Diagonal Stability of Uncertain Interval Systems

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Abstract: In this paper we consider the problem of diagonal stability of interval systems. We investigate the existence and evaluation of a common diagonal solution to the Lyapunov and Stein matrix inequalities for third order interval systems. We show that these problems are equivalent to minimax problem with polynomial goal functions. We suggest an interesting approach to solve the corresponding game problems. This approach uses the opennes property of the set of solutions. Examples show that the proposed method is effective and sufficiently fast.

1 INTRODUCTION

Consider state equation

$$\dot{x} = Ax$$

where $x = x(t) \in \mathbb{R}^n$ and $A = (a_{ij})\ (i, j = 1, 2, \ldots, n)$ is $n \times n$ matrix. In many control system applications each entry a_{ij} can vary independently within some interval. Such systems are called interval systems. In other words $\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}$ where $\underline{a}_{ij}$, $\overline{a}_{ij}$ are given. Denote the obtained interval family by $\mathcal{A}$, i.e.

$$\mathcal{A} = \{A = (a_{ij}) : \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, (i, j = 1, 2, \ldots, n)\}. \qquad (1)$$

Interval matrices have many engineering applications. Due to its natural tie with robust control system analysis and design, several approach have involved for the stability analysis of interval matrices (see (Barmish, 1994; Rohn, 1994; Bhattacharyya et al., 1995; Liberzon and Tempo, 2004; Pastravanu and Matcovschi, 2015; Yıldız et al., 2014)).

We are looking for the existence and evaluation of a common diagonal Lyapunov function which guarantees diagonal stability of interval systems. In other words we investigate the problem of existence of a diagonal matrix $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ with $x_i > 0$ and with the property

$$A^T D + DA < 0 \quad \text{for all} \quad A \in \mathcal{A} \qquad (2)$$

where the symbol "T" stands for the transpose and "$<$" means negative definiteness.

Diagonal stability have many applications and this problem has been considered in many works (see (Arcat and Sontag, 2006; Johnson, 1974; Ziolko, 1990; Kaszkurewicz and Bhaya, 2000; Khalil, 1982; Pastravanu and Matcovschi, 2015; Oleng and Narendra, 2003; Büyükköroğlu, 2012; Yıldız et al., 2014) and references therein).

An algebraic characterization of necessary and sufficient conditions for the existence of a diagonal Lyapunov function for a single third order matrix has been derived in (Oleng and Narendra, 2003). The algorithm submitted in (Pastravanu and Matcovschi, 2015) for a common diagonal solution of interval matrix family is not effective since it uses complicated bilinear matrix inequalities and the solver PENBMI.

2 COMMON DIAGONAL SOLUTION FOR 3×3 INTERVAL SYSTEMS

In this section for 3×3 interval family we give necessary and sufficient condition for the existence of Hurwitz common diagonal solution and the corresponding solution algorithm.

Consider 3×3 interval family

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$$\mathcal{A} = \left\{ A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} : \right.$$
$$\left. a_i \in [a_i^-, a_i^+],\ (i = 1, 2, \ldots, 9) \right\}. \quad (3)$$

Without loss of generality all 3×3 positive diagonal matrices $\operatorname{diag}(x_1, x_2, x_3)$ with $x_i > 0$ $(i = 1, 2, 3)$ may be normalized to have the form

$$D = \operatorname{diag}(t, 1, s) = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$$

with $t > 0$, $s > 0$.

Problem 1. Is there $D = \operatorname{diag}(t, 1, s)$ with $t > 0$, $s > 0$ such that

$$A^T D + DA < 0 \quad (4)$$

for all $a_i \in [a_i^-, a_i^+]$ $(i = 1, 2, \ldots, 9)$.

We write

$$A^T D + DA = \begin{bmatrix} 2ta_1 & ta_2 + a_4 & ta_3 + sa_7 \\ ta_2 + a_4 & 2a_5 & sa_8 + a_6 \\ ta_3 + sa_7 & sa_8 + a_6 & 2sa_9 \end{bmatrix}$$

The matrix inequality (4), i.e. the negative definiteness of $A^T D + DA$ is equivalent to the following

- ***i)*** $a_1 < 0$
- ***ii)*** $(a_2 t + a_4)^2 - 4a_1 a_5 t < 0$
- ***iii)*** $d_0(t, a_1, \ldots, a_9) + d_1(t, a_1, \ldots, a_9)s + d_2(t, a_1, \ldots, a_9)s^2 < 0$.

The functions d_i $(i = 1, 2, 3)$ are low order polynomials and can be explicitly evaluated.

i) is satisfied for all $a_1 \in [a_1^-, a_1^+]$ if and only if $a_1^+ < 0$. The problem of existence of a common t satisfying *ii*) for all (a_1, a_2, a_4, a_5) is equivalent to the existence of a common diagonal solution for 2×2 family $\begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix}$ and has been investigated in (Yıldız et al., 2014). There whole interval of common t (in the case of nonempty) has been calculated. If this interval is empty then there is no common $D = \operatorname{diag}(t, 1, s)$ satisfying (4). Assume that this interval (α, β) of common t is nonempty. Then the existence of a common $D = \operatorname{diag}(t, 1, s)$ means that there exist $t \in (\alpha, \beta)$ and $s > 0$ such that *iii*) is satisfied for all $(a_1, a_2, \ldots, a_9)$. This problem is a game problem. Indeed denote the left-hand side of *iii*) by $f(t, s, a_1, \ldots, a_9)$. Then *iii*) is equivalent to the following minimax inequality

$$\inf_{t \in (\alpha,\beta), s > 0} \ \max_{(a_1, \ldots, a_9)} f(t, s, a_1, \ldots, a_9) < 0. \quad (5)$$

Solve the game problem (5) is difficult in general, this game has no a saddle point due to nonconvexity of the function f.

We suggest the following interesting approach to solve (5) numerically. This approach is based on the openness property of the solution set of (4) . In other words the following proposition is true.

Proposition 2.1. *If there exist a common* $D = \operatorname{diag}(t_*, 1, s_*)$ *then there exist intervals* $[t_1, t_2]$ *and* $[s_1, s_2]$ *which contain* t_* *and* s_* *respectively such that the matrix* $D = \operatorname{diag}(t, 1, s)$ *is a common solution for all* $t \in [t_1, t_2]$, $s \in [s_1, s_2]$.

Due to this proposition we suggest the following algorithm for a common diagonal solution.

Algorithm 1. *Let the interval family (3) be given.*

- ***i)*** *Using the results on* 2×2 *interval systems from (Yıldız et al., 2014) calculate the interval* (α, β) *for* t.
- ***ii)*** *Determine an upper bound* $\bar{s}$ *for the variable s from the positive definiteness condition of a suitable submatrix of* $-(A^T D + DA)$.
- ***iii)*** *Divide the interval* $[\alpha, \beta]$ *into k equal parts* $[\alpha_i, \beta_i]$ *and the interval* $[0, \bar{s}]$ *into m equal parts* $[s_j^-, s_j^+]$.
- ***iv)*** *On each box*

  $$[\alpha_i, \beta_i] \times [s_j^-, s_j^+] \times [a_1^-, a_1^+] \times \cdots \times [a_9^-, a_9^+]$$

  consider the maximization of the polynomial function $f(t, s, a_1, \ldots, a_9)$.
  If there exist indices i_* *and* j_* *such that the maximum is negative then stop. The whole interval* $[\alpha_{i_*}, \beta_{i_*}] \times [s_{j_*}^-, s_{j_*}^+]$ *defines family of common diagonal solutions.*

As can be seen the above game problem (5) is reduced to a finite number of maximization problems in which low order multivariable polynomials are maximized over boxes. These optimizations can be carried out by Maple program or by the Bernstein expansion. The following examples shows that Algorithm 1 is sufficiently effective.

Example 2.1. Consider the interval family

$$\begin{bmatrix} -4 & q_1 & 1 \\ 1 & -4 & q_2 \\ q_3 & 1 & -5 \end{bmatrix}$$

where $q_1 \in [2, 3]$, $q_2 \in [1, 2]$ and $q_3 \in [1, 2]$. We obtain

$$A^T D + DA = \begin{bmatrix} -8t & q_1 t + 1 & t + q_3 s \\ q_1 t + 1 & -8 & s + q_2 \\ t + q_3 s & s + q_2 & -10s \end{bmatrix}.$$

The 2×2 leading principal minor gives

$$64t - (q_1 t + 1)^2 > 0 \Rightarrow 64t > (q_1 t + 1)^2,$$

$$\max_{q_1 \in [2,3]} (q_1 t + 1)^2 = (3t + 1)^2 < 64t,$$

$$9t^2 - 58t + 1 < 0, \quad t \in (0.0173, 6.427).$$

Hence $64t - (q_1 t + 1)^2 > 0$ for all $t \in (0.0173, 6.427)$, $q_1 \in [2,3]$.

The positive definiteness of the submatrix

$$\begin{bmatrix} 8 & -(s+q_2) \\ -(s+q_2) & 10s \end{bmatrix}$$

gives $80s - (s+q_2)^2 > 0$ or $\max_{q_2}(s+q_2)^2 < 80s$ or $(s+2)^2 < 80s$ or $s^2 - 76s + 4 < 0$ and the upper bound $\overline{s} = 80$ is suitable.

We divide the intervals $[0.0173, 6.427]$ and $[0, 80]$ into 20 and 200 equal parts respectively. In Figure 1, it is shown the family of boxes on which the determinant $\det(A^T D + DA)$ is negative for all $q_1 \in [2,3]$, $q_2 \in [1,2]$ and $q_3 \in [1,2]$.

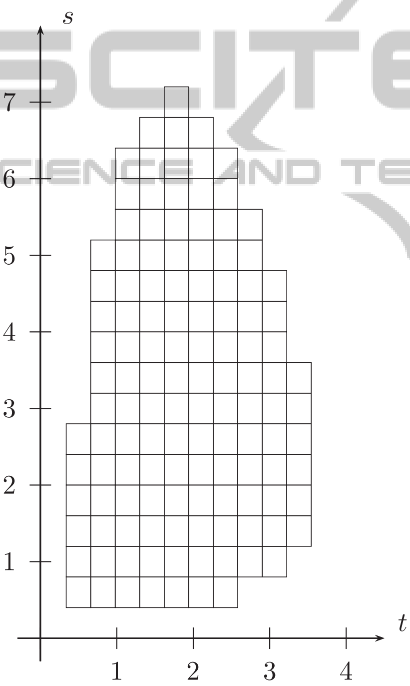

Figure 1: Each (t,s) from each box gives common diagonal solution $D = \operatorname{diag}(t,1,s)$.

Example 2.2. Consider the interval family

$$\begin{bmatrix} -3 & q_1 & -5 \\ q_2 & -2 & 1 \\ q_3 & q_4 & -6 \end{bmatrix}$$

where $q_1 \in [1,2]$, $q_2 \in [1,2]$, $q_3 \in [4,6]$ and $q_4 \in [-3,-1]$. We obtain

$$A^T D + DA = \begin{bmatrix} -6t & q_1 t + q_2 & -5t + q_3 s \\ q_1 t + q_2 & -4 & q_4 s + 1 \\ -5t + q_3 s & q_4 s + 1 & -12s \end{bmatrix}.$$

Again

$$24t - (q_1 t + q_2)^2 > 0 \Rightarrow 24t > (q_1 t + q_2)^2,$$

$$\max_{q_1 \in [1,2],\ q_2 \in [1,2]} (q_1 t + q_2)^2 = (2t+2)^2 < 24t,$$

$$t^2 + 2t + 1 < 0, \quad t \in (2 - \sqrt{3}, 2 + \sqrt{3}).$$

Hence $24t - (q_1 t + q_2)^2 > 0$ for all $t \in (0.268, 3.732)$, $q_1 \in [1,2]$ and $q_2 \in [1,2]$.

The value $\overline{s} = 50$ is acceptable. We divide the intervals $[0.268, 3.732]$ and $[0, 50]$ into 50 and 200 equal parts respectively. Figure 2 gives all boxes on which $\det(A^T D + DA)$ is negative for all $q_1 \in [1,2]$, $q_2 \in [1,2]$, $q_3 \in [4,6]$ and $q_4 \in [-3,-1]$.

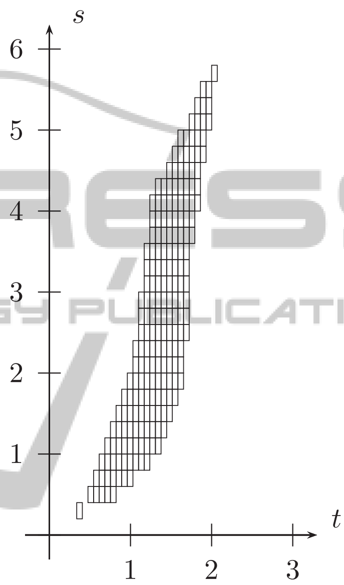

Figure 2: Each (t,s) from each box gives common diagonal solution $D = \operatorname{diag}(t,1,s)$.

It should be noted that the sufficient condition from (Pastravanu and Matcovschi, 2015, Theorem 1) is not satisfied for this example since for the matrix U from Theorem 1 the maximum real eigenvalue is positive.

3 DISCRETE SYSTEMS (SCHUR STABILITY)

Common diagonal stability of discrete interval systems is equivalent to the existence of a positive diagonal matrix D which satisfies the following matrix inequality

$$A^T DA - D < 0 \text{ for all } A \in \mathcal{A} \tag{6}$$

where $\mathcal{A}$ is given by (1).

The case $n = 2$ has been solved in (Yıldız et al., 2014). In the case of $n = 3$ taking again $D =$

diag$(t,1,s)$ we get

$$A^TDA - D =$$

$$\begin{bmatrix} ta_1^2+a_4^2+sa_7^2-t & ta_1a_2+a_4a_5+sa_7a_8 & ta_1a_3+a_4a_6+sa_7a_9 \\ ta_1a_2+a_4a_5+sa_7a_8 & ta_2^2+a_5^2+sa_8^2-1 & ta_2a_3+a_5a_6+sa_8a_9 \\ ta_1a_3+a_4a_6+sa_7a_9 & ta_2a_3+a_5a_6+sa_8a_9 & ta_3^2+a_6^2+sa_9^2-s \end{bmatrix}$$

From the principal minors condition the negative definiteness of the above matrix is equivalent to three polynomial inequalities.

Denote

$$\begin{aligned} f_1(t,s,a_1,a_4,a_7) &= t(a_1^2-1)+sa_7^2+a_4^2, \\ f_2(t,s,a_1,\ldots,a_8) &= \text{Minus } 2\times 2 \\ &\quad \text{leading principal minor}, \\ f_3(t,s,a_1,\ldots,a_9) &= \text{Determinant}. \end{aligned}$$

Then the problem (6) is equivalent to the following: Is there a positive pair (t,s) such that

$$f_1 < 0, \quad f_2 < 0, \quad f_3 < 0 \tag{7}$$

for all $(a_1,a_2,\ldots,a_9)$.

Now we can suggest the following algorithm.

Algorithm 2. *Let* 3×3 *interval family (3) be given.*

i) *Using the results on* 2×2 *interval systems from (Yıldız et al., 2014) calculate the interval* (α,β) *for the variable* t.

ii) *Determine an upper bound* $\overline{s}$ *for s.*

iii) *Divide the interval* $[\alpha,\beta]$ *into k equal parts* $[\alpha_i,\beta_i]$ *and the interval* $[0,\overline{s}]$ *into m equal parts* $[s_j^-,s_j^+]$.

iv) *On each box*

$$[\alpha_i,\beta_i]\times[s_j^-,s_j^+]\times[a_1^-,a_1^+]\times\cdots\times[a_9^-,a_9^+]$$

consider the maximization of the polynomial functions f_i $(i=1,2,3)$.

If there exist indices i_* *and* j_* *such that the maximum each of three functions* f_k $(k=1,2,3)$ *is negative then stop. The whole interval* $[\alpha_{i_*},\beta_{i_*}]\times[s_{j_*}^-,s_{j_*}^+]$ *defines family of common diagonal solutions.*

Example 3.1. Consider the interval family

$$\begin{bmatrix} -0.5 & 0.3 & q_1 \\ q_2 & -0.3 & -0.6 \\ -0.2 & q_3 & 0.1 \end{bmatrix}$$

where $q_1\in[-0.2,0.4]$, $q_2\in[-1,0]$ and $q_3\in[0,0.2]$.

We obtain

$$A^TDA - D =$$

$$\begin{bmatrix} -0.75t+q_2^2+0.04s & -0.15t-0.3q_2-0.2q_3s & -0.5q_1t-0.6q_2-0.02s \\ -0.15t-0.3q_2-0.2q_3s & 0.09t+q_3^2s-0.91 & 0.3q_1t+0.1q_3s+0.18 \\ -0.5q_1t-0.6q_2-0.02s & 0.3q_1t+0.1q_3s+0.18 & q_1^2t-0.99s+0.36 \end{bmatrix}.$$

We get $[\alpha,\beta]=[1.3494,7.5833]$ and $\overline{s}=20$. Divide the intervals $[1.3494,7.5833]$ and $[0,20]$ into 20 and 50 equal parts respectively. In Figure 3, it is shown the family of boxes on which all three functions f_k $(k=1,2,3)$ are negative for all $q_1\in[-0.2,0.4]$, $q_2\in[-1,0]$ and $q_3\in[0,0.2]$.

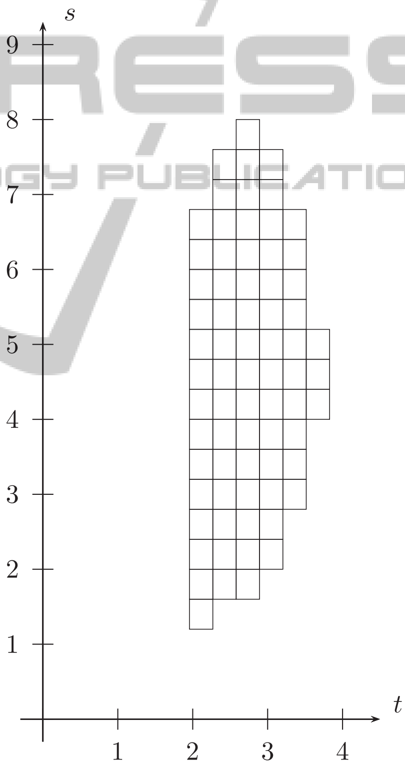

Figure 3: Each (t,s) from each box gives common diagonal solution $D=\text{diag}(t,1,s)$.

4 CONCLUSIONS

In this paper we consider the problem of diagonal stability of interval systems. The proposed approach is based on finding common diagonal Lyapunov functions. Both Hurwitz (continuous case) and Schur (discrete case) stability are considered for third order systems. We suggest an interesting approach to solve the corresponding game problems.

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