gök et al., 2008)).*

6 CONCLUDING REMARKS

Selections of an interval game are very important, since they do not contain any additional uncertainty. On the top of that, selection-based classes and strong core and imputation have crucial property that although we deal with uncertain data, all possible outcomes preserve important properties. In case of selection classes it is preserving superadditivity, supermodularity etc. In case of strong core it is an invariant of having particular stable payoffs in each selection. Furthermore, "weak" concepts like $\mathcal{SC}$ are important as well since if $\mathcal{SC}$ is empty, no selection has stable payoff.

The importance of studying selection-based classes instead of the existing classes using $\succeq$ operator can be further illustrated by the following two facts:

- Classes based on weakly better operator may contain games with selections that do not have any link with the properties of their border games and consequently no link with the name of the class. For example, superadditive interval games may contain a selection that is not superadditive.
- Selection-based classes are not contained in corresponding classes based on weakly better operator. Therefore, the results on existing classes are not directly extendable to selection-based classes.

Our results provide an important tool for handling cooperative situations involving interval uncertainty which is a very common situation in various OR problems. Some of the specific applications of interval games were already examined. See (Alparslan-Gök et al., 2014) (Alparslan-Gök et al., 2009a) (Alparslan-Gök et al., 2013) for applications to mountain situations, airport games and forest situations, respectively. However, these papers do not use a selection-based approach and therefore to study implications of our approach to them can be an interesting theme for another research.

To further study properties of selection-based classes is a possible topic. Fruitful direction can be an extension of the definition of stable set to interval games in a selection way. For example, one could examine a union or an intersection of sets of stable sets for each selection. Studying nucleolus and other concepts in interval games context may be interesting as well.

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