

Horizontal Stereoscopic Display based on Homologous Points

Bruno Eduardo Madeira^{1,3}, Carlos Frederico de Sá Volotão², Paulo Fernando Ferreira Rosa¹
and Luiz Velho³

¹*Departement of Computer Engineering, Instituto Militar de Engenharia, Rio de Janeiro, Brazil*

²*Departement of Cartographic Engineering, Instituto Militar de Engenharia, Rio de Janeiro, Brazil*

³*Visgraf Laboratory, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brazil*

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Abstract: In this paper we establish the relation between camera calibration and the generation of horizontal stereoscopic images. After that, we introduce a new method that handles the problem of generating stereoscopic pairs without using calibration patterns, instead we use the correspondence of homologous points. The method is based on the optimization of a measure that we call Three-dimensional Interpretability Error, which has a simple geometric interpretation. We also prove that this optimization problem has four global minima, one of which corresponds to the desired solution. After that, we present techniques to initialize the problem avoiding the convergence to a wrong global minimum. Finally, we present some experimental results.

1 INTRODUCTION

The stereoscopic technology is getting more and more common nowadays, as a consequence this kind of technology is becoming cheaper and widely accessible to people in general, (de la Riviere, 2010; Yoshiki Takeoka, 2010).

Most stereoscopic applications use simple adaptations of non-stereoscopic concepts in order to give the observer the sense of depth. This is true, for example, in the case of 3D movies where two versions are usually released, one to be watched in a stereoscopic movie theater and other to be watched in a normal theater.

We are exploring the use of stereoscopic technology changing the usual paradigm that tries to give the observer the “Sense of Depth” to the new paradigm that gives the observer the “Sense of Reality”. We call Sense of Reality when besides giving a sense of depth to the image, the setting is presented in such a way that it is compatible with real objects in the real world. Normal 3D movies do not implement the “Sense of Reality” because of the following reasons:

- The screen is limited, thus, points in the border can be shown without the stereo correspondence. It is not a problem if the whole scene is “inside” the screen, but it is a problem if the scene is over the screen.

- The objects presented in a movie are usually floating in space, because the scene is not grounded to the real world floor.
- Many scenes usually present a very large range of depth, which cannot be exhibited by the current stereoscopic technology.
- The zoom parameter of the camera is usually chosen in order to capture the scene in the same way as a regular movie, which in consequence magnifies portions of the scene.

The above aspects make it difficult for the observer to believe that the content, although presented in 3D, is actually real. To be physically plausible the content presented in the screen must make sense when viewed as part of the environment that surrounds it. This goal can be achieved by making four changes to the stereoscopic system:

- **Presenting the 3D Stereo Content on an Horizontal Support Leveling the Floor with the Screen.**
It establishes a link between virtual objects and the screen. This link makes the result more reliable compared to the exhibition of virtual objects flying in front of a vertical screen.
- **Not Presenting a Scene Whose Projected Points in the Border of the Screen are Closer to the Observer than the Screen.**

If a 3D point on the left or right border of the screen is closer to the observer than the screen, then one of its correspondent stereoscopic projections will not be exhibited due to the screen limitation. That means that it will generate a stereoscopic pair that does not correspond to a 3D scene. If the stereoscopic projections of an object cross the top border, but do not cross the laterals, then the scene will not be well accepted by the observer either, although the stereoscopic pair corresponds to a 3D scene. In this case, the problem is that the border limitation corresponds to a 3D cut in the object, that makes the top of the projection be perfectly aligned with the top border of the screen. Besides the fact that the 3D cut makes the scene odd, there is the fact that the alignment between the border and the cut implies that the observer had to be placed in a very specific position in order to be able to see it, it means that the stereoscopic projections are images that do not satisfy the generic-viewpoint assumption (Marr, 1982), that can cause interpretation problems, such as presented in Figure 1-b. Finally, if the stereoscopic projections cross the bottom border, then they will suffer from the same problems as those that cross the top border, plus the fact that they will correspond to floating objects.

- **Constraining the Scale of the Scene Based on Some Physical Reference.**

It can be achieved by changing the cinematography technique. For example, 3D stereo movies adopt the classic film language used for 2D films. As a consequence, it employs different framing techniques, such as close-ups, medium and long shots that cause the objects in a scene to change size relative to the screen. This practice impairs the sense of reality with the physical world. Such problem is avoided by establishing a fixed scaled correspondence between the displayed scene and the real environment.

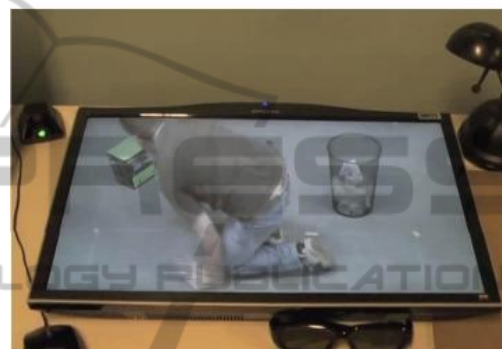
- **Restricting the Field of View to Encompass the Objects in the Scene.**

In standard 3D stereo movies, the fact that the cameras are positioned parallel to the ground implies in a wide range of depth, including elements far from the center of interest of the scene. Conversely, in stereoscopic images produced for display over a table the camera will be oriented at an oblique angle in relation to the ground, which limits the maximum depth of the scene and favors the use of stereoscopic techniques.

In short, in order to produce the “Sense of Reality” it is necessary to use a stereoscopic display disposed



(a)



(b)

Figure 1: (a) An image that satisfies all requirements to produce the “Sense of Reality”. (b) An image whose border produces a cut in the scene that affect the tridimensional interpretation.

in a horizontal position, taking care with the scene setup. For example, Figure 1-a illustrates a case in which all the requirements to produce the “Sense of Reality” are satisfied.

The idea of generating stereoscopic images for being displayed in an horizontal surface is not new. Many devices that use Computer Graphics for generating horizontal stereoscopic images have already appeared in the scientific literature, such as presented in (Cutler et al., 1997), (Leibe et al., 2000), (Ericsson and Olwal, 2011) and (Hoberman et al., 2012). On the other hand, the case of horizontal stereoscopic images generated by Image Processing is a subject not much discussed. Our bibliographic research shows that it has firstly appeared in the patent literature in (Wester, 2002) and (Aubrey, 2003) in a not very formal treatment. As far as we know, the first scientific paper that handled this problem formally is (Madeira and Velho, 2012). This paper shows that the generation of horizontal stereoscopic images can be interpreted as a problem of estimation and application of homographies, and it proposes the use of Computer Vision techniques to estimate them. It uses the establishment

of 3D-2D correspondences between 3D points of a calibration pattern and their respective 2D projections over images.

We have not found any reference about a method for generating horizontal stereoscopic pairs without using calibration patterns, so we decided to attack this problem just using homologous points, giving more flexibility to the user. We solved it minimizing a measure that we call by Three-dimensional Interpretability Error, which has a simple geometric interpretation. We also prove that this optimization problem has four global minima, one of which corresponds to the desired solution. After that, we present techniques to initialize the optimization process avoiding the convergence to a wrong global minimum.

We have tested the method and achieved good results.

2 HOMOGRAPHIES AND CAMERA PARAMETERS

Lets consider the problem of generating a stereoscopic pair designed for being presented over an horizontal surface.

Suppose that there is a camera in an oblique position capturing the projection p1 of an object over an horizontal surface (Figure 2). We need to find a way

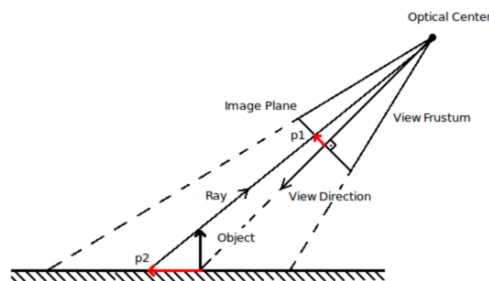


Figure 2: The rays emitted by the object and passing through the eye have the same color as the correspondent point in the horizontal surface.

to compute the projection p2, that corresponds to the projection of the object using the same optical center as the one used for capturing p1 but replacing the projective plane by the horizontal surface. It makes the rays emitted by the object and passing through the eye have the same color as the correspondent point in the horizontal surface, thus the eye sees the same image whether it came from the real object or from p2.

It is easy to notice, by examining Figure 3, that if a set of points in a scene is projected by a cam-

era over a set of collinear projections, then they keep collinear if we maintain the optical center in the same place and change the position of the projection plane. It happens because the rays whose intersection generate these projections must be coplanar, and if the optical center is unchanged they still have to be used for defining the projections over the plane in the new position. Since the rays are coplanar, the intersection of them with any plane must be collinear.

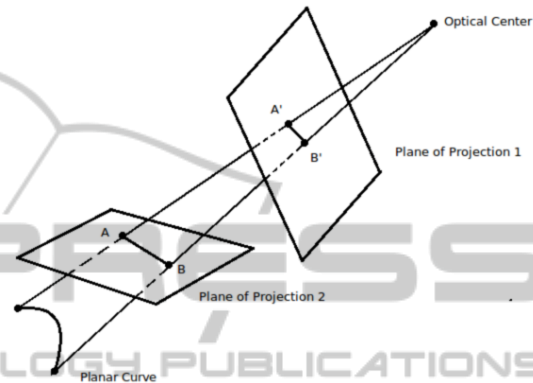


Figure 3: This example shows a curve whose projection over a projection plane is collinear. As a consequence, it is also collinear if we change the projection plane and keep the optical center unchanged.

This result implies that there is a homography relating the coordinates of projections, measured over the images captured by the cameras pointed to the object to be captured, and the coordinates of the projections, made by using the same optical center as center of projection and using the planar support as projection plane. It means that, the projections p1 and p2, presented in Figure 2, are related by a homography.

Lets suppose that there is a 3D coordinate system located on the horizontal plane, such that the x-axis and y-axis are on the plane. Lets consider that the camera used for capturing the image of the object over the plane is defined in this coordinate system by a projective transform $T : \mathbb{R}P^3 \rightarrow \mathbb{R}P^2$ given by

$$T = K \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{pmatrix},$$

where K is the matrix of intrinsic parameters.

We can establish a homography H between the horizontal plane and the image plane by restricting the domain of T to the xy-plane. More specifically,

$$H = K \begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{pmatrix}.$$

Each possible choice of x-axis and y-axis on the horizontal plane will define a different homography. The choices that are adequate for generating the stereoscopic effect are the ones that will make homologous points have the same y-coordinate. It happens when the x-axis is parallel to the line passing through the eyes of the observer and the y-axis is orthogonal (Figure 4).

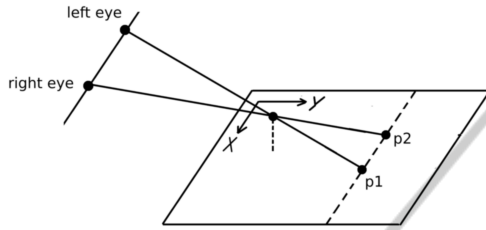


Figure 4: An adequate choice of x and y axis. The x-axis is parallel to the line passing through the eyes of the observer, and the homologous points p1 and p2 have the same y-coordinate.

This link between the camera model and homographies shows that the problem of finding the homographies appropriated for generating horizontal stereoscopic pairs can be solved by calibrating the camera using an adequate coordinate system over the horizontal plane.

3 THE THREE-DIMENSIONAL INTERPRETABILITY ERROR

It is easy to notice that any stereoscopic pair prepared for being observed in a horizontal position must satisfy the following constraints.

1. homologous points that are leveled to the horizontal surface must be coincident.
2. homologous points that are not leveled to the horizontal surface must have the same y-coordinate.

In order to fix notation, we assume that the homologous pairs of points that correspond to 3D points leveled to the horizontal surface are points of Type I, and the ones that are not leveled are points of Type II. And we also assume that the 3D point is classified in the same group as its respective homologous pair. The Figure 5 illustrates the constraints related to each type.

We define the Three-dimensional Interpretability Error as a measure of how the constraints related to points of Types I and II are being satisfied. More precisely, let's suppose that $\{(\mathbf{p}_1, \hat{\mathbf{p}}_1), \dots, (\mathbf{p}_n, \hat{\mathbf{p}}_n)\}$ is a set of homologous pairs of Type I and

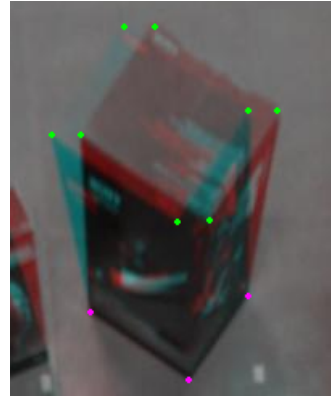


Figure 5: An anaglyph of a cube prepared for being presented in horizontal position. The green dots are homologous pairs of Type II. Each pair is formed by points with the same y-coordinate. The pink dots are coincident homologous points, they are homologous pairs of Type I.

$\{(\mathbf{q}_1, \hat{\mathbf{q}}_1), \dots, (\mathbf{q}_m, \hat{\mathbf{q}}_m)\}$ is a set of homologous pairs of Type II. We define the Three-dimensional Interpretability Error as

$$\alpha \sum_{i=1}^n \|\mathbf{p}_i - \hat{\mathbf{p}}_i\|^2 + \beta \sum_{j=1}^m (\mathbf{q}_j - \hat{\mathbf{q}}_j)_y^2,$$

where $\alpha \in \mathbb{R}$ defines the importance of the constraints of Type I, and $\beta \in \mathbb{R}$ defines the importance of the constraints of Type II.

4 THE IMPORTANCE OF THE INTRINSIC PARAMETERS

It is obvious that any stereoscopic pair presented horizontally must have the Three-dimensional Interpretability Error equals to zero, for any considered set of homologous point. A non obvious question is:

If we find homographies that make a pair of captured images have the Three-dimensional Interpretability Error equals to zero, can we assume that the generated stereoscopic pair represents the 3D scene correctly ?

The answer is: No. If H_1 and H_2 produce a stereoscopic pair with Three-dimensional Interpretability Error equal to zero, then MH_1 and MH_2 also generate a stereoscopic pair with Three-dimensional Interpretability Error equal to zero, if

$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\alpha_1 \in \mathbb{R} - \{0\}$, $\alpha_2 \in \mathbb{R} - \{0\}$ and $\alpha_1 \neq \alpha_2$.

As presented in Section 2, the correct estimation of homographies corresponds to the correct calibration of the camera pair, but the possibility of assigning α_1 and α_2 to different values means that the intrinsic parameters of the camera are not well defined, such as presented in the Figure 6. It means that we cannot calibrate the intrinsic parameters by minimizing the Three-dimensional Interpretability Error.

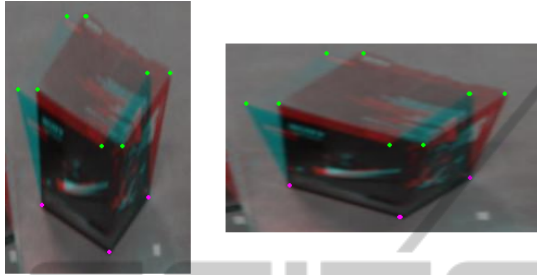


Figure 6: Both pictures present stereoscopic pairs whose Three-dimensional Interpretability Error is zero. The difference between them is the intrinsic parameters of the cameras related to each applied homography.

5 FOUR GLOBAL MINIMA

In the previous section we concluded that it is not sufficient to find the pair of homographies that makes the Three-dimensional Interpretability Error equal to zero in order to generate the correct horizontal stereoscopic pair, because we can find different results related to different choices of intrinsic parameters.

Now we present a Theorem that shows that if we fix the correct intrinsic parameters, there are just 4 different pairs of homographies that make the Three-dimensional Interpretability Error equal to zero. It is important because it means that, if we know the intrinsic parameters, and if we choose a parametrization for the cameras that fix them, we can estimate the homographies minimizing the Three-dimensional Interpretability Error. We just have to initiate the optimization process sufficiently close to the correct minimum in order to avoid the convergence to one of the 3 incorrect solutions.

Theorem 1. *Lets suppose that at least 4 pairs of homologous points of Type I and 2 pairs of homologous points of Type II are known. If there is a pair of homographies H_1 and H_2 that makes the points of Type I be coincident and that makes the points of Type II have the same y-coordinate then, keeping the intrinsic parameters unchanged, there are exactly other 3 pairs of homographies, defined with an ambiguity of translation, that also do it. Moreover, these pairs have the*

form WH_1 and WH_2 , where W can be:

$$\begin{pmatrix} 1 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof

H_1 and H_2 are homographies that make the homologous points satisfy their constraints. Lets suppose that W_1H_1 and W_2H_2 also do this.

Lets suppose that $\{(\mathbf{x}_1, \hat{\mathbf{x}}_1), \dots, (\mathbf{x}_4, \hat{\mathbf{x}}_4)\}$ are 4 pairs of homologous points of Type I. Since H_1 e H_2 make the homologous points of Type I be coincident we have that, $\exists \mathbf{y}_1, \dots, \mathbf{y}_4 \in \mathbb{R}P^2$ such that:

$$H_1\mathbf{x}_i = H_2\hat{\mathbf{x}}_i = \mathbf{y}_i, \text{ where } i \in \{1, 2, 3, 4\}. \quad (1)$$

W_1H_1 and W_2H_2 also make the homologous points of Type I be coincident thus:

$$W_1\mathbf{y}_i = W_2\mathbf{y}_i, \text{ where } i \in \{1, 2, 3, 4\}. \quad (2)$$

Since the range of 4 points by W_1 and W_2 are equals we have, by the Fundamental Theorem of Projective Geometry, that $W_1 = W_2$.

In order to fix the notation, lets define the homography W by:

$$W = W_1 = W_2. \quad (3)$$

Lets $(\mathbf{x}_5, \hat{\mathbf{x}}_5)$ and $(\mathbf{x}_6, \hat{\mathbf{x}}_6)$ be two pairs of homologous points of Type II.

Because H_1 and H_2 map homologous points of Type II over points with the same y-coordinate we have that $\{H_1\mathbf{x}_5, H_1\mathbf{x}_6, H_2\hat{\mathbf{x}}_5, H_2\hat{\mathbf{x}}_6\}$ define the vertices of a trapezium with two sides parallel to the x-axis, as shown in Figure 7.

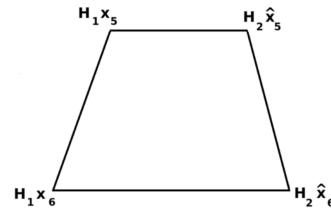


Figure 7: Trapezium whose sides defined by the vertices $H_1\mathbf{x}_5$ and $H_2\hat{\mathbf{x}}_5$ and the side defined by the vertices $H_1\mathbf{x}_6$ and $H_2\hat{\mathbf{x}}_6$ are parallel to the x-axis.

Since WH_1 and WH_2 also map homologous points of Type II over points with the same y-coordinate, it is necessary that W maps the trapezium over other trapezium with sides parallel to the x-axis. It means that, $\exists \lambda \in \mathbb{R}$ such that

$$W(1, 0, 0)^T = (\lambda, 0, 0)^T. \quad (4)$$

Therefore, W has the form

$$W = \begin{pmatrix} \lambda & a & d \\ 0 & b & e \\ 0 & c & f \end{pmatrix}. \quad (5)$$

WH_1 and WH_2 must be homographies that correspond to cameras whose intrinsic parameters are the same as the one related to the homographies H_1 and H_2 . In other words, lets consider the vectors $\mathbf{t}, \mathbf{t}' \in \mathbb{R}^3$ and the rotation matrices $R = (\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3)$ and $R' = (\mathbf{r}'_1 \mathbf{r}'_2 \mathbf{r}'_3)$ such that

$$H_1^{-1} = K(\mathbf{r}_1 \mathbf{r}_2 \mathbf{t}) \quad (6)$$

and

$$H_2^{-1} = K(\mathbf{r}'_1 \mathbf{r}'_2 \mathbf{t}'), \quad (7)$$

it must exist vectors $\hat{\mathbf{t}}, \hat{\mathbf{t}}' \in \mathbb{R}^3$ and rotation matrices $\hat{R} = (\hat{\mathbf{r}}_1 \hat{\mathbf{r}}_2 \hat{\mathbf{r}}_3)$ and $\hat{R}' = (\hat{\mathbf{r}}'_1 \hat{\mathbf{r}}'_2 \hat{\mathbf{r}}'_3)$ such that

$$H_1^{-1} W^{-1} = K(\hat{\mathbf{r}}_1 \hat{\mathbf{r}}_2 \hat{\mathbf{t}}) \quad (8)$$

and

$$H_2^{-1} W^{-1} = K(\hat{\mathbf{r}}'_1 \hat{\mathbf{r}}'_2 \hat{\mathbf{t}}'). \quad (9)$$

Thus, we have

$$(\mathbf{r}_1 \mathbf{r}_2 \mathbf{t}) = (\hat{\mathbf{r}}_1 \hat{\mathbf{r}}_2 \hat{\mathbf{t}}) \begin{pmatrix} \lambda & a & d \\ 0 & b & e \\ 0 & c & f \end{pmatrix} \quad (10)$$

and

$$(\mathbf{r}'_1 \mathbf{r}'_2 \mathbf{t}') = (\hat{\mathbf{r}}'_1 \hat{\mathbf{r}}'_2 \hat{\mathbf{t}}') \begin{pmatrix} \lambda & a & d \\ 0 & b & e \\ 0 & c & f \end{pmatrix}. \quad (11)$$

From the equations 10 and 11 we have

$$\mathbf{r}_1 = \lambda \hat{\mathbf{r}}_1 \quad (12)$$

and

$$\mathbf{r}'_1 = \lambda \hat{\mathbf{r}}'_1, \quad (13)$$

from which we conclude that $\lambda = 1$ or $\lambda = -1$.

Lets consider the case $\lambda = 1$. The case $\lambda = -1$ can be analyzed analogously. In this case we have that

$$\mathbf{r}_1 = \hat{\mathbf{r}}_1 \quad (14)$$

and

$$\mathbf{r}'_1 = \hat{\mathbf{r}}'_1. \quad (15)$$

The vector $\hat{\mathbf{r}}_2$ is orthogonal to $\hat{\mathbf{r}}_1$, as a consequence, from the equation 14 we have that it is also orthogonal to the vector \mathbf{r}_1 . That means, $\exists m_1, m_2 \in \mathbb{R}$ such that

$$\hat{\mathbf{r}}_2 = m_1 \mathbf{r}_2 + m_2 \mathbf{r}_3. \quad (16)$$

Analogously, $\exists m'_1, m'_2 \in \mathbb{R}$ such that

$$\hat{\mathbf{r}}'_2 = m'_1 \mathbf{r}'_2 + m'_2 \mathbf{r}'_3. \quad (17)$$

We have that $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a base to \mathbb{R}^3 , as well as $\{\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3\}$. So $\exists k_1, k_2, k_3, k'_1, k'_2, k'_3 \in \mathbb{R}$ such that:

$$\hat{\mathbf{t}} = k_1 \mathbf{r}_1 + k_2 \mathbf{r}_2 + k_3 \mathbf{r}_3 \quad (18)$$

and

$$\hat{\mathbf{t}}' = k'_1 \mathbf{r}'_1 + k'_2 \mathbf{r}'_2 + k'_3 \mathbf{r}'_3. \quad (19)$$

From the equations 10 and 11 we have

$$\mathbf{r}_2 = a \mathbf{r}_1 + b(m_1 \mathbf{r}_2 + m_2 \mathbf{r}_3) + c(k_1 \mathbf{r}_1 + k_2 \mathbf{r}_2 + k_3 \mathbf{r}_3) \quad (20)$$

and

$$\mathbf{r}'_2 = a \mathbf{r}'_1 + b(m'_1 \mathbf{r}'_2 + m'_2 \mathbf{r}'_3) + c(k'_1 \mathbf{r}'_1 + k'_2 \mathbf{r}'_2 + k'_3 \mathbf{r}'_3). \quad (21)$$

As a consequence, we have that

$$a + ck_1 = 0, \quad (22)$$

$$a + ck'_1 = 0, \quad (23)$$

$$bm_1 + ck_2 = 1, \quad (24)$$

$$bm'_1 + ck'_2 = 1, \quad (25)$$

$$bm_2 + ck_3 = 0 \quad (26)$$

$$bm'_2 + ck'_3 = 0. \quad (27)$$

Now we will show that $m_2 = 0$ and $m'_2 = 0$.

Lets suppose, by contradiction, that $m_2 \neq 0$. From the equations 22 and 23 we conclude that

$$ck_1 = ck'_1, \quad (28)$$

thus $c = 0$ or $k_1 = k'_1$.

If $c = 0$, we have from the equation 26 that $b = 0$, which contradicts the equation 24, which states that $bm_1 = 1$.

If $k_1 = k'_1$ then

$$\langle \hat{\mathbf{t}}, \mathbf{r}_1 \rangle = k_1 = k'_1 = \langle \hat{\mathbf{t}}', \mathbf{r}'_1 \rangle. \quad (29)$$

Lets assume that $\mathbf{c}_1 \in \mathbb{R}^3$ is the optical center of the camera related to the homography WH_1 , and $\mathbf{c}_2 \in \mathbb{R}^3$ is the optical center related to the homography WH_2 . We have that

$$\begin{aligned} \langle \hat{\mathbf{t}}, \mathbf{r}_1 \rangle &= \langle \hat{\mathbf{t}}, \hat{\mathbf{r}}_1 \rangle = \langle -\hat{R} \mathbf{c}_1, \hat{\mathbf{r}}_1 \rangle = \langle -\mathbf{c}_1, \hat{R}^T \hat{\mathbf{r}}_1 \rangle \\ &= \langle -\mathbf{c}_1, (1, 0, 0)^T \rangle = (-\mathbf{c}_1)_x. \end{aligned} \quad (30)$$

We can rewrite the equation 29 as

$$(\mathbf{c}_1)_x = (\mathbf{c}_2)_x. \quad (31)$$

Since two points of Type II are mapped over points with the same y-coordinate, it is necessary that $(\mathbf{c}_1)_y = (\mathbf{c}_2)_y$ and $(\mathbf{c}_1)_z = (\mathbf{c}_2)_z$, then we conclude that

$$\mathbf{c}_1 = \mathbf{c}_2. \quad (32)$$

Because the optical centers are equal, it follows that the images of the stereoscopic pair captured by the cameras must be related by a homography, which is a contradiction with the fact that the images have been captured by cameras whose optical centers were located in different places. It means that $m_2 = 0$. A similar reasoning can be used to show that $m'_2 = 0$.

From the equations 16 and 17 we conclude that

$$\hat{\mathbf{r}}_2 = \mathbf{r}_2 \text{ or } \hat{\mathbf{r}}_2 = -\mathbf{r}_2 \quad (33)$$

and

$$\hat{\mathbf{r}}'_2 = \mathbf{r}'_2 \text{ or } \hat{\mathbf{r}}'_2 = -\mathbf{r}'_2. \quad (34)$$

From the equations 12 and 13 we have, because λ can be assigned to 1 or -1 , that

$$\hat{\mathbf{r}}_1 = \mathbf{r}_1 \text{ or } \hat{\mathbf{r}}_1 = -\mathbf{r}_1 \quad (35)$$

and

$$\hat{\mathbf{r}}'_1 = \mathbf{r}'_1 \text{ or } \hat{\mathbf{r}}'_1 = -\mathbf{r}'_1. \quad (36)$$

So W must have one of the following formats:

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} -1 & 0 & * \\ 0 & -1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, it is clear that, if H_1 and H_2 satisfy the constraints of point of Type I and II, then all these four options for W make WH_1 and WH_2 also do it. Thus, it is necessary and sufficient that W takes one of these four formats.

6 THE LEAST SQUARE PROBLEM

Lets assume that the matrix of intrinsic parameters K is known. In this section we define a least square problem for finding the extrinsic parameters that minimize the Three-dimensional Intertepretability Error.

Lets suppose that two images I_1 and I_2 are captured by a pair of cameras, and $\{(\mathbf{u}_1, \mathbf{v}_1), \dots, (\mathbf{u}_n, \mathbf{v}_n)\}$ is a set of points of Type I and $\{(\mathbf{u}_{n+1}, \mathbf{v}_{n+1}), (\mathbf{u}_{n+2}, \mathbf{v}_{n+2}), \dots, (\mathbf{u}_m, \mathbf{v}_m)\}$ is a set of points of Type II, such that $\mathbf{u}_i \in I_1$ and $\mathbf{v}_i \in I_2$.

Lets assume that the extrinsic parameters of the camera used for capturing I_1 is R_1 and \mathbf{t}_1 and for capturing I_2 is R_2 and \mathbf{t}_2 where

$$R_1 = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix},$$

$$\mathbf{t}_1 = (\beta_1, \beta_2, \beta_3)^T,$$

$$R_2 = \begin{pmatrix} r'_{11} & r'_{12} & r'_{13} \\ r'_{21} & r'_{22} & r'_{23} \\ r'_{31} & r'_{32} & r'_{33} \end{pmatrix}$$

and

$$\mathbf{t}_2 = (\delta_1, \delta_2, \delta_3)^T.$$

We define the objective function by

$$\alpha \sum_{i=1}^n \|W_1 \mathbf{u}_i - W_2 \mathbf{v}_i\|^2 + \beta \sum_{i=n+1}^m (W_1 \mathbf{u}_i - W_2 \mathbf{v}_i)_y^2$$

where

$$W_1^{-1} = K \begin{pmatrix} r_{11} & r_{12} & \beta_1 \\ r_{21} & r_{22} & \beta_2 \\ r_{31} & r_{32} & \beta_3 \end{pmatrix}$$

and

$$W_2^{-1} = K \begin{pmatrix} r'_{11} & r'_{12} & \delta_1 \\ r'_{21} & r'_{22} & \delta_2 \\ r'_{31} & r'_{32} & \delta_3 \end{pmatrix}.$$

We can solve this problem using the Levenberg-Marquardt algorithm. We find the extrinsic parameters R_1 , R_2 and \mathbf{t}_2 fixing the vector \mathbf{t}_1 . If we did not fix \mathbf{t}_1 neither \mathbf{t}_2 then the value of the objective function would reduce to zero when $(\mathbf{t}_1)_z \rightarrow \infty$ and $(\mathbf{t}_2)_z \rightarrow \infty$. Besides that, by fixing \mathbf{t}_1 we do not reduce the generality of the solution, because we just define the position and the scale of one image of the stereoscopic pair.

We highlight that an appropriate parametrization of the space of rotations is required for solving this problem. In our experiments we chose a parametrization based on an axis-angle representation.

7 FINDING THE INITIAL PARAMETERS

We can find the initial extrinsic parameters for the least square problem defined in the previous section using the following process:

1. Take two samples R_1 and R_2 from the space of rotations.
2. Use R_1 and R_2 and two pairs of homologous points to find the translations \mathbf{t}_1 and \mathbf{t}_2 .

We repeat this process with different choices for R_1 and R_2 , and we select the extrinsic parameters $\{R_1, R_2, \mathbf{t}_1, \mathbf{t}_2\}$ that make the Three-dimensional Intertepretability Error have the minimum value. This process explores the fact that the space of rotation is limited, thus it can be sampled.

We must avoid getting samples R_1 and R_2 that are too far from the expected correct solution. We must keep in mind that, by the Theorem 1, there are 3 wrong pairs of rotations that also minimize the Three-dimensional Intertepretability Error. Fortunately the wrong rotations are far away from the correct ones (180 degrees).

In the next section we explain how to find \mathbf{t}_1 and \mathbf{t}_2 using two pairs of homologous points, and assuming that R_1 and R_2 are defined.

8 USING HOMOLOGOUS POINTS TO FIND THE TRANSLATIONS

Let suppose that I_1 and I_2 are images, $(a, b)^T \in I_1$ and $(c, d)^T \in I_2$ are the coordinate in pixels of the homologous points of Type I that correspond to a point m_1 on the horizontal plane, and that $(e, f)^T \in I_1$ and $(g, h)^T \in I_2$ are another pair of homologous points of Type I that correspond to a point m_2 on the horizontal plane.

We want to find the vectors $\mathbf{t} = (t_1, t_2, t_3)^T \in \mathbb{R}^3$ and $\mathbf{t}' = (t'_1, t'_2, t'_3)^T \in \mathbb{R}^3$ that correspond to the translations used for capturing I_1 and I_2 .

Lets define H_1 and H_2 as the homographies that map points with coordinates measured on the horizontal surface into pixels in the images I_1 and I_2 , respectively. That means

$$H_1 = (\mathbf{h}_1 \mathbf{h}_2 \mathbf{h}_3) = K \begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{pmatrix} \quad (37)$$

and

$$H_2 = (\mathbf{h}'_1 \mathbf{h}'_2 \mathbf{h}'_3) = K \begin{pmatrix} r'_{11} & r'_{12} & t'_1 \\ r'_{21} & r'_{22} & t'_2 \\ r'_{31} & r'_{32} & t'_3 \end{pmatrix}. \quad (38)$$

We can choose any point over the horizontal plane to be the origin of the coordinate system used for defining the camera parameters. Lets assume that m_1 is this point. By doing this, the point whose coordinates are $(0, 0)^T$ is mapped by H_1 over the pixel $(a, b)^T$ in the image I_1 , and is mapped by H_2 over the pixel $(c, d)^T$ in the image I_2 . It means that

$$(\mathbf{h}_1 \mathbf{h}_2 \mathbf{h}_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \quad (39)$$

and

$$(\mathbf{h}'_1 \mathbf{h}'_2 \mathbf{h}'_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} c \\ d \\ 1 \end{pmatrix}, \quad (40)$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are scalars that must be found.

Thus

$$\mathbf{h}_3 = \lambda_1 (a, b, 1)^T$$

and

$$\mathbf{h}'_3 = \lambda_2 (c, d, 1)^T.$$

It means that, \mathbf{t} and \mathbf{t}' are defined up to the scale factors λ_1 and λ_2 , because

$$\mathbf{t} = K^{-1} \mathbf{h}_3 \quad (41)$$

and

$$\mathbf{t}' = K^{-1} \mathbf{h}'_3. \quad (42)$$

We use the other pair of homologous points to calculate λ_1 and λ_2 . Since \mathbf{t} and \mathbf{t}' are defined with an ambiguity of one scale factor (Hartley and Zisserman, 2004), we just expect to calculate $\frac{\lambda_2}{\lambda_1}$.

Lets define

$$P = \begin{pmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \mathbf{p}_3^T \end{pmatrix} \quad (43)$$

as the inverse of the homography

$$\begin{pmatrix} a \\ \mathbf{h}_1 \mathbf{h}_2 & b \\ 1 \end{pmatrix},$$

and lets

$$Q = \begin{pmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \mathbf{q}_3^T \end{pmatrix} \quad (44)$$

be the inverse of the homography

$$\begin{pmatrix} c \\ \mathbf{h}'_1 \mathbf{h}'_2 & d \\ 1 \end{pmatrix}.$$

It is easy to notice that P and Q can be calculated, because we are assuming that the matrix of intrinsic parameters K and the rotations related to each camera are known.

We have that

$$H_1^{-1} = \begin{pmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \frac{1}{\lambda_1} \mathbf{p}_3^T \end{pmatrix} \quad (45)$$

and

$$H_2^{-1} = \begin{pmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \frac{1}{\lambda_2} \mathbf{q}_3^T \end{pmatrix}. \quad (46)$$

Applying the homography H_1^{-1} over $(e, f)^T$ and H_2^{-1} over $(g, h)^T$ we must obtain the same point on the horizontal plane. This means that, there is a scalar $\lambda_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} \mathbf{p}_1^T \\ \mathbf{p}_2^T \\ \frac{1}{\lambda_1} \mathbf{p}_3^T \end{pmatrix} \begin{pmatrix} e \\ f \\ 1 \end{pmatrix} = \lambda_3 \begin{pmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \frac{1}{\lambda_2} \mathbf{q}_3^T \end{pmatrix} \begin{pmatrix} g \\ h \\ 1 \end{pmatrix}. \quad (47)$$

There follows from the equation in the first line that

$$\lambda_3 = \frac{\langle \mathbf{p}_1, (e, f, 1)^T \rangle}{\langle \mathbf{q}_1, (g, h, 1)^T \rangle}. \quad (48)$$

From the equation in the third line we have that

$$\frac{1}{\lambda_1} \langle \mathbf{p}_3, (e, f, 1)^T \rangle = \frac{\lambda_3}{\lambda_2} \langle \mathbf{q}_3, (g, h, 1)^T \rangle. \quad (49)$$

Replacing the equation 48 in the equation 49 we find that

$$\frac{\lambda_2}{\lambda_1} = \left(\frac{\langle \mathbf{p}_1, (e, f, 1)^T \rangle}{\langle \mathbf{p}_3, (e, f, 1)^T \rangle} \right) \left(\frac{\langle \mathbf{q}_3, (g, h, 1)^T \rangle}{\langle \mathbf{q}_1, (g, h, 1)^T \rangle} \right). \quad (50)$$

9 FINDING THE SIZE OF THE OUTPUT

We know that any calibration process performed using just the information of homologous points has an ambiguity of scale (Hartley and Zisserman, 2004). As a consequence, in the previous section we just could find homographies that generate stereoscopic pairs with an ambiguity of size.

This problem can be solved, for example, if the following extra information is available:

- The distance l between the optical centers of the cameras, in both poses, used for capturing the stereoscopic pair.
- The distance between two points m_1 and m_2 in the horizontal plane, that correspond to two identifiable pixels p_1 and p_2 .

The scale of the output must be chosen in such a way that the ratio of the distance between p_1 and p_2 and the distance between the eyes becomes equal to the ratio of the distance between the points m_1 and m_2 and the distance l .

For example, let's suppose that the distance l is $65m$, and the distance between m_1 and m_2 is $20m$. Since the distance between the eyes of a person is about $6.5cm$, there follows that the output scale must be chosen in such a way that the distance between p_1 and p_2 become $2cm$.

If any geometric information about the scene is available, the scale can be adjusted by a trial and error method until a good perceptual result be achieved.

10 EXPERIMENTS WITH SYNTHETIC DATA

We made 630 experiments in order to find the homographies, minimizing the Three-dimensional Interpretability Error, using synthetic cameras and points. It means that the projections used were perfectly calculated by the computer. The initialization method used for the Levenberg-Marquardt algorithm was the one described in the previous sections.

The experiments were divided into 15 groups whose poses of the synthetic cameras used in the calculation of projections were the same.

In each group, we calculated the distance between a pair of reference homographies, found using 10 points of Type I and 10 points of Type II, and homographies calculated by using combinations with a smaller number of points.

We know that the reference homographies were found by a determined optimization problem, because

they were calculated using more points than the sufficient condition defined by the Theorem 1. In order to guarantee that the the Levenberg-Marquardt algorithm converges to the correct global minimum, we limited the rotations used in the sampling processes of synthetic cameras and the rotations used during the initialization of the optimization. We did this, because we need to be confident that the sector of the space of rotations considered contains only one pair of cameras with Three-dimensional Interpretability Error equals to zero.

We measured the distance between the pairs of homographies using the formula

$$\|H_1 - H'_1\| + \|H_2 - H'_2\|,$$

where H_1 and H_2 are the homographies that are being compared to the reference homographies H'_1 and H'_2 . The norm considered to a matrix is its largest eigenvalue. Since the matrix representation of homographies are defined up to a scale factor, we put all the homographies in the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & 1 \end{pmatrix}$$

before applying the formula.

The results of the 15 experiments are presented in the tables of the Appendix. In each table, the cells' values are the distance between the reference homographies and the solution calculated using a different combination of points of Type I and II defined by the cell's position. The number of points of Type I is presented in the left of the table, and the number of points of Type II is presented in the top.

We joined the information of all 15 tables in the Table 1. Each cell of this table correspond to the amount of tables from the Appendix whose correspondent cell's value is below 10^{-4} , which is the threshold chosen to consider that the solution agrees with the reference homographies.

We read the number of points of Type I and II in the border of the Table 1 following the same logic of the tables in the Appendix.

In order to analyze the Table 1 we must take into a count that:

1. If there is a 0 in a cell, it means that any of the 15 considered solutions agrees with the reference solution. Thus, the amount of points of Type I and II related to the cell's position is, probably, not enough for making the optimization problem well defined.
2. If there is a number different from 0 in a cell, it means that there is an agreement between a solution and the reference. If this number is large,

we can conclude that it happened in many tables, meaning that probably the solution of the optimization problem is well defined for the amount of points of Type I and II related to the cell's position. This number can be different from 15, because a local minimum can be found, once we are using a sparse sampling in the initialization, since we had to solve hundreds of optimization problems.

Table 1: Each cell of this table corresponds to the amount of tables from the Appendix whose correspondent cell's value is below 10^{-4} , which is the threshold chosen to consider that the the solution agrees with the reference homographies.

	0	1	2	3	4	5	6
2	0	0	0	0	0	11	15
3	0	0	0	10	12	13	13
4	0	10	11	12	14	14	14
5	0	12	14	14	14	13	13
6	0	12	13	14	14	13	13
7	0	14	15	15	15	14	15

By analyzing the Table 1, we conclude that the experimental result is in agreement with the Theorem 1. But we discover that, probably, 4 points of Type I and 2 points of Type II is not a minimal combination for solving the problem of finding the correct homographies by minimizing the Three-dimensional Interpretability Error. Moreover, we establish the Conjecture 1.

Conjecture 1. *The minimal combinations of points that make the Theorem 1 still valid are:*

- 2 points of Type I and 5 points of Type II;
- 3 points of Type I and 3 points of Type II;
- 4 points of Type I and 1 point of Type II.

11 EXPERIMENT WITH REAL IMAGES

We made some experiments using real images. In Figure 8 we present a pair of images captured by two cameras. We use colored dots to identify the homologous points used for estimating the homographies. The pink dots correspond to points of Type I and the green dots are points of Type II.

Figure 9 shows the stereoscopic pair generated by the application of the homographies estimated using the methodology described in the previous sections. And Figure 10 shows the result presented over a horizontal display. It gives the idea of the user perception

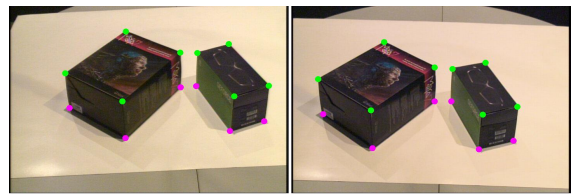


Figure 8: Two pictures used for generating an horizontal stereoscopic pair. There are pink dots over the homologous points of Type I, and green dots over the points of Type II.



Figure 9: Stereoscopic pair generated using the method described in this paper.



Figure 10: One image of a stereoscopic pair being presented over an horizontal display. This picture gives the idea of the user perception.

(only one image of the stereoscopic pair is being presented).

The scale of the output images was adjusted by the user using a trial and error method.

12 CONCLUSION AND FUTURE WORKS

We presented a new method for generating horizontal stereoscopic pairs using images captured by cameras. Our method is not based on the use of calibration patterns, such as the method presented in (Madeira and Velho, 2012). It is based on the establishment of correspondences between homologous points, which

gives more flexibility to the user.

An important property of our method is that it finds the best solution considering a metric that has an intuitive geometric interpretation, the Three-dimensional Interpretability Error, which is defined in this paper.

We also proved a theorem that establishes a sufficient condition to the use of our method, and a conjecture that support other conditions.

Finally, we believe that this paper and (Madeira and Velho, 2012) show that it may be possible to build a new and interesting theory of horizontal stereoscopy based on the deformation of images, instead of using a rendering process. This theory would be made of results from Computer Vision, such as done in (Madeira and Velho, 2012), or by new results, inspired in Computer Vision, established using Projective Geometry and Optimization, such as the ones presented in this paper. Some problems that this new theory could treats are:

1. Find good methods to initiate the Levenberg-Marquardt algorithm that minimize the Three-dimensional Interpretability Error.
2. Prove or disprove the Conjecture 1.
3. Find methods to estimate the 3D error of the scene presented to the user when the capture process is not perfect. For example, if the camera centers are not parallel to the horizontal surface used as reference.
4. Find the best deformation that the stereoscopic pair must suffer in order to try to compensate the movement of the user's head, although this problem does not have an exact solution.
5. Define new metrics different from the Three-dimensional Interpretability Error.

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APPENDIX

There follows the 15 tables generated by the experiments made using synthetic data described in the Section 10.

Table 2.

	0	1	2	3	4	5	6
2	1.4003	0.7885	0.2338	0.2883	0.2766	0.0000	0.0000
3	0.3507	0.2133	0.1625	0.0000	0.0000	0.0000	0.0000
4	0.4124	0.1775	0.2610	0.2837	0.0000	0.0000	0.0000
5	0.2216	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.1871	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0759	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 3.

	0	1	2	3	4	5	6
2	0.3246	0.3358	0.1134	0.1088	0.1270	0.0000	0.0000
3	0.1132	0.1134	0.0593	0.0000	0.0000	0.0000	0.0000
4	0.0388	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0464	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0218	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0718	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.

	0	1	2	3	4	5	6
2	0.4397	0.4398	0.2474	0.2468	0.2138	0.0000	0.0000
3	0.2942	0.1217	0.0953	0.0000	0.0000	0.0000	0.0000
4	0.2262	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.2652	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.2412	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.2355	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 5.

	0	1	2	3	4	5	6
2	0.2615	1.0214	0.3444	0.3502	0.2016	0.0000	0.0000
3	0.1957	0.0951	0.3514	0.3514	0.0000	0.0000	0.0000
4	0.6002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.1848	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.2007	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1914	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 6.

	0	1	2	3	4	5	6
2	0.6248	0.4991	0.5126	0.5434	0.0463	0.0000	0.0000
3	0.4613	0.4961	0.6431	0.0000	0.0000	0.0000	0.0000
4	0.8455	0.4403	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.4626	0.4934	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.7626	0.4005	0.6388	0.0000	0.0000	0.0000	0.0000
7	0.8050	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 7.

	0	1	2	3	4	5	6
2	0.4310	0.4333	0.2562	0.3135	0.1940	0.5502	0.0000
3	0.3730	0.3674	0.6479	0.0000	0.0000	0.0000	0.0000
4	0.1067	0.7464	0.7723	0.0000	0.0000	0.0000	0.0000
5	0.1345	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.1487	0.7467	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1093	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 8.

	0	1	2	3	4	5	6
2	0.7874	0.3198	0.3140	0.3392	0.1825	0.0000	0.0000
3	0.1879	0.1711	0.7936	0.8844	0.7696	0.7654	0.6834
4	0.1550	0.0000	0.7679	0.7452	0.5967	0.6757	0.6654
5	0.5634	0.5935	0.6346	0.6295	0.4998	0.5237	0.6706
6	0.1909	0.6858	0.6874	0.6871	0.4199	0.4277	0.5338
7	0.0024	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 9.

	0	1	2	3	4	5	6
2	0.8925	0.4310	0.5367	0.2543	0.2095	0.0000	0.0000
3	0.2532	0.1614	0.1681	0.2341	0.3068	0.0000	0.0000
4	0.1835	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.2148	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.2102	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1822	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 10.

	0	1	2	3	4	5	6
2	0.7426	0.8303	0.3292	0.3646	0.2573	0.0000	0.0000
3	0.4350	0.2784	0.5071	0.4037	0.4607	0.4367	0.4211
4	0.0307	0.0000	0.3957	0.3928	0.0000	0.0000	0.0000
5	0.2708	0.0000	0.0000	0.0000	0.0000	0.3964	0.4045
6	0.0613	0.0000	0.0000	0.0000	0.0000	0.3962	0.4069
7	0.4015	0.4405	0.0000	0.0000	0.0000	0.3949	0.0000

Table 11.

	0	1	2	3	4	5	6
2	0.6068	0.3028	0.4622	0.1847	0.1440	0.0000	0.0000
3	0.4653	0.4556	0.4395	0.0000	0.0000	0.0000	0.0000
4	0.6014	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.4418	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.4010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.4034	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 12.

	0	1	2	3	4	5	6
2	0.9409	0.1904	0.3584	0.2534	0.0562	0.4134	0.0000
3	0.3599	0.3445	0.3425	0.0000	0.0000	0.0000	0.0000
4	0.1694	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.2569	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.2307	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.2254	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 13.

	0	1	2	3	4	5	6
2	0.6323	0.8084	0.3247	0.4999	0.1544	0.3315	0.0000
3	0.3686	0.3772	0.5905	0.0000	0.0000	0.0000	0.0000
4	0.0112	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0355	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0395	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 14.

	0	1	2	3	4	5	6
2	2.2263	0.3821	0.5053	0.2740	0.2094	0.0000	0.0000
3	0.2677	0.2671	0.2695	0.0000	0.0000	0.0000	0.0000
4	0.6059	0.4744	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0126	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0281	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.1253	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 15.

	0	1	2	3	4	5	6
2	0.6504	0.4855	0.2401	0.2383	0.0609	0.0000	0.0000
3	0.0698	0.0594	0.0452	0.0000	0.0000	0.0000	0.0000
4	0.0480	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0375	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0739	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0564	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 16.

	0	1	2	3	4	5	6
2	1.2309	0.1480	0.1603	0.1065	0.0950	0.2002	0.0000
3	0.2407	0.3813	0.1260	0.2328	0.0000	0.0000	0.0000
4	0.0807	0.2838	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.4762	0.5174	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0086	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0059	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000