Second Degree of Freedom of Elastic Objects
Adjustable Poisson's Ratio for Mass Spring Models

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Abstract: In this paper, we show how to construct mass spring models for the representation of homogeneous isotropic elastic materials with adjustable Poisson's ratio. Classical formulation of elasticity on mass spring models leads to the result, that while materials with any value of Young's modulus can be modeled reliably, only fixed value of Poisson's ratio is possible. We show how to extend the conventional model to overcome this limitation.

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1 INTRODUCTION

Computer graphics community has been using mass spring models (MSMs) for the representation of deformable objects since the earliest attempts to accommodate elastic solids in computer generated animations. While MSMs were popular because of their low implantation complexity, the link between their physical properties and spring-network parameters has never been properly established. This led to belief that MSMs cannot represent elastic objects accurately and that the models do not converge to certain solutions upon mesh refinement (Van Gelder, 1998; Nealen et al., 2006). Consequently more physically accurate techniques such as finite element method (FEM) gained popularity.

The accuracy of the description of an elastic object is, however, not a problem in MSM representations. The standard lattice based models used in physics, mechanical engineering and other related fields offer a description of elastic solids, which is as accurate as the limitations of linear elasticity theory allow it to be (Ladd and Kinney, 1997; Ostoja-Starzewski, 2002; Kot et al., 2014). There is however a limitation to what can be modeled with MSMs. The classical models allow to obtain any value of Young's modulus E, however Poisson's ratio ν is limited to $1/4$ for volumetric objects and $1/3$ for 2D meshes. This allows to freely adjust the stiffness of an object, but is not sufficient to replicate all types of materials. In this work we show how to efficiently extend conventional MSM to overcome this limitation and obtain a reliable model of any homogeneous isotropic solid.

2 MASS SPRING MODELS

If we consider a homogeneous isotropic solid, its elastic properties are defined by exactly two parameters (elastic moduli). Classically Young's modulus E and Poisson's ratio ν are the popular pair. The Young's modulus is the ratio of stress to strain measured along the same axis under an uniaxial stress condition, that is, it gives the resistance to directional stretching/compression. The Poisson's ratio is the ratio of transverse to axial strain (denotes to what degree material expands in one direction when compressed in another). Depending on the application, besides E and ν, other moduli are often used such as bulk modulus K, or Lamé parameters λ and μ. In any description only two of them are independent and providing a link between spring-network parameters, and a chosen pair of the elastic moduli is sufficient to describe elastic properties of the MSM.

In case of 2D MSM, an isotropic homogeneous structure can be obtained with hexagonal lattice (Fig. 1) (Ostoja-Starzewski, 2002). All the springs have the same spring coefficient k and the relation between the spring coefficient and the Lamé constants for a such network are given by

$$\lambda = \mu = \frac{3}{4\sqrt{3}}k, \tag{1}$$

from which it follows that $E = \frac{2}{\sqrt{3}}k$ and $\nu = \frac{1}{3}$. Springs are assumed to be of a unit length.

Similarly a volumetric isotropic solid can be constructed with cubic lattice MSM (Ladd and Kinney,

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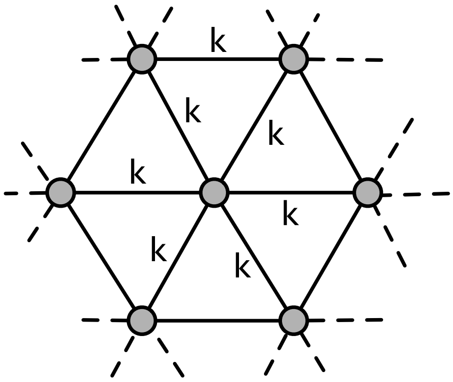

Figure 1: Hexagonal lattice.

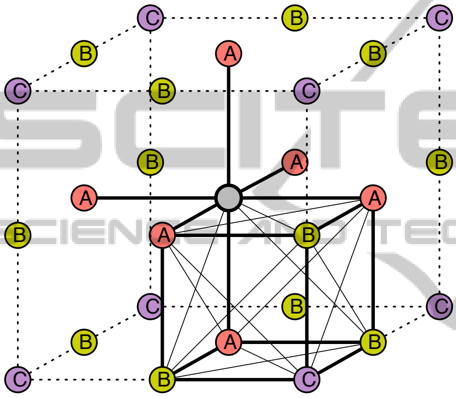

Figure 2: Cubic lattice. The nearest neighbors of a node are classified in three groups A, B and C. They correspond to the potential connections along cube edges, face diagonals and cube diagonals.

1997). The spring connections are present between nearest neighbors (A) and second nearest neighbors (B) (Fig. 2) and all have the same spring coefficient k. The elastic moduli for such network are given by

$$E = 2.5\frac{k}{a} \qquad \nu = 1/4, \tag{2}$$

where a is the length of an edge of an elemental cube.

The models discussed above have a limitation on the value of Poisson's ratio, which comes from a well-established result of the continuum mechanics. If the constituents of a material interact with the central forces dependent upon distance alone the Poisson's ratio of a homogeneous and isotropic material is identically 1/4 (or 1/3 for a 2D system) (Love, 1906; Lakes, 1991). Other values of the Poisson's ratio can be obtained by incorporating non-central forces into the model, e.g. the angular terms, or forces that do not depend on distance alone or by introducing anisotropy (Lakes, 1991).

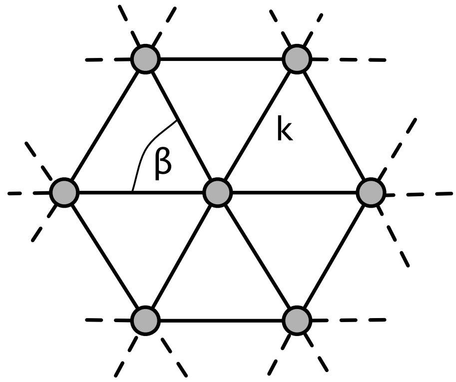

Figure 3: Hexagonal lattice with normal as well as angular springs.

An example of such extension is the Kirkwood model of an isotropic 2D material (Ostoja-Starzewski, 2002). It is based on a hexagonal lattice and introduces additional angular springs, which provide a resisting force when an angle between two coinciding springs changes from its neutral value of 60° (Fig. 3). The Poisson's ratio is given by

$$\nu_{2D} = \left(1 - \frac{3\beta}{2ka^2}\right) \Big/ \left(3 + \frac{3\beta}{2ka^2}\right), \tag{3}$$

where a is a length of an edge of a triangle, k is a spring stiffness coefficient of regular springs and β of angular ones.

Similar modifications are possible in case of 3D materials and the problem of constructing MSM-like models capable of representing isotropic materials with arbitrary values of Poisson's ratio have been explored before by a number of researchers (Baudet et al., 2007; Lakes, 1991; Zhao et al., 2011). An overview of well established techniques can be found in (Sahimi, 2003). Typically additional degrees of freedom are added to the model by incorporating beams, angular springs and other custom three or four node connections which react not only to stretching, but shearing or applying toque as well. Introducing such elements allows to obtain wide range of materials, however as volumetric objects are inherently more complex than two dimensional structures, the description of their properties tends to be more complex as well. Analytical solutions are in most cases too difficult to derive and obtaining desirable properties of such models often involves a numerical minimization and parameter fitting, which tunes the model to the particular problem at hand. This fact, together with increased computational and memory costs, makes such models less suitable for practical applications.

In contrast in this work we will explore an ap-

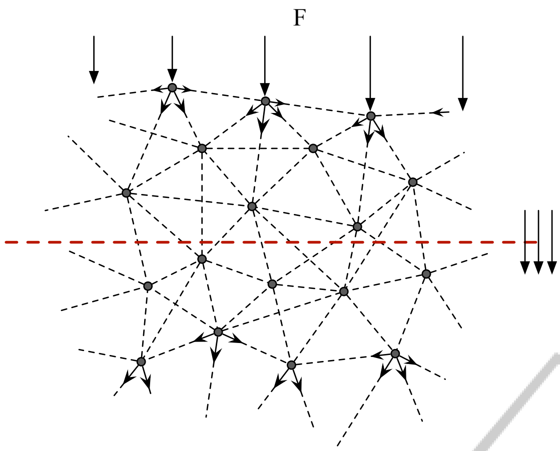

Figure 4: Momentum flow through a simple MSM.

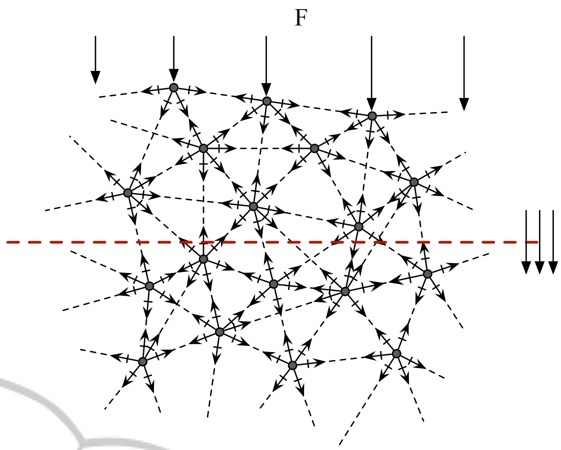

Figure 5: Dispersive momentum flow.

proach of constructing MSM which allows to achieve arbitrary values of ν, without introducing additional structures into the model. Our approach will in fact make use only of nodes and their relative pairwise distances.

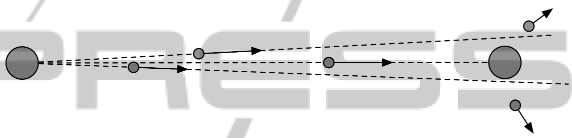

Figure 6: Momentum dispersion.

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3 EXTENDED MASS SPRING MODELS

Simple MSMs described in Section 2 can be viewed as a network in the Fig. 4. Forces in such networks act along straight lines between particles, and as we know, this leads to $\nu = 1/4$, which may be considered a geometric property of the space (the way distance to neighboring nodes changes, when we change the position of a node).

We generalize this model by introducing additional phenomenon. When two nodes approach, they start repelling each other as usual, and additionally each of them starts radiating momentum in "random" directions.

To visualize how we can justify such behavior, let us consider a particle-carrier interaction model in which we assume, that constituents of the matter (particles) interact with each other by means of exchanging carriers (which in turn can be considered a small particles). Carriers interact with particles (and with each other) by means of central forces (elastic collisions), therefore this model still uses central forces as most basic means of interaction. Carriers however have a finite velocity so the transfer of momentum does not happen instantaneously; the big particles move as well. Having this in mind, we may assume that a carrier shot form one particle, may not hit the other particle with a sniper's precision. In such case it will disperse in a "random" direction and will not come back to the first particle (Fig. 6). If a particle is bombarded from all directions uniformly, the dispersed carriers will also appear to be radiating from the particle uniformly (even if the actual distribution of dispersion angles is not trivial).

Figure 5 illustrates the dissipative part of the interaction, where each node absorbs the incoming carriers and radiates them uniformly in all directions. Such network will behave like a gas or fluid with no viscosity.

We will model this phenomenon using MSMs in the following way. When two nodes approach each other causing increase of momentum flow between them, only a fraction of the flow will participate in the direct exchange of momentum between these two nodes and appear as a regular force of the spring $F^{\mu} = -\kappa^{\mu}\Delta L$. The rest of the flow will be distributed uniformly to all springs connected to a node (scaled to the length of a spring), causing dissipation of momentum $F^{*} = -q\kappa^{\mu}\Delta L$. Parameter q denotes the ratio between dissipative momentum flow, and the direct one.

Material modeled in such way is now characterized by two parameters – μ gives the strength of direct interactions, and $q[\mu]$ of the dispersive ones.

The standard expression for a stress in an elastic body

$$\sigma_{ij} = K\delta_{ij}\varepsilon_{kk} + 2\mu(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}), \qquad (4)$$

can be rewritten, by treating our material as a superposition of Cauchy's isotropic structure, in which

$\lambda = \mu$ with a fluid in which $\sigma_{ij} = B\varepsilon_{kk}\delta_{ij}$, where $B\varepsilon_{kk}$ corresponds to pressure.

$$\sigma_{ij} = \mu\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + B\delta_{ij}\varepsilon_{kk}, \qquad B = \lambda - \mu \quad (5)$$

$$\sigma_{ij} = \frac{5}{3}\mu\delta_{ij}\varepsilon_{kk} + 2\mu(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}) + B\delta_{ij}\varepsilon_{kk} \quad (6)$$

The direct interactions will correspond to μ parameter, dispersive ones to B. Both can be controlled, which allows to achieve other values of Poisson's ratio:

$$\nu = \frac{B+\mu}{2(B+2\mu)}, \quad (7)$$

$$\nu = \frac{1+\frac{5}{3}Q}{2(2+\frac{5}{3}Q)}, \qquad Q = \frac{B}{\frac{5}{3}\mu}, \quad (8)$$

where in a perfect structure $Q = 2q$, and it denotes the ratio between dispersed and not dispersed carriers. The multiplication by the factor of 2 is introduced for convenience (one spring connects two nodes).

4 TESTS

We have applied our technique to Ladd's cubic lattice models and verified that we are able to obtain any value of Poisson's ratio with their use.

In order to estimate numerically the values of E and ν in an MSM, the following numerical experiment has been performed: a block of an elastic material was compressed by applying static displacement to its opposite ends along x direction, and the resulting deformation was measured both for random and cubic MSMs. The Young's modulus and Poisson's ratio are related to the elastic response of such a system by

$$E = \frac{F/A}{\Delta x/L_x}, \qquad \nu = \frac{\Delta y}{\Delta x}$$

where F is the reaction force, A the cross-sectional area of the block (in YZ plane), and Δx and Δy are the deformations of the block along x and y directions respectively. The initial block dimensions were $70a_0 \times 15a_0 \times 15a_0$ (where a_0 is an arbitrary unit of length) and the base spring constant has been set to k_0. The static displacement in x direction was imposed on all the nodes on the boundary of the block.

The results are presented in Fig. 7 and they confirm that any value of Poisson's ratio can be achieved with a very high accuracy. Figure 8 shows visual examples of a few chosen materials with different ν. Although the presented shapes are very simple, the tests

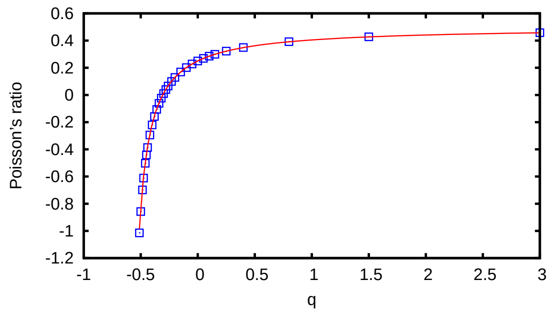

Figure 7: Poisson's ratio in Cubic lattice MSM. Red curve represents a theoretical prediction. Blue points are the results of measurements.

show that our extended MSM is capable of modeling bulk materials accurately. Given sufficiently high resolution any complex shape can be modeled reliably.

5 CONCLUSIONS

In this article we showed an approach of constructing MSM which allows to achieve arbitrary values of ν, without introducing additional structures into the MSM. We demonstrated that by incorporating the concept of momentum dispersion additional degree of freedom can be added to the classical MSM, allowing it to freely represent homogeneous isotropic materials characterized by two different constants (in contrast to one constant description given by classical MSM). Because our method does not introduce additional elements to the MSM itself, the memory costs remain unchanged and the computational costs rise by much lower degree when compared with methods which do introduce additional elements to the MSM.

This makes our method useful for real time applications involving deformable elastic objects of any kind (e.g. in computer games), and especially attractive for simulating fracture or crack propagation (peridynamics) – the application for which MSMs are generally considered to be better suited than FEM based approaches, but suffer from the fact that they cannot represent all types of materials.

The simplest implementation of our model, when used with explicit time integrator, requires additional iteration through all the springs in the system, effectively doubling the computational time when compared to the simple MSM. Some improvements are however possible; for instance the value of momentum dispersed on the nodes in the 'previous' frame could be used to approximate the value in the current frame, which eliminates the need of additional iteration through the springs, but may influence the stability of the simulation. However, in case of quasi-static

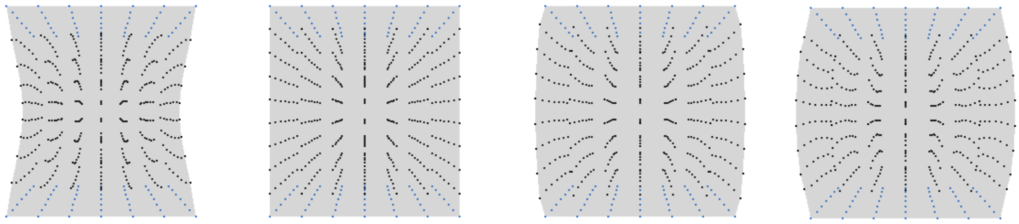

Figure 8: A block of material with dimensions $2 \times 3 \times 2$ compressed in y direction to 75% of its natural length. Different values of Poisson's ratio, from the left: -1, 0, 0.25, 0.47.

simulation (such as point based integration techniques (Bender et al., 2013)), it should not affect the stability.

Additionally the accuracy of our models is expected to be higher than that of techniques involving parameter fitting and the analytical description of our model is provided. This makes our MSM an attractive starting point for developing more advanced models (e.g. for anisotropic materials). In our model we have assumed that the dispersion of the force happens uniformly in all directions. In MSMs it will translate into equal redistribution of the incoming force to all the springs connected with a node (scaled by the length). Because redistribution is isotropic, so will be the elastic properties of the material. However by introducing non uniform dispersion mechanisms it should be possible to achieve non isotropic behaviors without extensive modifications of the current model. Such modifications may be a good direction for the future work, as they would allow to efficiently model organic tissues e.g. for surgical simulations, the application for which MSMs are still actively used, but once again suffer from the lack of a mathematical model that would allow to express their elastic properties.

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