Finding Maximal Quasi-cliques Containing a Target Vertex in a Graph

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Abstract: Many real-world phenomena such as social networks and biological networks can be modeled as graphs. Discovering dense sub-graphs from these graphs may be able to find interesting facts about the phenomena. Quasi-cliques are a type of dense graphs, which is close to the complete graphs. In this paper, we want to find all maximal quasi-cliques containing a target vertex in the graph for some applications. A quasi-clique is defined as a maximal quasi-clique if it is not contained by any other quasi-cliques. We propose an algorithm to solve this problem and use several pruning techniques to improve the performance. Moreover, we propose another algorithm to solve a special case of this problem, i.e. finding the maximal cliques. The experiment results reveal that our method outperforms the previous work both in real and synthetic datasets in most cases.

1 INTRODUCTION

Graphs have been used to model lots of real-world applications for decades. For instance, biological networks, social networks, and financial domains can be modeled using graphs. In a graph, vertices represent objects and edges represent the relationships among objects. Finding dense sub-graphs around certain important vertices is an interesting problem in the graph research. In the web network graph, (Gibson, 2005) observe that a dense sub-graph can correspond to spam link farms. In social networks or blogospheres, the specific vertices can be assigned as *leaders* or *bloggers* to advertise new products or to lead fashions, as observed by (Agarwal, 2008) and (Goyal, 2008). In the biology, (Fratkin, 2006) and (Langston, 2005) discover regulatory motifs in genomic DNA. (Zou, 2007) find terrorist groups in a terrorist network by matching a specified structure in the corresponding graph.

Suppose that there is a terrorist network built by an official security department. In the corresponding graph, each vertex corresponds to a terrorist and each edge denotes the partnership between two individuals. Through a long time investigation, polices aim at a terrorist as one of the suspects of a terror attack. In order to identify the whole terrorist group, dense sub-graphs containing the target vertex corresponding to the suspect need to be found. We measure whether a sub-graph is close enough by checking whether it meets the definition of a *quasi-clique*.

A graph is defined as a *clique* if an edge exists in any pair of the vertices in the graph. Different types of cliques, such as *maximal cliques* and *maximum cliques* have been addressed. To model real applications by graphs, incomplete situations need to be considered. The concept of quasi-clique has therefore been proposed. A quasi-clique represents an *almost clique* as defined in (Liu, 2008). A graph is a quasi-clique if the degree of each vertex is larger than or equal to $\lceil \gamma \times (N-1) \rceil$, where γ is a parameter between 0 and 1 and N is the number of vertices in the graph. In this paper, we address a new problem on finding maximal quasi-cliques from a graph, which contain a specific target vertex. The maximal quasi-clique in a graph is a quasi-clique not contained by any other quasi-cliques.

Given a graph, the search space of finding quasi-cliques from the graph is equivalent to the power set of the number of vertices in it. In order to efficiently find all maximal quasi-cliques of a target vertex, we design several pruning strategies to reduce the search space. In addition, we modify the Quick

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algorithm proposed in (Liu, 2008) for a comparison with our proposed method, which is originally designed for finding maximal quasi-cliques. Moreover, we also propose an algorithm to efficiently solve the special case on $\gamma = 1$. Two synthetic datasets and one real dataset are used to test the proposed methods, and the experiment results demonstrate that our methods are better than the modified Quick algorithm in most cases.

The remainder of this paper is organized as follows. The related works are reviewed in Section 2. Then, the preliminaries are given in Section 3. The modified Quick algorithm and our methods are detailed in Section 4. Thereafter, the performance evaluation on the proposed methods is presented in Section 5. Finally, Section 6 concludes this work.

2 RELATED WORKS

The dense graph problems have been adopted on a variety of applications, such as finding thematic groups, organizing social events, and tag suggestion (Sozio, 2010), (Tsourakakis, 2013). A *Clique*, also known as *complete graph*, is a typical dense graph, in which vertices are all connected to each other. The problem of finding a clique with a given size k in a graph is NP-complete. In addition, to find all of the maximal cliques is more difficult. (Du, 2006) have studied the techniques to enumerate all maximal cliques in a complex network. For general undirected graphs, Xiang et al. propose a color-based technique to compute an upper bound of the size of cliques in (Xiang, 2013). If two vertices have different colors, it means that no edge exists between those two vertices. Since cliques are complete graphs, the number of colors in the graph represents the possible size of maximal clique able to be found. A partitioning algorithm is designed in (Xiang, 2013), which computes the maximum clique on MapReduce using a branch and bound search. (Zou, 2010) combine the maximal clique problem and the top-k query. They assume that the graph data is generally interfered in reality. This kind of graphs is called *uncertain graphs*. In an uncertain graph, vertices and edges have their own weights for representing the probabilities of existence. When they confirm that a sub-graph is a clique, its corresponding score is calculated from the weights of vertices and edges. Then, we can use the score to prune some other vertices, which cannot form other cliques with larger scores.

On the other hand, researchers consider *quasi-cliques*, another type of dense graphs, which have different definitions for different studies. (Tsourakakis, 2013) define the threshold for the number of edges in a quasi-clique, and mention that each vertex need connect to most other vertices in a quasi-clique. (Brunato, 2007) formulate two parameters to define the quasi-clique. The first one determines the number of neighbors of each vertex in a quasi-clique, and the second one determines the number of edges in the quasi-clique. (Abello, 2002) and (Liu, 2008) have the similar definition for quasi-cliques, which is based on the degree of each vertex in the same sub-graph. (Abello, 2002) propose an algorithm for finding a single maximal quasi-clique. (Liu, 2008) propose the *Quick* algorithm for finding maximal quasi-cliques in a graph. The basic idea of this Quick algorithm is to use the depth-first order to explore the search space. Then, they use several pruning techniques to reduce the execution time. We illustrate the detailed steps of the Quick algorithm in Section 4 as a comparison of our method.

3 PRELIMINARIES

In this section, we describe the notations and terms to be used in this paper, and formally define the problem on finding maximal quasi-cliques for a target vertex in a graph. Given a *simple graph* $G = (V, E)$, where V denotes a set of vertices and E denotes a set of edges to represent objects and the relationships among objects, respectively. That is, if any two objects have a relationship, an edge between the two corresponding vertices exists. An edge is denoted using a form of (u, v) where $u, v \in V$. $|V|$ and $|E|$ denote the number of vertices and the number of edges in a graph, respectively. $N_G(v) = \{u | \forall(u, v) \in E\}$ denotes the *neighbors* of a vertex v in G. $|N_G(v)|$ therefore denotes the degree of v in G. $dist_G(u, v)$ denotes the distance between the vertex u and the vertex v, which equals the minimum number of edges to traverse from u to v in G. $G' = (V', E')$ is a sub-graph of $G = (V, E)$ when $V' \subset V$, $E' \subset E$, and for any u and v in V', if $(u, v) \in E$, then $(u, v) \in E'$. In the following discussion, we also use a set of vertices to represent the corresponding sub-graph.

Definition 1 (Quasi-Clique): Given a sub-graph $G' = (V', E')$ of G, where $V' \subset V$ and $E' \subset E$, G' is defined as a quasi-clique of v with respect to a parameter γ, denoted $QC(\gamma, v)$, where $v \in V$ and $0 < \gamma \leq 1$, if G' satisfies the following three conditions. 1) $v \in V'$. 2) G' is connected, which means at least a path exists between any two vertices in V'. 3) $|N_{G'}(v)|$ needs to equal or exceed $\lceil(|V'| - 1) \times \gamma\rceil$, $\forall v \in V'$, where $\lceil(|V'|$

$-1) \times \gamma\rceil$ is named the *degree threshold* and denoted $deg_\gamma(V')$.

Example 1: As shown in Figure. 1, let the target vertex be v_1 and γ be 0.5. $G' = (V', E')$, where $V' = \{v_1, v_2, v_3, v_5\}$ and $E' = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_5), (v_3, v_5)\}$ is a quasi-clique QC (0.5, v_1), since G' is connected, and for all vertices $v \in V'$, $|N_{G'}(v)| \geq deg_\gamma(V')$ ($= \lceil(4-1) \times 0.5\rceil = 2$).

Definition 2 (Maximal Quasi-Clique): Given a sub-graph $G' = (V', E')$ of G and G' is a quasi-clique of v with respect to a parameter γ, where $v \in V'$; G' is defined as a *maximal quasi-clique* of v with respect to γ if G' is not a sub-graph of any other quasi-cliques of v with respect to γ.

Example 2: As shown in Figure. 2, let γ be 0.6 and the target vertex be v_2, according to Definition 1, $G' = (V', E')$, where $V' = \{v_1, v_2, v_3, v_4\}$ and $E' = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$ is a quasi-clique $QC(0.6, v_2)$. Since no other quasi-cliques $QC(0.6, v_2)$ contain G', G' is a maximal quasi-clique of v_2 with respect to 0.6.

Given a graph $G = (V, E)$, a parameter $\gamma \in (0, 1]$, and a target vertex $v \in V$, the problem of finding maximal quasi-cliques for v in G is to discover all the sub-graphs G' where G' is a maximal quasi-clique of v with respect to γ.

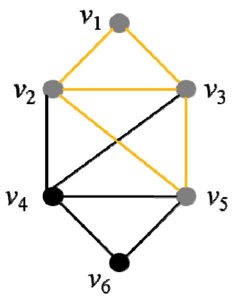


Figure 1: (Left) G' is a $QC(0.5, v_1)$.

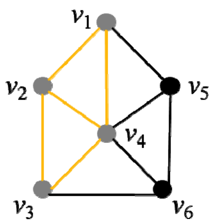


Figure 2: (Right) G' is a $QC(0.6, v_2)$.

4 APPROACHES TO FINDING MAXIMAL QUASI-CLIQUES FOR A TARGET VERTEX

In this section, the solutions on finding maximal quasi-cliques for a target vertex are detailed. In Section 4.1, we discuss the Quick algorithm proposed in (Liu, 2008) and describe how to modify it to solve our problem. This modification is used to compare with our method in the experiments. Then, we describe our solutions in Section 4.2.

4.1 The Quick Algorithm

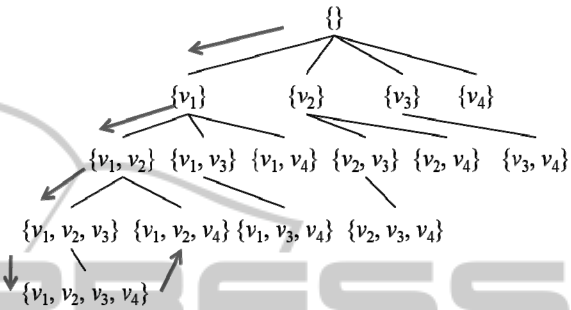


Figure 3: The depth-first search tree of a group G.

Since any sub-graphs of $G = (V, E)$ may have chances of being quasi-cliques, the search space of finding quasi-cliques is equivalent to the power set of V. The Quick algorithm proposed by (Liu, 2008) uses *depth-first search* to find quasi-cliques. An example of a depth-first search tree of a graph G with four vertices $\{v_1, v_2, v_3, v_4\}$ is shown in Figure. 3. Each node in the tree is associated with a sub-graph which contains a set of vertices and the corresponding edges in G. Moreover, the search order follows the order of the vertex id, that is, the sub-graphs with the smallest vertex id v_2 are traversed after those with the smallest vertex id v_1. Notice that, if the smallest vertex ids of two sub-graphs are the same such as $\{v_1, v_2\}$ and $\{v_1, v_3\}$, we compare the second smallest vertex id to decide the search order and so on. As shown in Fig. 3, for each internal node N in the tree, its children contain an additional vertex and moreover, this additional vertex must be with a larger vertex id than all of the vertex ids of the vertices in N. For example, $\{v_1, v_2\}$ is one of the children of $\{v_1\}$. The vertex used to *extend* an internal node related to the sub-graph G' is called a *candidate vertex* of G'. For instance, in Figure. 3, let $G' = (V', E')$ where $V' = \{v_2, v_3\}$, the candidate vertex of G' is v_4. The set of candidate vertices of G' is denoted $CV(G')$, e.g., $CV(G') = \{v_3, v_4\}$ while $V' = \{v_1, v_2\}$. During traversing the whole depth-first search tree, some lemmas used in the Quick Algorithm to prune the candidate vertices are discussed in the following.

Lemma 1: (Liu, 2008) Given a sub-graph $G' = (V', E')$ of G, let $ex\_deg_{\min}(G') = \min(|N_G(v)|)$, $\forall v \in V'$. If G''

$= (V'', E'')$ is a quasi-clique extended from G', then $|V''| \leq \lfloor ex\_deg_{min}(G')/\gamma \rfloor + 1$.

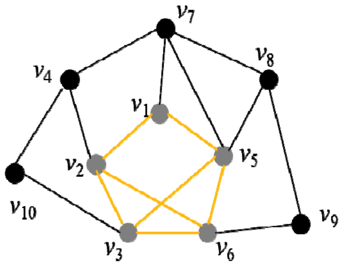


Figure 4: G' is a $QC(0.5, v_1)$.

$ex\_deg_{min}(G')$ is the minimum degree of the vertices in V', considering the edges in G. Since the vertex degree in a quasi-click should be large enough, i.e. at least $\lceil(|V''| - 1) \times \gamma\rceil$ for G'' to be a quasi-clique, according to Lemma 1, the number of vertices to be added to the sub-graph G' to form a quasi-clique G'' is limited to be no larger than $\lfloor ex\_deg_{min}(G')/\gamma\rfloor + 1 - |V'|$, denoted $U(G')$ (U for upper bound). Once the number of vertices being added to G' is larger than $U(G')$, the newly generated sub-graph G'' cannot be a quasi-clique.

Example 3: As shown in Figure. 4, let γ be 0.5 and the target vertex v_1. The sub-graph $G' = (V', E')$, where $V' = \{v_1, v_2, v_3, v_5, v_6\}$ and $E' = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_2, v_6), (v_3, v_5), (v_3, v_6), (v_5, v_6)\}$ is a quasi-clique $QC(0.5, v_1)$. Also, $ex\_deg_{min}(G') = 3$, and $U(G') = \lfloor 3/0.5\rfloor + 1 - 5 = 2$.

Lemma 2: (Liu, 2008) Given a sub-graph $G' = (V', E')$ of G and G' is not a quasi-clique, let $in\_deg_{min}(G') = \min(|N_{G'}(v)|)$, $\forall v \in V'$. If $G'' = (V'', E'')$ is a quasi-clique extended from G', then $|V''| \geq |V'| + n$, where n = the minimal value in $\{i \mid in\_deg_{min}(G') + i \geq \lceil\gamma \times (|V'| + i - 1)\rceil\}$.

$in\_deg_{min}(G')$ is the minimum degree of the vertices in V', considering the edges in G'. If G' is not a quasi-clique, $|V''|$ should be large enough for G'' to be a quasi-clique. According to Lemma 2, the number of vertices to be added to the sub-graph G' to form a quasi-clique G'' is limited to be no smaller than the minimal value in $\{i \mid in\_deg_{min}(G') + i \geq \lceil\gamma \times (|V'| + i - 1)\rceil\}$, denoted $L(G')$ (L for lower bound). Once the number of vertices being added to G' is smaller than $L(G')$, the newly generated sub-graph G'' cannot be a quasi-clique.

Example 4: As shown in Figure. 5, let γ be 0.6 and the target vertex v_1. The sub-graph $G' = (V', E')$ is not a quasi-clique $QC(0.6, v_1)$, where $V' = \{v_1, v_2, v_3, v_5, v_6\}$ and $E' = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_3, v_6), (v_5, v_6)\}$. Also, we have $in\_deg_{min}(G') = 2$, and $L(G') = 1$. G' is extended to form G'' by adding v_{10} as shown in Figure. 6. Since G'' is a quasi-clique, $|V''| \geq |V'| + L(G')$.

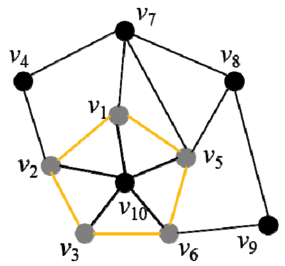


Figure 5: G' is not a $QC(0.6, v_1)$.

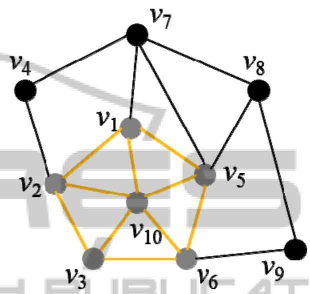


Figure 6: G'' is a $QC(0.6, v_1)$.

Definition 3 (*critical vertices*) (Liu, 2008)): Given a sub-graph $G' = (V', E')$ of $G = (V, E)$, if there is a vertex $u \in V'$ such that $|N_{G'}(u)| < \lceil(|V'| - 1) \times \gamma\rceil$, then u is defined as a *critical vertex* of G'. The set of the critical vertices of G' is denoted $CritV(G')$.

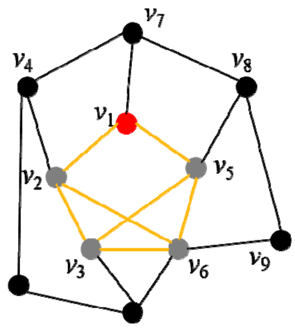


Figure 7: G' is not a $QC(0.6, v_1)$.

Example 5: As shown in Figure. 7, let γ be 0.6 and the target vertex v_1. The sub-graph $G' = (V', E')$, where $V' = \{v_1, v_2, v_3, v_5, v_6\}$ and $E' = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), (v_2, v_6), (v_3, v_5), (v_3, v_6), (v_5, v_6)\}$ is not a quasi-clique $QC(0.6, v_1)$. The vertex v_1 is a *critical vertex* of G' since $|N_{G'}(v_1)| = 2 < \lceil(5 - 1) \times 0.6\rceil = 3$.

Lemma 3: (Liu, 2008) Given a sub-graph $G'' = (V'', E'')$ which is extended from $G' = (V', E')$ where $|V''| > |V'|$ and G' has some critical vertices. If G'' is a quasi-clique then at least $\lceil(|V''| - 1) \times \gamma\rceil$ - $|N_{G'}(u)|$ of the neighbor vertices of u must be contained in V''-V', $\forall\, u \in CritV(G')$.

Proof. Assume that a quasi-clique $G'' = (V'', E'')$ extended from $G' = (V', E')$ and there is a critical vertex u with N neightbor vertices in V''- V', where $N < \lceil(|V''| - 1) \times \gamma\rceil$ - $|N_{G'}(u)|$. Then, the degree of u in G'', i.e., $|N_{G''}(u)|$, is equal to $N + |N_{G'}(u)| < \lceil(|V''| - 1) \times \gamma\rceil$. By Condition 3 of quasi-cliques, if G'' is a quasi-clique, $|N_{G''}(v)| \geq \lceil(|V''| - 1) \times \gamma\rceil$, $\forall\ v \in V''$. A contradiction occurs. Accordingly, G'' is not a quasi-clique.

The above three lemmas are used in (Liu, 2008) to prune candidate vertices for each sub-graph before they are extended. The detailed proofs are described in (Liu, 2008). We focus on the quasi-cliques regarding a target vertex. The Quick algorithm can be modified to solve our problem as follows. The target vertex v is used as the root node of the depth-first search tree in (Liu, 2008). Then, we renumber the other vertices in $V - \{v\}$ and apply the original Quick algorithm. This modified Quick algorithm will be used to compare with our solutions in the experiments.

4.2 The Target-extending Algorithm

Given a graph $G = (V, E)$, a target vertex $v \in V$ and a parameter γ, any subsets of $V - \{v\}$ and v may form a quasi-clique of v with respect to γ if G is connected. Therefore, the search space of finding maximal quasi-cliques for a target vertex in G is the power set of V.

Our baseline algorithm is described as follows. We set the target vertex v as the root node to form a sub-graph and select the neighbors of v to extend this sub-graph. We use the neighbors of v to generate combinations by the exhaustive method and then extend the root node to form the new sub-graphs using adding these combinations as shown in Figure. 8. We detail the whole extending process as follows, by which, maximal quasi-cliques for v can be found if they exist. In the extending process, a vertex being processed to extend a sub-graph G' to a new sub-graph G'' is called the *extending vertex* of G', and the set of the extending vertices denoted $EV(G')$. For example, initially, the target vertex v is the extending vertex. The neighbors (with vertex ids larger than the extending vertices) of the extending vertices of G' will be considered to extend the sub-graph G', called the *candidate vertices* of G', and the set of the candidate vertices of G' denoted $CV(G')$. For example, while v is the extending vertex, the set of the candidate vertices is $\{v_2, v_3\}$. If G' adds some candidate vertices to extend to $G'' = (V'', E'')$, then these candidate vertices of G' will become the extending vertices of G'' for a further extension.

For example, in Figure 8, to extend the sub-graph G' denoted $\{v, v_3\}$, we have $EV(G') = \{v_3\}$ and $CV(G') = \{v_4\}$. Repeat this extending step until no vertex can be added to form a new sub-graph, or all vertices have been used.

Example 6: Given a sub-graph $G' = (V', E')$ which only contains the target vertex v. Assume that the neighbors of v are $\{v_1, v_2, v_3\}$, which form $CV(G')$. We generate the combinations of $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_1, v_2, v_3\}$ from $CV(G')$, and then add to G' to form the new sub-graphs.

To avoid enumerating the whole search tree of the target vertex, in the following, we present some pruning strategies. Lemmas 1-3 mentioned above are also used in our solution. However, Lemma 1 needs to be modified to match our baseline method, thus generating the following Lemma 4.

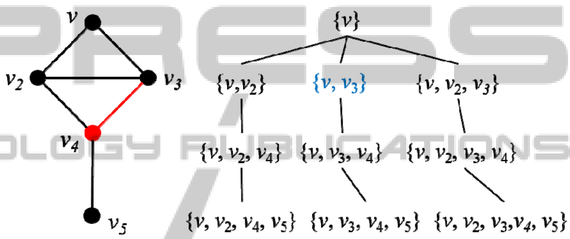


Figure 8: A graph G and the corresponding search tree of our baseline algorithm.

Lemma 4: Given a sub-graph $G' = (V', E')$ of G, let $in\_deg_{\min}(G')$ be $\min(|N_G(v)|)$, $\forall v \in V' - EV(G')$, $ex\_deg_{\min}(G')$ be $\min(|N_G(u)|)$, $\forall u \in EV(G')$, and $deg_{\min}(G')$ be $\min(in\_deg_{\min}(G'), ex\_deg_{\min}(G'))$. If $G'' = (V'', E'')$ is a quasi-clique extended from G', $|V''| \leq \lfloor deg_{\min}(G') / \gamma\rfloor + 1$.

Proof. Assume that a quasi-clique G'' extended from G' exists and $|V''| > \lfloor deg_{\min}(G') / \gamma\rfloor + 1$. Then, there must be a vertex $u \in V'$, with the degree $|N_{G''}(u)| = deg_{\min}(G')$. We know that $|N_{G''}(u)| < \lceil(|V'| - 1) \times \gamma\rceil$. By Condition 3 of quasi-cliques, if G'' is a quasi-clique, $|N_{G''}(u)| \geq \lceil(|V'| - 1) \times \gamma\rceil$, $\forall\ v \in V'$. A contradiction occurs. Accordingly, G'' is not a quasi-clique.

For each sub-graph $G' = (V', E')$, we can compute $U(G')$ and $L(G')$ from Lemma 4 and Lemma 2, respectively. These two boundaries $U(G')$ and $L(G')$ can help to prune the combinations being extended from a sub-graph G' if the number of vertices of the combinations is larger than $U(G')$ or less than $L(G')$. Suppose that a vertex u in V' is a critical vertex of G'. If G' can be extended to form a quasi-clique $G'' = (V'', E'')$, at least one of the neighbors of u belongs to $\{V'' - V'\}$. In addition, if u is not the extending vertex of G', G' cannot be extended to form a quasi-clique by Lemma 3. By applying Lemmas 2-4 to our baseline algorithm, the number of sub-graphs can be reduced and the depth of the search tree can be limited.

Definition 4 ($Hop_{G'}(v)$): Given a sub-graph $G' = (V', E')$ of G and let the target vertex be v, $Hop_{G'}(v)$ denotes the maximum length of the shortest distances between the target vertex v and all vertices $u \in V'$, i.e. $\max(dist_{G'}(u, v))$, $\forall\ u \in V'$, where $dist_{G'}(u, v)$ is the shortest distance between u and v in G'.

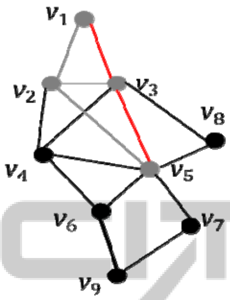


Figure 9: (Left) $Hop_{G'}(v_1)$ is equal to 2 in a $QC(0.6, v_1)$ G'.

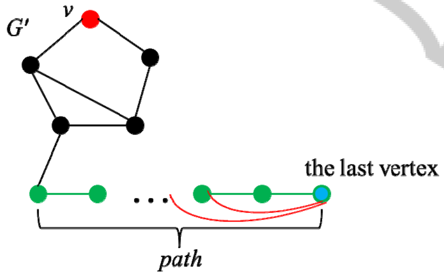


Figure 10: (Right) G'' is a $QC(0.2, v)$.

Example 7: As shown in Figure. 9, let γ and the target vertex be 0.6 and v_1, respectively. The sub-graph $G' = (V', E')$, where $V' = \{v_1, v_2, v_3, v_5\}$ and $E' = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_5), (v_3, v_5)\}$ is a quasi-clique $QC(0.6, v_1)$. $Hop_{G'}(v_1) = 2$ is the maximum length of the shortest distances between the target vertex v_1 and $\{v_2, v_3, v_5\}$ in G'.

Given a sub-graph $G' = (V', E')$ of G and the parameter γ, let the target vertex be v. There are $U(G')$ vertices able to be added to G' to form a quasi-clique $G'' = (V'', E'')$. We use $Fdist(G')$ to denote the maximum length of the shortest distances between v and u, for all $u \in V''$.

Lemma 5: Given a sub-graph $G' = (V', E')$ of G and let the target vertex be v, if $G'' = (V'', E'')$ is a quasi-clique extended from G', $dist_G(v, u)$ is equal to or less than $Fdist(G')$, for all $u \in V''$.

Proof. If we want to find the quasi-clique $QC(\gamma, v)$ extended from G', we can add at most $U(G')$ vertices into the sub-graph G' by Lemma 4. Consider the connecting relation shown in Figure. 10. Given $U(G')$ vertices and $U(G') - 1$ edges, we use these vertices and edges to form a simple path as the path shown in Figure. 10. Obviously, the distance of any two vertices in the path is maximized as the path shown in Fig. 10 is the minimal requirement of a connected graph. Since we need to satisfy the requirement of $deg_{\min}(G')$, the last vertex in the path needs to connect to the other vertices as the arc lines in Figure. 10. We add an edge between G' and the path to form G''.

Suppose that a vertex w exists to be added to form a quasi-clique G'' and $dist_G(v, w) > Fdist(G')$. The vertex w must connect to the last vertex to form a longer path due to $dist_G(v, w) > Fdist(G')$. Therefore, the number of vertices of G'' becomes $|V'| + U(G') + 1$. By Lemma 4, $U(G')$ is the upper bound which denotes the number of vertices can be added to G' to form a quasi-clique. Therefore, G'' is not a quasi-clique. A contradiction occurs. Accordingly, if $G'' = (V'', E'')$ is a quasi-clique extended from G', $dist_G(v, u)$ is equal to or less than $Fdist(G')$, for all $u \in V''$.

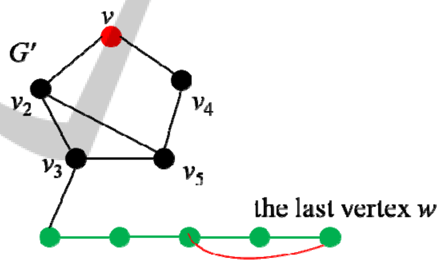


Figure 11: G' is a QC(0.2, v).

From different situations of G', $Fdist(G')$ has the following three cases. (Case 1) If $U(G') \geq deg_{\min}(G')$, $Fdist(G') = Hop_{G'}(v) + U(G') - deg_{\min}(G') + 1$. (Case 2) If $U(G') < deg_{\min}(G')$ and $U(G') + |EV(V')| \geq deg_{\min}(G')$, $Fdist(G') = Hop_{G'}(v) + 1$. (Case 3) If $U(G') + |EV(V')| \leq deg_{\min}(G')$, the sub-graph G' cannot be extended to form a quasi-clique and $Fdist(G') = -1$.

Example 8 (Case 1): As shown in Figure. 11, let γ and the target vertex be 0.2 and v, respectively. The sub-graph $G' = (V', E')$, where $V' = \{v, v_2, v_3, v_4, v_5\}$ and $E' = \{(v, v_2), (v, v_4), (v_2, v_3), (v_2, v_5), (v_3, v_5), (v_4, v_5)\}$ is a quasi-clique $QC(0.2, v)$. The extending vertices of G' are v_3 and v_5. By Lemma4, $U(G') = \lfloor deg_{\min}(G') / \gamma \rfloor + 1 - |V'| = \lfloor 2 / 0.2 \rfloor + 1 - 5 = 6$. Since $U(G')$ is larger than $deg_{\min}(G')$, there are enough new vertices able to connect to the last vertex w to let $N_{G''}(w) \geq deg_{\min}(G')$. Therefore, $Fdist(G') = dist_G(v, w) = Hop_{G'}(v) + U(G') - deg_{\min}(G') + 1 = 7$.

Example 9 (Case 2): As shown in Figure. 12, let γ and the target vertex be 0.4 and v, respectively. The sub-graph $G' = (V', E')$, where $V' = \{v, v_2, v_3, v_4, v_5\}$ and $E' = \{(v, v_2), (v, v_4), (v_2, v_3), (v_2, v_5), (v_3, v_5), (v_4, v_5)\}$ is a quasi-clique $QC(0.4, v)$. The extending vertices of G' are v_3 and v_5. $|EV(V')| = 2$. $deg_{\min}(G') = 2$ is bigger than $U(G') = \lfloor 2 / 0.4 \rfloor +1 - 3 = 1$ and less than $U(G') + |EV(V')| = 3$. Since there are not enough new vertices able to connect to the last vertex w, w needs to connect to the extending vertices of G' to let $|N_{G''}(w)| \geq deg_{\min}(G')$. Therefore, $Fdist(G') = Hop_{G'}(v) + 1 = 2$.

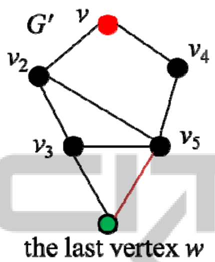


Figure 12: (Left) G' is a $QC(0.4, v)$.

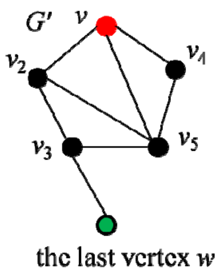


Figure 13: (Right) G' is a $QC(0.4, v)$.

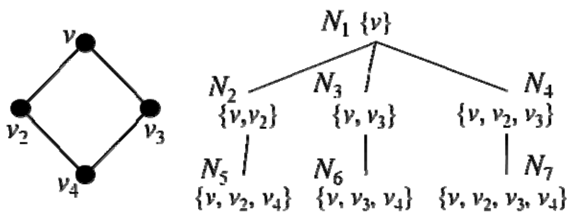


Figure 14: The search tree for $\{v, v2, v3, v4\}$.

Example 10 (Case 3): As shown in Figure. 13, let γ and the target vertex be 0.4 and v, respectively. The sub-graph $G' = (V', E')$, where $V' = \{v, v_2, v_3, v_4, v_5\}$ and $E' = \{(v, v_2), (v, v_4), (v, v_5), (v_2, v_3), (v_2, v_5), (v_3, v_5), (v_4, v_5)\}$ is a quasi-clique $QC(0.4, v)$. The extending vertex of G' is v_3 and $U(G') = \lfloor 2 / 0.4 \rfloor + 1 - 3 = 1$. $deg_{\min}(G') = 2$ is equal to $U(G') + |EV(V')|$. Since there are not enough neighbors of w in G', the sub-graph G' cannot be extended to form a larger quasi-clique. Therefore, we set $Fdist(G') = -1$.

Algorithm 1: The Target-Extending algorithm.

Input: A graph $G = (V, E)$, a target vertex v_p, and a parameter γ.
Output: A result list RL, the set of maximal quasi-cliques of v_p with respect to γ in G.

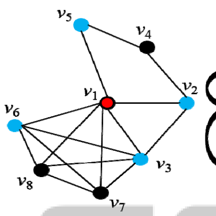
```
Keep G into a two-dimensional array D[|V|][|V|].
D[i][j] = 1 means v_i and v_j are adjacent.
RL = ϕ and dist = 0
Put v_p into the vertex set A
for j = 1 to |V| do
  if D[p][j] = 1 then
    Put v_j into the set of candidate vertices CV(A).
Put v_p into the set of extending vertices EV(A)
Recursive function RF(A, CV(A), EV(A), dist)
  Compute the upper bound U(A) from Lemma 4
  Compute the lower bound L(A) from Lemma 2
  Select the critical vertices for A and put into
  CritV(A) by Lemma 3
  for each vertex v_u in CritV(A) do
    if v_u ∈ (A − EV(A))
      Return
  Compute Fdist(A) by Lemma 5.
  if |A| = the maximal size A may have (from
  Lemma 4) and dist > Fdist(A) then
    return
  else
    Put CritV(A) into A
    for i = L(A) to U(A) do
      From EV(A) we choose i vertices to form a
      combination and in this selection, at least one
      vertex in CritV(A) should be contained. All
      of these combinations generated are
      individually merged with A and then put into
      S.
    for each vertex set U in S do
      if U is a quasi-clique QC(γ, p) then
        Add U to RL and update CV(U) and EV(U)
        RF(U, CV(U), EV(U), dist + 1)
Return RL
```

Lemmas 2-5 can be used in our methods to reduce the search space. The corresponding pruning strategies are named Strategies 2-5. Since we only focus on the maximal quasi-cliques, a sub-graph $G' = (V', E')$ need not be checked whether it is a $QC(\gamma, v)$ if we can find another quasi-clique G'' to contain G' earlier. Therefore, we verify the sub-graphs with the larger sizes and extend them in the search tree as early as possible to find the large enough quasi-clique quickly. This strategy is called Strategy 6 in the following discussion. As shown in Figure. 14, we first check whether the sub-graph corresponding to N_4 is a quasi-clique, and then move to a larger sub-graph corresponding to N_7. If the sub-graph corresponding to N_7 is a $QC(\gamma, v)$, then the sub-graphs contained in $\{v, v_2, v_3, v_4\}$ need not be checked as they have no chances to be the maximal quasi-cliques. By combining the baseline algorithm with Strategies 2-6, the Target-Extending algorithm is proposed. The pseudo code of this algorithm is shown in Algorithm 1.

4.3 A Special Case

The quasi-clique G' is a complete graph when γ is equal to 1. We only need to focus on the cliques containing the target vertex in G. In fact, all vertices in the cliques are the 1-hop neighbors of the target vertex in G. We design another algorithm for the special case on $\gamma = 1$, based on this concept.

4.3.1 The Target-clique Algorithm



A_i	$CS(A_i)$
$A_1\{v_1\}$	$v_3(5), v_6(4), v_2(3), v_5(2)$
$A_2\{v_1, v_3\}$	$v_6(4), v_2(3)$
$A_3\{v_1, v_5\}$	ϕ
$A_4\{v_1, v_3, v_6\}$	ϕ
$A_5\{v_1, v_3, v_2\}$	ϕ

Figure 15: The illustration of the Target-Clique Algorithm.

Given a graph $G = (V, E)$ and a target vertex $v \in V$, first, we put v into a vertex set A_1 and put the neighbors of v in G to the candidate set $CS(A_1)$. Each vertex in $CS(A_1)$ has a corresponding flag cv equal to 0 initially, which shows whether the vertex is checked. The vertices in $CS(A_1)$ are sorted in a descending order of their degree in G. Second, we select a vertex u with cv equal to 0 from the first of the sorted $CS(A_1)$, put it into a new vertex set A_2, and merge A_2 with A_1. Then, we create a new candidate set $CS(A_2)$ which collects vertices adjacent to u in $CS(A_1)$. Those vertices are the common neighbors of u and v in G. cv of u is set to 1 in $CS(A_1)$.

Algorithm 2: The Target-Clique algorithm.

```
Input: A graph G = (V, E), a target vertex v_p
Output: A result list RL, the set of maximal quasi-cliques
of v_p with respect to γ in G.
Keep G into a two-dimensional array D[|V|][|V|].
D[i][j] = 1 means v_i and v_j are adjacent.
RL = φ
Put v_p into the vertex set A1
for j = 1 to |V| do
  if D[p][j] == 1 then
    Put v_j into CS(A1) and set v_j.cv = 0
Recursive function RF(A1, CS(A1))
  if |CS(A1)| ≤ 0 then
    Put A1 ∪ CS(A1) into RL and return
  else
    Sort all vertices in CS(A1) into a decreasing
    order of degree in G
    for each vertex v_i in CS(A1) do
      if v_i.cv = 0 then
        Copy A1 and v_i into a new vertex set A2
        Set v_i.cv = 1
      for each vertex v_j in CS(A1) do
        if D[i][j] = 1 then
          Add the vertex v_j into CS(A2)
    RF(A2, CS(A2))
    return RL
```

We repeat the second step and create the new vertex set A_i until all the corresponding values become 1 in $CS(A_1)$. A_i merging with $CS(A_i)$ is a clique that we want. We need not check whether the obtained cliques are contained by some others because this case will not be produced by our method. The pseudo codes of the Target-Clique algorithm are shown in Algorithm 2.

Example 11: As shown in Figure. 15, given a graph $G = (V, E)$, let the target vertex be v_1, the table shows the steps of the Target-Clique algorithm. We add v_1 to A_1 and $CS(A_1)$ collects the neighbors of v_1 in G. Thereafter, $CS(A_1)$ is sorted according to the degree to have the order list of $<v_3, v_6, v_2, v_5>$. We select v_3 to join A_1 to form A_2 and $CS(A_2)$ collects the common neighbors of v_1 and v_3 in G, that is $<v_6, v_2>$. The vertex set A_2 is not an answer if $CS(A_2)$ contains more than one vertex. Then, we select v_6 to join with A_2 to form A_4 but $CS(A_4)$ is empty. The vertex set A_4 is a clique we demand because there are no common neighbors of v_1, v_3 and v_6, and A_4 is not contained by any other clique. Finally, we obtain three maximal cliques $\{v_1, v_5\}$, $\{v_1, v_3, v_6\}$, and $\{v_1, v_3, v_2\}$, which contain the target vertex v_1 in the graph G.

5 EXPERIMENTS

In this section, a series of experiments are performed to evaluate our approach and the experiment results are also presented and analyzed.

5.1 Experiment Setup

Table 1: The description of the experiment factors.

Factors	Default	Range	Description
number of vertices	5K	4K-8K	number of vertices in the graph
average degree	20	5-25	average degree of vertices (the first dataset)
average degree	300	100-500	average degree of vertices (the second dataset)
Γ	0.5	0.1-0.9	parameter of quasi-cliques

Table 2: The description of the real data.

Vertices	Edges	Average degree	Maximum degree
8,298	100,764	24	743

Since there are no approaches focusing on finding maximal quasi-cliques from a graph, which contain a specific target vertex, we compare the proposed algorithms with the Quick algorithm. We use two synthetic datasets for testing the proposed algorithms. The first dataset is used to test the methods for quasi-cliques and the second dataset is used to test the Target-Clique algorithm. To generate a synthetic graph $G = (V, E)$, we first generate a sufficient amount of vertices and randomly add edges between any two vertices to make the sum of edges equal to $N \times D / 2$, were N is the number of vertices and D is the average degree of vertices, both of which are experiment factors. All of the experiment factors are descripted in Table 1. Moreover, we also use a real dataset named *Wikipedia vote network* in the experiments, which is related to a social network graph and obtained from Stanford Large Network Dataset Collection (https://snap.stanford.edu/data/). Its description is shown in Table 2. All of the proposed algorithms are implemented using C++ and performed on a PC with the Intel i5-3210M 2.50GHz CPU, 8 GB of memory, and under the windows7 64bits operating system. To obtain a result point shown in the experiment, we perform the process ten times to compute the average value. For easily showing the experiment results, we use a few symbols to indicate the baseline algorithm and pruning lemmas. For example, the baseline algorithm plus Strategy 2 and Strategy 3 is denoted B+23.

5.2 Experiment Results

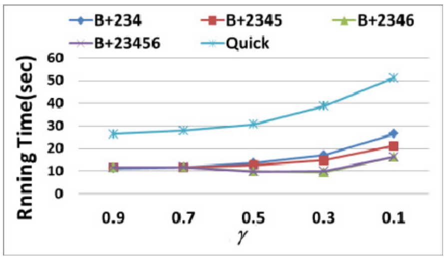


Figure 16: The running time on varying γ (the first synthetic dataset).

The running time of the methods for quasi-cliques on the synthetic dataset is shown in Figures. 16-19. The running time on varying γ is shown in Figure. 16. As can be seen, our method is always better than the modified Quick algorithm. The pruning strategy from Lemma 5 works well when γ is small. Since the large quasi-cliques may be found quickly, we can ignore numerous small sub-graphs contained by the large quasi-cliques. The larger γ is, the more the sub-graphs need to be checked whether they are contained by other quasi-cliques, reducing the pruning capability of Lemma 5. Similarly, when Strategy 6 is used, finding the large quasi-clique in the very beginning can reduce the needing of checking sub-graphs, making the running time to be further reduced.

The running time on varying the average degree of vertices is shown in Figure. 17. While the average degree of vertices increases, the running time of our method and that of the modified Quick algorithm both exponentially grow. Under the condition of the small average degree, our method is better than the modified Quick algorithm. This is because the modified Quick algorithm needs to consider the combinations of the target vertex and the other vertices in the first layer of the depth-first search tree. However, we only consider the combinations of the neighbors of the target vertex. Accordingly, we generate fewer combinations. The running time on varying the number of vertices is shown in Figure. 18. The number of vertices causes little impact to the running time of our method, since more vertices connecting to the target vertex need to be considered with the growth of the total number of vertices.

The running time on the real data is shown in Figure. 19. As can be seen, our method is still better than the modified Quick algorithm. The pruning strategy from Strategy 6 works well in this dataset. In the experiments, we can see that B+23456 outperforms the modified Quick algorithm. In most cases, the pruning capability of Strategy 6 is better than that of Lemma 5.

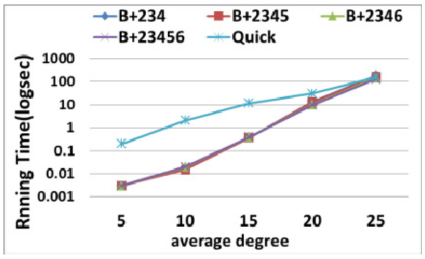


Figure 17: The running time on varying average degree (the first synthetic dataset).

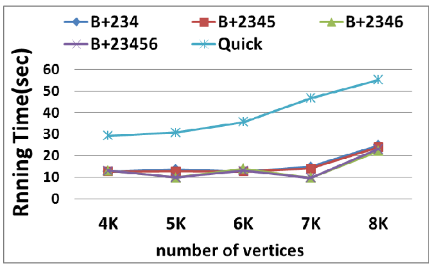


Figure 18: The running time on varying number of vertices (the first synthetic dataset).

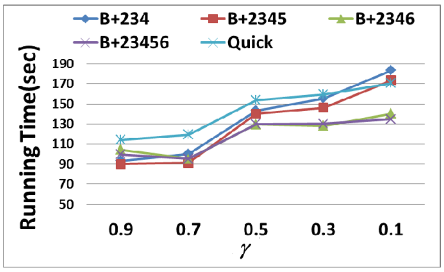


Figure 19: The running time of the real dataset.

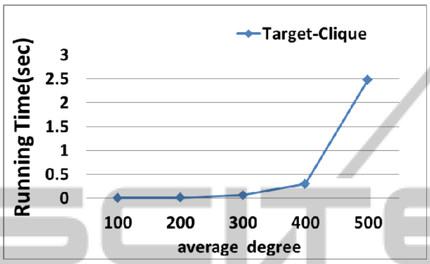


Figure 20: The running time on varying γ (the second synthetic dataset).

The running time of the Target-Clique algorithm on the second synthetic dataset is shown in Figures. 20-21. The running time will exponentially grow with the increase of the average degree. The number of the possible vertex combinations will increase with the increase of the vertices. However, the number of vertices will not significantly affect the running time of the Target-Clique algorithm. This is because we only consider the neighbors of the target vertex, which may or may not be affected by the number of vertices.

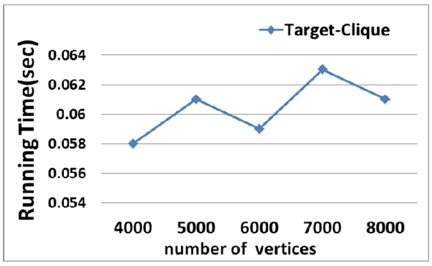


Figure 21: The running time of varying number of vertices (the second synthetic dataset).

6 CONCLUSIONS

In this paper, we solve the problem of finding maximal quasi-cliques for a target vertex. Given a graph $G = (V, E)$, a parameter $\gamma \in (0, 1]$ and a target vertex $v \in V$, we find all of the maximal quasi-cliques of v with respect to γ in G. We propose an algorithm to solve this problem and use five pruning techniques to improve the performance. These techniques compute the maximum size and minimum size of each sub-graph of G based on the degrees of relevant vertices. The containment relations between sub-graphs are also considered, thus making most of the sub-graphs to be pruned before quasi-clique checking. Moreover, we modify the Quick algorithm (Liu, 2008) to solve our problem for a comparison with our method. The experiment results, using a real and two synthetic datasets, demonstrate that the pruning techniques are effective and our algorithm outperforms the modified Quick algorithm in most cases.

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