Freezing Method Approach to an Asymptotic Stability of the Discrete-time Oscillator Equation

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Abstract: The presented research work considers stability criteria of second-order differential equation. The second-order discrete-time oscillator equation is obtained from discretization of second order continuous-time equation using the forward difference operator. The stability criteria are drawn with freezing method and are presented in the terms of the equation coefficients. Finally, an illustrative example is shown.

1 INTRODUCTION

In the literature, there are known research work describing a model called the simple harmonic oscillator (Buonomo and di Bello, 1996), (Hyland and Bernstein, 1987). The simple harmonic oscillator is expressed by the formula:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = 0, \tag{1}$$

where:

- $y(t)$ is a measure of the displacement from the equilibrium point at a given time;
- m is the mass;
- k is the spring parameters;
- c is the friction parameter.

The generalization of the equation (1) is a damped linear oscillator defined by the following formula:

$$\ddot{y}(t) + a(t)\dot{y}(t) + \omega^2 y(t) = 0, \tag{2}$$

where:

- the spring constant ω is positive;
- the damping coefficient $a(t)$ is continuous and nonnegative for $t \geq 0$.

The equation (2) is one of the most famous model using to describe many physical phenomena (Sugie and et al, 2012). Moreover, this equation is also known (Smith, 1961) as:

$$\ddot{y}(t) + a(t)\dot{y}(t) + y(t) = 0. \tag{3}$$

In (Graef and Karsai, 1996), the damped impulsive equation is analysed and authors studied the stability of this model. Instead, authors of (Yan and Zhao, 1998) researched the stability of oscillator equation with delays. In this case, the mathematical model was described by the first-order linear delay impulsive differential equation.

In (Zhang and Cheng, 1995) researchers focuses on several oscillation criteria for a related neutral first-order difference equation with delay. Authors of (Yu and Cheng, 1994), (Kulikov, 2010), (Yu, 1998) consider stability criteria of the first-order difference equation with various kind of delays. They use, inter alia, the Lyapunov function method.

In our research, we concentrate on stability of second-order difference equation which we obtain by a certain discretization of the equation (3).

For the discretization of the equation (3), it has been used the so-called forward difference operator and denoted by Δ. Its formal definition is as follows (Agarwal and et al, 2005):

Definition 1. *The first forward difference operator is expressed by following formula*

$$\Delta x(k) = x(k+1) - x(k) \tag{4}$$

and the second forward difference operator Δ^2 is defined as

$$\Delta^2 x(k) = \Delta(\Delta x(k)) = \Delta x(k+1) - \Delta x(k). \tag{5}$$

Using the Definition 1, the equation (3) can be discretized in the following way (see (Agarwal and et al, 2005), page 3):

$$\Delta(\Delta(y(n))) + a(n)\Delta(y(n)) + y(n) = 0. \tag{6}$$

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Subsequently, the forward difference is used and we get:

$$y(n+2)+y(n+1)(a(n)-2)- \\ y(n)(a(n)-2)=0. \quad (7)$$

$$y(n+1)+y(n)(a(n-1)-2)- \\ y(n-1)(a(n-1)-2)=0. \quad (8)$$

Substituting

$$q(n)=a(n-1)-2, \quad (9)$$

the equation (8) will obtained the following form:

$$y(n+1)=-q(n)y(n)+q(n)y(n-1). \quad (10)$$

The main objective of our further consideration is the stability problem for the equation (10) with initial conditions $y(0)=y_0$, $y(1)=y_1$, where $(q(n))_{n\in\mathbb{N}}$ is a sequence of real numbers. We will say that equation (10) is asymptotically stable if for all initial conditions (y_0,y_1) the corresponding solution tends to zero. In the paper we will use the following notation:

- $\mathbb{R}^{s\times s}$ is the set of all s by s real matrices;
- I_s is the identity matrix of size s;
- $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm;
- A^T means the transposition of the matrix A.

The idea, which we use to obtain stability criteria for the equation (10) is called in the literature (Bylov and et al, 1966), (Desoer, 1970), (Gil and Medina, 2001), (Medina, 2008), (Czornik and Nawrat, 2010) the freezing method. For the discrete linear time-varying system given by

$$x(n+1)=D(n)x(n) \quad (11)$$

the main conception of the freezing method is to freeze the matrices $D(n)$ and consider system

$$x(l+1)=D(n)x(l). \quad (12)$$

The stability of all equations (12) usually does not imply the stability of the system (11) (see (Ge and Sun, 2005), (Czornik and Nawrat, 2006)). The stability criteria for (11) are usually a combination of the assumption of the stability of the system (12) and constraints on variation of the parameter $D(n)$ of (11). The above-mentioned approach is presented in (Desoer, 1970), (Gil and Medina, 2001) and (Czornik and Nawrat, 2010).

2 MAIN RESULT

Let us introduce the following notation

$$A(n)=\begin{bmatrix} -q(n) & q(n) \\ 1 & 0 \end{bmatrix} \quad (13)$$

and

$$B(q)=\begin{bmatrix} -q & q \\ 1 & 0 \end{bmatrix}. \quad (14)$$

Moreover for the solution $(y(n))_{n\in\mathbb{N}}$ of (10) denote

$$x(n)=\begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \quad (15)$$

for $n=1,2,\ldots$. With this notation equation (10) may be rewritten in the following form

$$x(n+1)=A(n)x(n) \quad (16)$$

with initial condition

$$x(0)=\begin{bmatrix} y(1) \\ y(0) \end{bmatrix}. \quad (17)$$

If for all initial conditions $x(0)$ the solution of (16) tends to zero, then we will called (16) asymptotically stable. In the paper we will use the following facts.

Lemma 1. *(Khalil, 1995). If for (16) there exists a function $V:\mathbb{N}\times\mathbb{R}^2\to[0,\infty)$ such that*

1.

$$\|x\|^2\le V(n,x)\le C_1\|x\|^2$$

2.

$$V(n,x(n))-V(n,x(n+1))\le -C_2\|x(n)\|^2$$

for all $x\in\mathbb{R}^2$, $n\in\mathbb{N}$ and certain positive C_1, C_2, then (16) is asymptotically stable. The function V is called the Lyapunov function.

Lemma 2. *(Gajic and et al, 1995). For a matrix $A\in\mathbb{R}^{s\times s}$ the following conditions are equivalent:*

1. *matrix $A\in\mathbb{R}^{s\times s}$ has all eigenvalues in the open unit circle;*
2. *for each positive definite matrix $Q\in\mathbb{R}^{s\times s}$ there exists a positive definite matrix $P\in\mathbb{R}^{s\times s}$ such that the following Lyapunov equation is satisfied*

$$A^TPA-P=-Q; \quad (18)$$

3. *there are positive definite matrices P, $Q\in\mathbb{R}^{s\times s}$ such that (18) is satisfied.*
   Moreover if $A\in\mathbb{R}^{s\times s}$ has all eigenvalues in the unit circle then the solution of (18) is given by

$$P=\sum_{k=0}^{\infty}\left(A^T\right)^kQA^k. \quad (19)$$

Lemma 3. *(Horn and Johnson, 1985). For any matrix $A \in \mathbb{R}^{s\times s}$, $A = [a_{ij}]_{i,j=1,\ldots,s}$ we have*

$$\|A\| \leq s \max_{i,j=1,\ldots,s} |a_{ij}|. \tag{20}$$

Lemma 4. *All the eigenvalues of $B(q)$ lies in the open unit circle if and only if $q \in \left(-1, \frac{1}{2}\right)$ and then*

$$\|B(q)\| < \frac{\sqrt{5}+1}{2}. \tag{21}$$

Proof. Consider the Lyapunov equation (18) with $A = B(q)$ and $Q = I_2$ then the solution is given by

$$P = \frac{1}{2q^2+q-1}\left[\begin{array}{cc} 2q-2 & 2q^2 \\ 2q^2 & 2q^3+q-1 \end{array}\right]. \tag{22}$$

It is easy to check that the solution is positive definite if and only if $q \in \left(-1, \frac{1}{2}\right)$. The first conclusion follows now from point 3 of Lemma 2. We have

$$\|B(q)\| = \sqrt{q^2 + \frac{1}{2}\sqrt{4q^4+1} + \frac{1}{2}} \tag{23}$$

and using the standard method of calculus we may verify that the greatest value of the function $f : \left[-1, \frac{1}{2}\right] \to R$,

$$f(x) = \sqrt{x^2 + \frac{1}{2}\sqrt{4x^4+1} + \frac{1}{2}} \tag{24}$$

is

$$f(-1) = \frac{\sqrt{5}+1}{2}. \tag{25}$$

□

Lemma 5. *If*

$$\varepsilon \in \left(0, \frac{3}{4}\right), \tag{26}$$

$$q \in \left(-1+\varepsilon, \frac{1}{2}-\varepsilon\right) \tag{27}$$

and $P(q)$ is the solution of the Lyapunov equation (18) with $A = B(q)$ and $Q = I_2$, then

$$1 \leq \|P(q)\| \leq \frac{8-4\varepsilon}{-2\varepsilon^2+3\varepsilon}. \tag{28}$$

Proof. The most left inequality follows from the formula (19). As we noticed in the proof of Lemma (4) $P(q)$ is given by (22).
Since for

$$q \in \left(-1+\varepsilon, \frac{1}{2}-\varepsilon\right), \tag{29}$$

$$\left|2q^2+q-1\right| > -2\varepsilon^2+3\varepsilon, \tag{30}$$

therefore

$$\|P(q)\| \leq \frac{1}{-2\varepsilon^2+3\varepsilon}\left\|\begin{bmatrix} 2q-2 & 2q^2 \\ 2q^2 & 2q^3+q-1 \end{bmatrix}\right\|. \tag{31}$$

Using Lemma 3 we have

$$\|P(q)\| \leq \frac{2}{-2\varepsilon^2+3\varepsilon} \cdot \left[\max_{-1+\varepsilon<q<\frac{1}{2}-\varepsilon}\left\{|2q-2|, 2q^2, \left|2q^3+q-1\right|\right\}\right] = \frac{8-4\varepsilon}{-2\varepsilon^2+3\varepsilon}. \tag{32}$$

□

Theorem 6. *If for certain $\varepsilon \in \left(0, \frac{3}{4}\right)$ and $\eta \in (0,1)$ the sequence $(q(n))_{n\in\mathbb{N}}$ satisfies the following two conditions:*

1.

$$q(n) \in \left(-1+\varepsilon, \frac{1}{2}-\varepsilon\right) \tag{33}$$

2.

$$|q(n)-q(n-1)| \leq \frac{(1-\eta)\left(-2\varepsilon^2+3\varepsilon\right)^2}{\sqrt{2}\left(\sqrt{5}+1\right)(8-4\varepsilon)^2} \tag{34}$$

then the equation (10) is asymptotically stable.

Proof. Consider the solution $P(n)$ of (18) with $A = A(n-1)$ and $Q = I_2$. We will show that

$$V(n,x) = x^T P(n) x \tag{35}$$

is the Lyapunov function for (16). From Lemma 5 we know that

$$\|x\|^2 \leq V(n,x) \leq \alpha \|x\|^2, \tag{36}$$

where

$$\alpha = \frac{8-4\varepsilon}{-2\varepsilon^2+3\varepsilon}. \tag{37}$$

Let us estimate

$$R(n) = P(n+1) - P(n). \tag{38}$$

We have

$$A^T(n)R(n)A(n) - R(n) = -Q(n), \tag{39}$$

where

$$Q(n) = \left(A^T(n) - A^T(n-1)\right)P(n)A(n) + A^T(n-1)P(n)\left(A(n)-A(n-1)\right) \tag{40}$$

and according to (19)

$$R(n) = Q(n) + \sum_{k=1}^{\infty}\left(A^T(n)\right)^k Q(n) A^k(n). \tag{41}$$

Because

$$\left(A^T(n)\right)^k Q(n) A^k(n) \leq \|Q(n)\| \left(A^T(n)\right)^k A^k(n) \tag{42}$$

and by Lemmas 4 and 5

$$\|Q(n)\| \leq 2\|A(n) - A(n-1)\|\alpha\frac{\sqrt{5}+1}{2}, \quad (43)$$

then from (41) we have

$$R(n) \leq Q(n) + \|Q(n)\| \sum_{k=1}^{\infty} \left(A^T(n)\right)^k A^k(n) \leq$$

$$\|Q(n)\| I + \|Q(n)\| \sum_{k=1}^{\infty} \left(A^T(n)\right)^k A^k(n) =$$

$$\|Q(n)\| \left[I + \sum_{k=1}^{\infty} \left(A^T(n)\right)^k A^k(n)\right]. \quad (44)$$

By the definition of $P(n)$ we know that

$$P(n) = I + \sum_{k=1}^{\infty} \left(A^T(n)\right)^k A^k(n). \quad (45)$$

From (44) and (45) we obtain

$$\|R(n)\| \leq \|Q(n)\| \|P(n)\|. \quad (46)$$

Using (28) and (43) we may estimate $R(n)$ as follows

$$\|R(n)\| \leq \|A(n) - A(n-1)\| \alpha^2 \left(\sqrt{5}+1\right) =$$

$$\sqrt{2}|q(n) - q(n-1)| \alpha^2 \left(\sqrt{5}+1\right) \leq 1 - \eta. \quad (47)$$

The last inequality implies that

$$V(n+1, x(n+1)) - V(n, x(n)) \leq$$

$$-\eta \|x(n)\|^2. \quad (48)$$

Inequalities (36) and (48) show that $V(n,x)$ is the Lyapunov function for (16). By Lemma 1 we conclude that (16) is asymptotically stable what implies that (10) is asymptotically stable. □

Example 1. *Consider equation (10) with*

$$q(n) = \frac{\sin(\ln(\ln(n+12)))}{r}, \quad (49)$$

where $r > 0$. *Using Theorem 6 we will find the values of* r *such that the conditions (33) and (34) are satisfied with* $\varepsilon = 0.39$ *and certain* $\eta > 0$, *i.e.*

$$q(n) \in (-0.61, 0.11) \quad (50)$$

and

$$|q(n) - q(n-1)| \leq (1-\eta) \cdot 3.9494 \times 10^{-3} \quad (51)$$

with $\eta > 0$. *According to Lagrange theorem*

$$|q(n) - q(n-1)| = \left|f'(c)\right|, \quad (52)$$

where $c \in (n-1, n)$ *and*

$$f(x) = \frac{\sin(\ln(\ln(x+12)))}{r}. \quad (53)$$

Since

$$\frac{d}{dx}\frac{\sin(\ln(\ln(x+12)))}{r} =$$

$$\frac{1}{\ln(x+12)}\frac{\cos(\ln(\ln(x+12)))}{12r+rx}, \quad (54)$$

then

$$\left|f'(c)\right| \leq \frac{1}{(12r+rc)\ln(c+12)}. \quad (55)$$

It is easy to verify that for all $c > 0$ *and* $r > 8.5$ *there exists* $\eta > 0$ *such that*

$$\frac{1}{(12r+rc)\ln(c+12)} <$$

$$(1-\eta) \cdot 3.9494 \times 10^{-3}. \quad (56)$$

It is also clear that for $r > 10$ *the condition (50) is satisfied. Finally we conclude from Theorem 6 that equation (10) with* $q(n)$ *given by (49) is asymptotically stable.*

3 CONCLUSIONS

In this paper we have obtained the asymptotic stability criteria for discrete version of damped linear oscillator. These criteria are obtained using the freezing method. Typical situation in this method is that the stability condition is a combination of requirements about eigenvalues of certain matrices and variation of the parameters. In our case we were able to present the conditions in the terms of the original parameters of the equation (10) only.

In further work, the authors intend to study the stability of various type of equation (10) using different kind of exponents (Czornik et al., 2010a), (Czornik et al., 2010b), (Czornik and Niezabitowski, 2013a), (Czornik et al., 2013), (Czornik and Niezabitowski, 2013b), (Babiarz et al., 2015).

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