Round Colour Space for Pentachromacy
Circularity of Multidimensional Hue

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Abstract: We generalize previous results to dimension 5 and further. The geometry of the 5-hypercube $[0,1]^5$ gives a model for colour vision in the case of 5 photoreceptor types and a colour space corresponding to the combination of five primary lights. In particular, we focus on the (topologically spherical) boundary of the hypercube and on an equatorial sphere within the boundary, roughly orthogonal to the achromatic segment. In the polytopal and double-cone type spaces, we consider a tridimensional hue component; in the round Runge space we consider a 4-dimensional colourfulness component.

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1 INTRODUCTION

The three properties of colour, luminance, hue and chromatic saturation can be seen as geometric properties of points in the RGB cube; properties that depend on the position and orientattion of the points in the cube with respect to both the *black point* and the *achromatic segment* (Restrepo, 2011). This approach generalises to dimension 4 (Restrepo, 2012a), (Restrepo, 2012b), (Restrepo, 2013b), (Restrepo, 2013a) and to dimensions 5 and further. A model for colour vision in the case of 5 photoreceptor types, or a colour space corresponding to the additive combination of five primary lights is presented here. The approach allows to do pentachromatic colour image processing and the study of pentachromatic metamerism.

Pentachromacy is the case of the visual system of many animals, e.g. pigeons (David M. Hunt and Davies, 2009) and dragonflies and flies (Kelber, 2006).

For the visualization of multispectral images with five bands, pentachromacy is likewise relevant as pentachromatic colour processing followed by the RGB visualisation of three out of the five channels makes explicit important aspects of the image (Restrepo and Maldonado, 2015). Likewise, in the screen ilumination industry, it is also useful to have models for more than three primary lights (Shmuel Roth, 2010), (Roger P. A. Delnoij, 2012) in the visible spectrum.

2 GEOMETRY AND COLOUR

We extend the geometric characterisation of trichromatic colour to a model for pentachromatic colour that is based on the geometry of the hypercube $[0,1]^5$. This interpretation provides a basis for the processing and visualisation of pentachromatic images as well as a plausible model for the study of the colour vision systems of pentachromatic animals.

Let the interval $[0,1]$ model the set of possible intensities of each of five primary lights in an additive colour combination or, of the possible response levels of each of five photoreceptors. In this way the *cubic colour space* $[0,1]^5 \subset \mathbf{R}^5$ models the set of possible primary combinations or of photoreceptor responses; thus, *colours* are modelled as points in the hypercube. The points of $\mathbf{R}^5$ are denoted either as (v,w,x,y,z) or as $\mathbf{p} = (p_0, p_1, p_2, p_3, p_4)$. The position of the colour points, relative to an *equatorial* 3-sphere σ in the 4-spherical boundary $\Sigma := \partial[0,1]^5$ of the 5-cube; σ is the basis for the definition of tridimensional *pentachromatic hue* while Σ is the basis for the definition of 4-dimensional *pentachromatic kolor*.

The boundary Σ of the 5-cube is the set of the colour points having at least one 0-valued coordinate or one 1-valued coordinate. The points $(p_0, p_1, p_2, p_3, p_4) \in \Sigma$ are classified into $5 \times 2 = 10$ 4-cubes depending on which coordinate p_i is equal to 1 or equal to 0; e.g. $\{0vwxyz\}$ and $\{1vwxyz\}$. As a matter of fact, Σ is a PL (piecewise linear) 4-sphere,

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a topological $\mathbf{S}^4$, that in addition to these 10 4-cubes, can be *celled* into 40 3-cubes, 80 squares, 80 edges and 32 points; these numbers result as follows: by fixing 2 of the coordinates p_i and p_j of points in the 5-cube with values in $\{0,1\}$, you get the $2^2 \times \binom{5}{2} = 40$ 3-cubes of Σ; fixing 3 coordinates in $\{0,1\}$, you get the $2^3 \times \binom{5}{3} = 80$ squares of Σ; fixing 4 coordinates, $2^4 \times \binom{5}{4} = 80$ edges and, fixing all 5 coordinates, $2^5 = 32$ points or vertices.

Each of the 32 vertices of Σ, that is points $(p_0, p_1, p_2, p_3, p_4)$ with $p_i \in \{0,1\}$ is a vertex of $\binom{5}{2}$ = 10 3-cubes in the cell complex; e.g. (00000)$\in$ $\{00xyz\}$. Each of the 80 edges is an edge of $\binom{4}{2}$ = 6 3-cubes; for example, $\{v0000\} \subset \{v00yz\}$. Each of the 80 squares is a face of 3 3-cubes; for example, $\{vw000\} \subset \{vwx00\}, \{vw0y0\}, \{vw00yz\}$.

Out of the 40 3-cubes of Σ, 20 do not have neither of the points $\mathbf{s}$:= (0 0 0 0 0) or $\mathbf{w}$:= (1 1 1 1 1) as vertices (**w** and **s** stand for white and black (*schwartz*), respectively.) The union of these 20 cubes is an *equatorial* $\mathbf{S}^3$ for Σ, called the *hue sphere*, and denoted as σ. The points of σ are precisely those points of Σ having at least one coordinate at value 0 and at least one coordinate at value 1. The equatorial σ is *linked* in $\mathbf{S}^5 = \mathbf{R}^5 \cup \{\infty\}$ with the line (circle through ∞) through the points **s** and **w**. The *achromatic segment* is the line segment that joins **s** and **w** and is given by $\phi := \{\lambda\mathbf{s} + (1-\lambda)\mathbf{w} : \lambda \in [0,1]\}$; it consists of the colour points having equal coordinates. In a sense that is made precise below, loosely speaking we say that σ and ϕ are orthogonal.

The equatorial σ, of codimension 2 in $\mathbf{R}^5$, is used to define *pentachromatic hue* by giving coordinates to the points of $[0,1]^5$ on the basis of σ, i.e. by locating the colour point with respect to σ (and ϕ). There are other ways to define a pentachromatic hue as there are other closed 3-manifolds in the cell complex Σ, that are unions of 3-cubes of Σ but, in a sense, σ is the canonical choice.

In fact, since each of the 20 3-cubes of σ can be triangulated into 3! = 6 tetrahedra, according to the 6 possible orderings of the *free* 3 coordinates (there are two *fixed* coordinates p_i, p_j, with $p_i, p_j \in \{0,1\}$, in each such cube), each of the resulting 120 tetrahedra contains the points of σ having one of the 120 possible orderings of their (five) coordinates, where the minimal coordinate has value 0 and the maximal one has value 1. To each point in one such tetrahedron of σ, there corresponds a triangle of points in the hypercube having such an ordering of their coordinates, this time without the restriction of the minimal coordinate being 0 and the maximal being 1, and the union of such triangles is the 5-simplex of the points in the hypercube sharing such ordering of their coordinates. For example, the ordering $p_0 \leq p_1 \leq p_2 \leq p_3 \leq p_4$ corresponds to the tetrahedron $\{p_0 = 0 \leq p_1 \leq p_2 \leq p_3 \leq p_4 = 1\}$ in the cube $\{p_0 = 0, p_4 = 1\} \subset \sigma$. Thus, in this triangulation of σ into $20 \times 6 = 120$ tetrahedra, the interior of each tetrahedron precisely corresponds to each of the elements of the symmetric group S_5, which in turn corresponds as well to the interior of the 5-simplex of points in the 5-hypercube having that ordering of their coordinate values.

We say that a colour point $\mathbf{h} = [h_0, h_1, h_2, h_3, h_4]$ is a *hue* if at least one of its coordinates h_i is 0-valued and at least one of its coordinates h_j is 1-valued, that is, the points of σ are the possible *hues* of the colour points not on the achromatic segment ϕ, called *chromatic colour points*. The set of colour points having a given hue **h** is the *hue triangle* that is the *cone* of the achromatic segment ϕ and the hue point **h** given by $\{[\gamma+\beta h_0, \gamma+\beta h_1, \gamma+\beta h_2, \gamma+\beta h_3, \gamma+\beta h_4] : \beta+\gamma \leq 1, 0 \leq \beta, \gamma \leq 1\}$. There are 120 *hue families* into which the hue points $\mathbf{h} \in \sigma$ can be classified. All colours of a given hue family have the same ordering regarding the relative contributions from each of the photoreceptors, or primaries. An analogy with the trichromatic *RGB* case is for example the *family of the oranges*, where $R \geq G \geq B$. In the trichromatic case, a hue is either unary or binary, in the pentachromatic case, a hue may be unary, binary, trinary or quaternary, correspondingly depending on whether it is a vertex, or is in the interior of a segment, a triangle or a tetrahedron of the PL, equatorial, hue sphere σ. Hues at the boundary of a 3-cube of σ belong to exactly two families.

The hypercube $[0,1]^5$ is triangulated into 120 5-simplices by taking the *topological join* of the achromatic segment ϕ and each of the 120 tetrahedra in the triangulation of σ. (For our purposes, the join of two subsets A and B of $\mathbf{R}^N$ is the union of the line segments connecting points of A with points of B.) Thus, in each of these 5-simplices, the hues of the colour points belong to the same hue family.

To each chromatic colour point $\mathbf{p} = (p_0, p_1, p_2, p_3, p_4)$ there corresponds a unique *hue* $\mathbf{h} \in \sigma$, in fact, **p** belongs to the unique *hue triangle* having as one side ϕ and as opposing vertex the point $\mathbf{h} = (h_0, h_1, h_2, h_3, h_4) \in \sigma$ with coordinates

$$h_i = \frac{p_i - m}{\rho} \qquad (1)$$

where $m := \min\{p_i : i \in 5\}$ and $\rho := \max\{p_i : i \in 5\} - \min\{p_i : i \in 5\}$; for example, the hue of $\mathbf{p} = [0.1, 0.3, 0.5, 0.7, 0.9]$ is $\mathbf{h} = [0, 0.25, 0.5, 0.75, 1]$.

The connectivity of the cubes and tetrahedra of σ is topologically captured by a connectivity graph. Consider the graph having as nodes the tetrahedra (equivalently the elements of S_5) of the chromatic sphere σ, where two nodes are joined by a branch precisely when the two corresponding permutations differ by a transposition of two consecutive elements. This corresponds to two tetrahedra being connected by a triangular face. When a colour changes from a hue family to another, the two corresponding orderings differing by a transposition of two consecutive elements, we say that a *mild change* of hue family has occurred. We show below a table corresponding to a *Hamiltonian circuit* in the graph of hue families connected by mild changes, that cycles through the 120 families. In this way, we give a cyclic order to the hue families, analogous to the trichromatic case, where you can give a cyclic order to the 6 hue families and in fact (but this only in the trichromatic case) a cyclic order to the hues themselves.

3 ROUNDER COLOUR SPACES

For colour processing, it is better to transform the hypercube into a *rounder* space. Initially, we obtain a space of the "hexcone" type, that although not round is a starting point to obtain the rounder *double-cone type* space and a round ball type space. Such spaces are more intuitive and less prone, under colour transformations, to end up at "forbidden colours" due to a careless change of coordinates. A useful geometric technique to obtain rounder spaces is *spinning*.

The *spin* $\mathrm{Sp}(N,M,\mathbf{R}^3)$ in $\mathbf{R}^3$ of a subset N of the upper half plane of $\mathbf{R}^2$ *around* a set $M \subset \mathbf{R}^2$ of dimension 1 with parametrisation $M = \{[m_1(t), m_2(t)] : t \in P \subset \mathbf{R}^1\}$ is given by
$\mathrm{Sp}(N,M,\mathbf{R}^3) := \{[x, y.m_1(t), y.m_2(t)] : (x,y) \in N\}$

Usually, the coordinates m_i of M are taken to be in the interval [-1, 1]; for example, the spin of the triangle $\{[x,y] \in R^2 : x \in [-1,1], y = 1-|x|\}$ around a circle in $\mathbf{R}^3$, is the double cone $\{[x,(1-|x|)\cos(t),(1-|x|)\sin(t)] : x \in [-1,1], t \in [0,2\pi)\}$ and the spin of the same triangle around a hexagon is a hexcone. In a sense, you are spinning the *flag* of the upper half of $\mathbf{R}^2$, that contains the triangle, with *pole* the x axis.

Similarly, the *spin* $\mathrm{Sp}(N,M,\mathbf{R}^5)$ in $\mathbf{R}^5$ of a subset N of the upper half plane of $\mathbf{R}^2$ *around* a set $M \subset \mathbf{R}^4$, usually of dimension 3, with parametrisation $M = \{[m_1(t_1,t_2,t_3), m_2(t_1,t_2,t_3), m_3(t_1,t_2,t_3), m_4(t_1,t_2,t_3)] : (t_1,t_2,t_3) \in P \subset \mathbf{R}^k\}$ is given by
$\mathrm{Sp}(N,M,\mathbf{R}^5) := \{[v, w.m_1(t_1,t_2,t_3), w.m_2(t_1,t_2,t_3), w.m_3(t_1,t_2,t_3), w.m_4(t_1,t_2,t_3)] : (v,w) \in N, (t_1,t_2,t_3) \in P\}$.

The equatorial σ sits in $\mathbf{R}^5$ in codimension 2. By projecting σ onto the 4-hyperplane Π that is orthogonal to the achromatic segment ϕ you get again a PL $\mathbf{S}^3$, denoted σ^*, this time embedded (in Π) in codimension 1. The equatorial sphere σ has been made "flat", and each ray in Π from the origin intersects a circumscribing, round $\mathbf{S}^3$, which is key to getting a round hue space.

In fact, for this projection $\pi : \sigma \to \Pi$, $\pi(\sigma)$ is an *embedding* of σ. To see that it is injective, notice that $\pi(\mathbf{s}) = \pi([s_0,s_1,s_2,s_3,s_4]) = [s_0-k, s_1-k, s_2-k, s_3-k, s_4-k]$, where k is the average of the s_i's. Therefore, if $\pi({}_a\mathbf{s}) = \pi({}_b\mathbf{s})$ then ${}_a\mathbf{s} - {}_b\mathbf{s} = {}_b\mathbf{k} - {}_a\mathbf{k}$, that is, ${}_a\mathbf{s}$ and ${}_b\mathbf{s}$ differ by a constant vector, but since the points of σ have exactly one 0-valued coordinate and one 1-valued coordinate, the remaining coordinates being strictly between 0 and 1, the only possibility is that such constant vector be the origin.

The function $f : \mathbf{R}^5 \to \mathbf{R}^2$ with $(p_0,p_1,p_2,p_3,p_4) \mapsto (\mu,\rho)$, where ρ is the range of the coordinate values and $\mu := \frac{max\{p_i : i \in 5\} + min\{p_i : i \in 5\}}{2}$ is the midrange, bijects each of the hue triangles (whose union is $[0,1]^5$) to a canonical, isosceles "$\mu\rho$-triangle", also called the *luminance-saturation triangle* $T := f([0,1]^5)$. T is an isosceles triangle with base $\{(\mu,\rho) \in \mathbf{R}^2 : \mu \in [0,1], \rho = 0\}$ and height $\{(\mu,\rho) \in \mathbf{R}^2 : \mu = 0.5, \rho \in [0,1]\}$. We say that μ measures the *luminance* of a colour on each hue triangle and that ρ measures its *chromatic saturation*.

The 5-hypercube is the (union of the) collection of the hue triangles, each of which has as common side the achromatic segment ϕ and oposing vertex a point in σ; the fact that the hue triangles have diffrent shapes differentiates this construction of the hypercube $[0,1]^5$ from a spin.

The spin of the $\mu\rho$ triangle around σ^* is (analog to the the hexcone space) the 5D 120-cone $H := \mathrm{Sp}(T,\sigma^*,\mathbf{R}^5)$.

The spin of the triangle T around the standard (Euclidean, round) sphere $\mathbf{S}^3 \subset \mathbf{R}^4$ gives the double-cone type space. This space makes explicit the fact that, corresponding to maximal 1 and minimal 0 values of luminance (at the vertices of the cone), there is no hue nor chromatic saturation to be added.

By deforming the triangle T into a semicircle, D and spinning it around the round $\mathbf{S}^3$ of $\mathbf{R}^4$, you get a solid ball $\mathbf{B}^5$, called Runge space. The process of spinning adds the hue to the luminance and saturation of T. In Runge space, the hue and the luminance are made explicit but not so is the chromatic saturation; instead, we call the distance from the central middle gray the *kolorfulness* or vividness of a colour point

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and the corresponding point in the $\mathbf{S}^4$ boundary of the ball the *kolor* or chromaticity (defined for achromatic colours such as **s** and **w** as well) of the colour point. The kolor is the point of the 3-sphere at which the spin was made and the kolorfulness is the distance to the central point *middle gray* of the 5-ball.

4 PENTACHROMATIC HUE

In the trichromatic case, the hue is a *cyclic variable* (Restrepo and Estupinan, 2014); in the tetrachromatic case, the hue is bidimensional and spherical; in the pentachromatic case the hue is tridimensional, spherical. In both the tetrachromatic and the pentachromatic cases, it is not the hues but *families of hue* that can be cyclically ordered. There are 24 families of hue in the tetrachromatic case, and 120 families of hue in the pentachromatic case.

The ordering of the relative contributions of each of the five primaries giving rise to a coloured light beam can be seen as a broad property of its colour; colours are thus characterised as belonging to one of 120 possible *families of hue*. Likewise for the relative contributions of five photoreceptors with different photopigments at a small retinal spot.

In the case of human colour vision, the perceptual proper of uniqueness is made *explicit* (in Marr's jargon (Marr, 1982)) at V4 and not at the receptoral level nor in the retina, the thalamus or V1 (Zeki, 1993). In fact, since the cone responses overlap (as do the spectral responses of camera filters), it is not possible to have a stimulus eliciting response from only one cone type; instead of using the term uniqueness of hue, to such hypothetical case, we refer to as (receptoral) the *unariness* of a hue. Further retinal, and cortical processing differentiates further the hue in these hue families of a receptoral level; in our trichromatic case, the receptoral family $L \geq M \geq S$ is subdivided into 2 cortical-hue families: oranges and citrines (greenish yellows) separated by cortical, unique yellow ($L = M$). The cortical uniques and hue families are then unique red, reddish oranges, central orange, yellowish oranges, unique yellow, yellowish citrines, central citrine, greenish citrines and unique green.

Also in the trichromatic case, the vertices of the *hue hexagon*, an equatorial $\mathbf{S}^1$ in the PL $\mathbf{S}^2$ given by $\partial[0,1]^3$, called the *chromatic hexagon* in (Restrepo, 2011), belong either to {red green, blue} or to {yellow, violet, cyan} (at the receptoral level, yellow is a binary colour). With the modification of yellow to a unique and the corresponding addition of the binary families orange and cetrine, the hue hexagon becomes an octagon with vertices {red, yellow, green, blue} (the uniques) and segments {orange, cetrine, cyan, violet} (the binaries). In the trichromatic case, the *uniqueness* of a hue is the opposite of its *binariness*: the closer you get to a unique hue (red, green, yellos or blue) the farther you are from either of the *central binary* hues (orange, cetrine, violet or cyan). Geometrically in σ, define *uniqueness* as the distance to a closest unique and *binariness* as the distance to a closest central binary hue.

In the pentachromatic case, a vertex of σ is said to be a *unary* hue; at a point in the interior of an edge you have contributions from two primaries and the hue is said to be *binary*, on a triangle in the complex σ, you have contributions from 3 primaries and the hue is said to be *trinary* and inside (i.e. in the interior of) a tetrahedron of σ, the hue is said to be *quaternary*. There are 5 unary hues, e.g. [10000], which are the vertices of σ having 4 zero-valued coordinates, 10 binary hue families, e.g. [1*w*000] which are the segments of σ made of points having 3 0-valued ccordiantes; 10 trinary hue families, e.g. [1*wx*00], which are the triangles of σ of points having 2 0-valued coordinates and 5 quaternary hue families, e.g. [1*wxy*0], which are the tetrahedra of σ having 1 0-valued coordinate. Since a colour point with no zero coordinate is not a hue (not an element of σ), there are no pentary hues, such a colour has an achromatic component.

Every colour point can be writtten uniquely either as an achromatic colour e.g. $[0.1,0.1,0.1,0.1,0.1]$, or an achromatic colour plus a colour of a quaternary hue e.g. $[0.1,0.2,0.2,0.2,0.2]$, or an achromatic colour plus a colour of a quaternary hue plus a colour of a trinary hue, e.g. $[0.1,0.2,0.4,0.4,0.4]$, or an achromatic colour plus a colour of a quaternary hue plus a colour of a trinary hue plus a colour of a binary hue e.g. $[0.1,0.2,0.4,0.7,0.9]$, or as an achromatic colour plus colour of a quaternary hue plus a colour of a trinary hue plus a colour of a binary hue plus a colour of a unique hue as follows: [0.1, 0.2, 0.4, 0.7, 0.9] = [0.1, 0.1, 0.1, 0.1, 0.1] + [0, 0.1, 0.1, 0.1, 0.1] + [0, 0, 0.2, 0.2, 0.2] + [0, 0, 0, 0.3, 0.3] + [0, 0, 0, 0, 0.2], of hues, correspondingly, *undefined*, [01111], [00111], [00011] and [00001].

For a given a 5-tuple $\mathbf{q} = [q_0,q_1,q_2,q_3,q_4] \in [0,1]^5$ there is at least one (of the 120 possible) permutation $p : \{0,1,2,3,4\} \to \{0,1,2,3,4\}$ such that a nondecreasing ordering $q_{p(0)} \leq q_{p(1)} \leq q_{p(2)} \leq q_{p(3)} \leq q_{p(4)}$ results. As already said, in the equatorial 3-sphere σ, the set of 5-tuples so ordered by a given permutation is a tetrahedron; thus the 120 permutations p determine a triangulation of the hue sphere σ into 120 tetrahedra. Correspondingly, the 120 permutations determine a triangulation of $[0,1]^5$ into 120

5-simplices where each 4-simplex is the topological join of the achromatic segment and the corresponding tetrahedron in the triangulation of σ.

The permutations have the structure of a group; algebraically, they are the elements of the *symmetric group* S_5 and, perhaps not an obvious fact, its elements can be *cyclically sequenced* so that two consecutive permutations differ by a transposition of consecutive coordinates (Johnson, 1963). From the viewpoint of the wavelength domain, two consecutive coordinates of a colour are better related than another pair of ccordinates. The swapping of two consecutive colour coordinates is called a *mild, hue-family change*. Also, it so happens that the two tetrahedra of σ that correspond to such a permutation have exactly a triangular face in common; also, you can visit each tetrahedra of σ exactly once going from one tetrahedron to the next through such triangles. Equivalently stated, there is a Hamiltonian circuit in the Cayley graph of S_5 (Witte, 1982).

We say that each tetrahedron of σ determines a family of hues in the sense that in each tetrahedron the relative contributions of the primaries is fixed. Also, you have that the 120 hue families are cyclically ordered. It is the families of hue rather than the hues themsalves that are cyclically ordered, unlike the trichromatic case where the hue sphere is triangulated into 6 intervals (Restrepo, 2011), and each interval is linearly ordered, the points within each tetrahedron of σ do not have a linear order.

As we said in Section 2, the S^4 boundary Σ of $[0,1]^5$ consists of 10 4-cubes; each such 4-cube connects with 8 other 4-cubes via each 3-cube of its boundary. For example, the 4-cubes $\{v = 0\}$ and $\{z = 1\}$ connect via the 3-cube $\{v = 0, z = 1\}$. In the corresponding connectivity graph, having as nodes the 4-cubes and as branches the connecting 3-cubes, there are several Hamiltonian circuits.

Also, the 120 tetrahedra of the S^3 equatorial hue sphere σ can be grouped into 20 3-cubes; for example, the 3-cube $\{v = 0, w = 0\}$ groups the tetrahedra $\{v = 0, w = 0, x \leq y \leq z\}$, $\{v = 0, w = 0, y \leq x \leq z\}$, $\{v = 0, w = 0, y \leq z \leq x\}$, $\{v = 0, w = 0, z \leq y \leq x\}$, $\{v = 0, w = 0, z \leq x \leq y\}$ and $\{v = 0, w = 0, x \leq z \leq y\}$. That is, σ can also be *celled* into 20 3-cubes, on each of which, as in the trichromatic case, the colour points can be given the trichromatic attributes of hue saturation and value. Each 3-cube in σ connects with each of 6 other 3-cubes via a 2-square. For example, the 3-cubes $\{v = 1, x = 0\}$ and $\{x = 0, z = 1\}$ connect via the square $\{v = 1, x = 0, z = 1\}$; there are several Hamiltonian circuits in the graph corresponding to this cellular decomposition of σ into 3-cubes. in such a way that each 3-cube (the union of six tetrahedra) is visited exactly once and every pair of consecutive 3-cubes have a square (the union of two triangles) face in common.

We indicate an iterative procedure of obtaining Hamiltonian circuits in the graph of S_n. Let e_n denote the identity element of S_n. In Figure 1 the permutations of two elements are used to get the permutations of a set of three elements in such a way that you get a factorisation, into transpositions of consecutive elements, of e_3 from that of e_2. The symmetric group S_3 with three elements is generated by the transpositions $t_0 := (0,1)$ and $t_1 := (1,2)$. Denote the action of a transposition t acting on an ordered set, or triple $\mathbf{p} = [p_0, p_1, p_2]$ as $t(\mathbf{p})$, or simply $t\mathbf{p}$; thus, for example, (1, 2)[0, 1, 2] = [0, 2, 1]. From Figure 1, the permutations of $[0,1,2]$ are cyclically ordered as

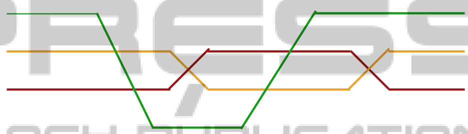

Figure 1: The lines mustard and red represent the symmetric group S_2. A new element is positioned at each possible position for after each transposition in the previous permutation to get S_3. The iteration of this gives a Hamilton circuit on each S_n.

[0,1,2]
[1,0,2] = (0,1).[0,1,2]
[1,2,0] = (1,2)(0,1).[1,2,0]
[2,1,0] = (0,1)(1,2)(0,1).[0,1,2]
[2,0,1] = (1, 2)(0, 1)(1, 2)(0, 1).[0,1,2]
[0,2,1] = (0,1)(1,2)(0,1)(1,2)(0,1).[0,1,2]
[0,1,2] = (1,2)(0,1)(1,2)(0,1)(1,2)(0,1).[0,1,2]

The last ordering being the initial one. Thus, at the last line you have a factorization of the identity e_3 of the symmetric group S_3 as

$e_3 = \Pi_{i=0}^{3!-1} t_i$ = (1,2)(0,1)(1,2)(0,1)(1,2)(0,1)

into 6 transpositions of consecutive elements, either (0,1) or (1,2). In fact, the 6 permutations

$\pi_j := \Pi_{i=0}^{j} t_i$, $j \in /0,5/$

of S_3 are thus cyclically ordered. We simplify further the notation by writing only the first element of each transposition in the factorisations, for example $e_3 =$ 1.0.1.0.1.0.

The 24 permutations in S_4 are likewise cyclically ordered. By interleaving the sequences of transpositions 0:=(0, 1), 1:=(1,2), 2:=(2,3), 2:=(2, 3), and its reverse, alternatively in between each of the 6 transpositions in the factorisation of $e_3 \in S_3$ and taking care of adding a one in the notation of the transpositions in e_3, after the end of an inserted sequence 2, 1, 0, but not after an inserted sequence 0, 1, 2, (extending the idea in Figure 1) you get

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$e_4 = \Pi_{i=0}^{4!-1} t_i$

$= [0.1.2.(0).2.1.0.(1{+}1).0.1.2.(0).2.1.0.(1{+}1).0.1.2.(0).2.1.0.(1{+}1)]^T$

where the transpositions in parenthesis are those corresponding to e_3; thus,

$e_4 = [0.1.2.0.2.1.0.2.0.1.2.0.2.1.0.2.0.1.2.0.2.1.0.2]^T$

where the T in Π^T indicates a reversing in the ordering of the factorization Π. A cyclic ordering of the elements of S_4 is then $\{\pi_j | j \in \{0,1,2,...23\}\}$, with $\pi_j = \Pi_{i=0}^{j} t_i$.

This procedure can be generalized. A factorization of $e_5 \in S_5$ into $5 \times 24 = 120$ transpositions of consecutive elements is

$e_5 = \Pi_{i=0}^{5!-1} t_i$

$= [\alpha\ 0\ \alpha^T\ 1'\ \alpha\ 2\ \alpha^T\ 0'\ \alpha\ 2\ \alpha^T\ 1'\ \alpha\ 0\ \alpha^T\ 2'\ \alpha\ 0\ \alpha^T\ 1'\ \alpha\ 2\ \alpha^T\ 0'\ \alpha\ 2\ \alpha^T\ 1'\ \alpha\ 0\ \alpha^T\ 2'\ \alpha\ 0\ \alpha^T\ 1'\ \alpha\ 2\ \alpha^T\ 0'\ \alpha\ 2\ \alpha^T\ 1'\ \alpha\ 0\ \alpha^T\ 2']^T$

where $\alpha = 0.1.2.3$ and a prime, as in n', denotes $n+1$, that is,

$= [\alpha\ 0\ \alpha^T\ 2\ \alpha\ 2\ \alpha^T\ 1\ \alpha\ 2\ \alpha^T\ 2\ \alpha\ 0\ \alpha^T\ 3\ \alpha\ 0\ \alpha^T\ 2\ \alpha\ 2\ \alpha^T\ 1\ \alpha\ 2\ \alpha^T\ 2\ \alpha\ 0\ \alpha^T\ 3\ \alpha\ 0\ \alpha^T\ 2\ \alpha\ 2\ \alpha^T\ 1\ \alpha\ 2\ \alpha^T\ 2\ \alpha\ 0\ \alpha^T\ 3]^T$

$= [0\ 1\ 2\ 3\ 0\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 1\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 0\ \alpha^T\ 3\ 0\ 1\ 2\ 3\ 0\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 1\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 0\ \alpha^T\ 3\ 0\ 1\ 2\ 3\ 0\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 1\ 0\ 1\ 2\ 3\ 2\ \alpha^T\ 2\ 0\ 1\ 2\ 3\ 0\ \alpha^T\ 3]^T$

$= [0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 3\ 0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 3\ 0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 1\ 0\ 1\ 2\ 3\ 2\ 3\ 2\ 1\ 0\ 2\ 0\ 1\ 2\ 3\ 0\ 3\ 2\ 1\ 0\ 3]^T$

and a ciclic ordering of the 120 elements of S_5 is $\{\pi_j | j \in \{0,1,2,...119\}\}$, with

$$\pi_j = \Pi_{i=0}^{j} t_i, j \in /0,119/ \tag{2}$$

5 CODING AND DECODING THE HUE

In the 120-cone space, the luminance and the chromatic saturation are coded in the $\mu\rho$ triangle while the hue is coded as a point in the PL hue sphere σ. This code of σ is the basis for the definition of hue in the $\mathbf{S}^3$ of the double-cone type space.

The cyclic ordering of S_5 given by Equation 2, together with the initial ordering $v \le w \le x \le y \le z$, denoted shortly as *vwxyz*, determine a cyclic ordering of the orderings of the 5-tuples $[v,w,x,y,z]$ of coordinates. Thus, an ordering of $[v,w,x,y,z]$ determines a number j given by the position in the list (see Table 1) of the cyclic ordering of S_5, and viceversa. Equation 1 gives the j-th ordering $\pi_j[vwxyz]$ of the 5-tuple; on the other hand, to recover the number j from a permutation of $[vwxyz]$, we proceed as follows.

Table 1: A cyclic ordering of the 120 orderings of the 5-tuple [*vwxyz*].

#	ordering	#	ordering	#	ordering
0	*vwxyz*	40	*vwzxy*	80	*vwyzx*
1	*wvxyz*	41	*wvzxy*	81	*wvyzx*
2	*wxvyz*	42	*wzvxy*	82	*wyvzx*
3	*wxyvz*	43	*wzxvy*	83	*wyzvx*
4	*wxyzv*	44	*wzxyv*	84	*wyzxv*
5	*xwyzv*	45	*zwxyv*	85	*ywzxv*
6	*xwyvz*	46	*zwxvy*	86	*ywzvx*
7	*xwvyz*	47	*zwvxy*	87	*ywvzx*
8	*xvwyz*	48	*zvwxy*	88	*yvwzx*
9	*vxwyz*	49	*vzwxy*	89	*vywzx*
10	*vxywz*	50	*vzxwy*	90	*vyzwx*
11	*xvywz*	51	*zvxwy*	91	*yvzwx*
12	*xyvwz*	52	*zxvwy*	92	*yzvwx*
13	*xywvz*	53	*zxwvy*	93	*yzwvx*
14	*xywzv*	54	*zxwyv*	94	*yzwxv*
15	*xyzwv*	55	*zxywv*	95	*yzxwv*
16	*xyzvw*	56	*zxyvw*	96	*yzxvw*
17	*xyvzw*	57	*zxvyw*	97	*yzvxw*
18	*xvyzw*	58	*zvxyw*	98	*yvzxw*
19	*vxyzw*	59	*vzxyw*	99	*vyzxw*
20	*vxzyw*	60	*vzyxw*	100	*vyxzw*
21	*xvzyw*	61	*zvyxw*	101	*yvxzw*
22	*xzvyw*	62	*zyvxw*	102	*yxvzw*
23	*xzyvw*	63	*zyxvw*	103	*yxzvw*
24	*xzywv*	64	*zyxwv*	104	*yxzwv*
25	*xzwyv*	65	*zywxv*	105	*yxwzv*
26	*xzwvy*	66	*zywvx*	106	*yxwvz*
27	*xzvwy*	67	*zyvwx*	107	*yxvwz*
28	*xvzwy*	68	*zvywx*	108	*yvxwz*
29	*vxzwy*	69	*vzywx*	109	*vyxwz*
30	*vxwzy*	70	*vzwyx*	110	*vywxz*
31	*xvwzy*	71	*zvwyx*	111	*yvwxz*
32	*xwvzy*	72	*zwvyx*	112	*ywvxz*
33	*xwzvy*	73	*zwyvx*	113	*ywxvz*
34	*xwzyv*	74	*zwyxv*	114	*ywxzv*
35	*wxzyv*	75	*wzyxv*	115	*wyxzv*
36	*wxzvy*	76	*wzyvx*	116	*wyxvz*
37	*wxvzy*	77	*wzvyx*	117	*wyvxz*
38	*wvxzy*	78	*wvzyx*	118	*wvyxz*
39	*vwxzy*	79	*vwzyx*	119	*vwyxz*

As a first step, *reduce* a given ordering of $[v,w,x,y,z]$ to an ordering of $[y,z]$ by deleting the letters v, w and x from the given ordering; for example, $zyxwv \mapsto zy$; then, notice if this corresponds to an even or an odd permutation (odd in the example) of *yz*; next, reduce the initial permutation by deleting

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only the letters v and w, obtaining a permutation of $[x,y,z]$; in the previous example, you get $zyxwv \mapsto zyx$ and notice if the reduced permutation ($xyz \mapsto zyx$ in the example) is even or odd (again odd, in the example) next notice the parity of the reduced-by-v permutation ($wxyz \mapsto zyxw$, even, in the example); finally, check the parity of the original (complete) permutation ($vwxyz \mapsto zyxwv$, even, in the example.) Notice also, as if going from a reduced ordering to a less reduced ordering, the *position* at which the added element is inserted; this position is sometimes measured from left to right and sometimes from right to left. If you are at an even ordering (e.g. 5432) the new element (e.g. 1) is assumed to be inserted from left to right (thus, in 54321, the 1 has been inserted at the 4-th position) while, if you are based at an odd ordering, the inserted element is assumed to have been inserted from right to left (thus, 54 being an odd permutation tells us that in 543, the 3 has been inserted at the 0-th position). These positions are the weight factors t_k in the computation of the number j of the permutation, that is,

$$j = \sum_{k=2}^{n} \frac{n!}{k!} t_k \tag{3}$$

where $n = 5$, and $t_k \in /0, k-1/$ is calculated as described above.

Given a colour point, for example [0.5, 0.4, 0.3, 0.2, 0.1], with Equation 3, we find that we are in the tetrahedron number 64 of σ; also, its luminance and its chromatic saturation are given by $\mu = \frac{0.1+0.5}{2} = 0.3$ and $\rho = 0.5 - 0.1 = 0.4$. Next, we find where in tetrahedron 64, the point [1, 0.75, 0.5, 0.25, 0] is (see Equation 1); that is, we compute the barycentric coordinates of point [1, 0.75, 0.5, 0.25, 0] in the tetrahedron with vertices [10000], [11000], [11100] and [11110] which, in a sense, are the barycentric coordinates of [0.75, 0.5, 0.25] in the tetrahedron with vertices [1000], [1100], [1110] and [1111] of $\mathbf{R}^4$. Since $[1, 0.75, 0.5, 0.25, 0] = \alpha[1000] + \beta[1100] + \gamma[1110] + \delta[1111]$ and $\alpha + \beta + \gamma + \delta = 1$, you can get α, β, γ and δ.

6 HUE PROCESSING

A simple yet useful way of processing pentachromatic images is to process separately the hue. The remaining pair of colour attributes of the luminance and the saturation can likewise be processed separately using e.g. exponential laws (A. Restrepo, 2009). The luminance and saturation can be jointly processed with *flows* of the points in the luminance-saturation triangle. Likewise, the kolor can be processed as a rotation or a homeomorphism of $\mathbf{S}^4$ and the kolorfulness with an exponential law, separately. In the remaining of this section we concentrate on the separate processing of the hue, by "PL rotations" of σ, a tool for the transformation of pentachromatic hue.

By a *PL rotation* of σ we mean the following. Initially, σ is projected on the 4-subspace Π of $\mathbf{R}^5$ that is orthogonal to ϕ. Call σ^* the embedding $\pi(\sigma)$. Then, Π is rotated; then, the rotated version of σ^* (a PL $\mathbf{S}^3$) is "fitted back" (see below) on σ^*. Then, σ^* is back projected to σ.

The projection π is linear and each tetrahedron of σ projects to a tetrahedron of σ^*. A point of σ such as $\mathbf{h} = [0, w, x, y, 1]$, with average $\eta = \frac{v+x+y}{5}$, has an image $\pi(\mathbf{h})$ in σ^* given by $\mathbf{h}^* = [-\eta, w-\eta, x-\eta, y-\eta, 1-\eta]$ which has zero average. Notice that the ordering of the coordinates of $\mathbf{h}$ is the same as that of $\mathbf{h}^*$. $\mathbf{h}^*$ defines the ray on $\mathbf{R}^5$ given by $\{t\mathbf{h}^* : t \in [0,\infty)\}$. To each ray in $\Pi \subset \mathbf{R}^5$ there corresponds a unique point of σ^* (and viceversa).

The the rigid motions of $\mathbf{S}^3 \subset \mathbf{R}^4$ (the rotations of $\mathbf{R}^4$) can coded with the help of unit quaternions, as in rot(s) = asb, with $|a| = |b| = 1$. The corresponding group is known as $SO(4)$. After $\subset \Pi$ is rotated, to each point $\mathbf{h}^* \in \sigma^*$ there corresponds a new point $\mathbf{h}^{**} \in \Pi$, not necessarily in σ^*. We find an appropriately corresponding in σ^* as follows. The ordering of the coordinates of $\mathbf{h}^{**}$ determines a tetrahedron of σ^* that contains the point intersection of the ray through $\mathbf{h}^{**}$ and σ^*; this intersection point is the a "rotated" version of $\mathbf{h}^*$. We compute the barycentric coordinates of this intersection point with respect to the vertices of the tetrahedron. Then we compute in the correspoding tetrahedron of σ, the corresponding barycentric combination, with the computed coordinates gives the "PL rotated" version of $\mathbf{h}$. The embedding of $\sigma^* = \pi(\sigma) \subset \Pi$ of σ has an inverse given by the addition of the constant vector such that the resulting largest coordinate has value 1 and the smallest one has value 0. For example, consider the hue $\mathbf{h} = [0, 0.4, 0.5, 0.6, 1]$ of average 0.5 and projection $\mathbf{h}^* = [-0.5, -0.1, 0, 0.1, 0.5]$ that is rotated to $\mathbf{g}^* = [0.5, 0.1, 0, -0.1, -0.5]$ via multiplication by the orthogonal matrix [00001; 00010; 00100; 01000; 10000]; the ray through $\mathbf{h}^{**}$ is in the terahedron of σ with vertices 11110, 11100, 11000, 10000 which it intersects at the same point $\mathbf{h}^{**} = \mathbf{g}^*$ and that is back projected to point [.5 .1 0 -.1 -.5] + [.5 .5 .5 .5 .5] = [1 .6 .5 .4 0]]

7 CONCLUDING REMARKS

A model for colour vision in the case of 5 photoreceptor types as well as a colour space corresponding to the additive combination of five primary lights are presented here. The approach allows to do pentachromatic colour image processing and the study of pentachromatic metamerism.

Pentachromacy is relevant in the study of the vision systems of many animals, e.g. pigeons (David M. Hunt and Davies, 2009) and dragonflies and flies (Kelber, 2006). Most mammals are dichromatic (dolphins are monochromatic) and old-world monkeys are trichromatic; the reduction in cone type of mammals is probably related to the fact that they evolved as nocturnal animals. Among vertebrates, only some rodents and marsupials take advantage of ultraviolet light. Light of short wavelengths is less absorbed by water and penetrates deeper in water.

By feeding three of the five channels of a pentachromatic image to the RGB inputs of a visualising device, useful information can be made explicit. This, combined with pentachroamtic colour processing should provide with a useful tool for the search of objects with given spectral surface reflectance in pentachromatic images.

The visualisation of multispectral images can be done in time; for this, continuous changes of hue (therefore of $SO(4)$) are useful (Lenz and Homma, 1996).

The colour vision of animals such as pigeons and dragonflies can be tested according to these colour components of luminance, saturation, 3D hue, kolor and kolorfulness with the models provided and a five-primary illuminating system that include two types of UV, in addition to RGB. Pentachromatic metamerism can also be studied along the lines of (Restrepo, 2014b), (Restrepo, 2014a). This has applications in illumination, photography and animal vision as well.

In the screen ilumination industry, it is also useful to have models for more than three primary lights (Shmuel Roth, 2010), (Roger P. A. Delnoij, 2012) in the visible spectrum. It is a fact that three lights cannot reproduce all colours, mainly when seen isolated; nevertheless when colours are seen in context, chromatic contrast gives the illusion of many more colours that those that can be projected in isolated form.

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