Revisiting Gradient Methods in Function Space
With Application to Rocket Trajectories

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Abstract: The gradient method in function space is revisited and applied to the problem of optimizing the trajectories of aerodynamically maneuvering rockets. The optimization objective may be the maximal range or the minimal control effort for a given range. The method is shown to provide an implementable and fast algorithm for a good approximation to the optimal solution. It does not require any non-linear programming solver, and can be straightforwardly programmed in a flight computer. The method can also be used to provide an initial guess for more precise techniques, thus accelerating the computational process.

1 INTRODUCTION

Numerical techniques for solving optimal control problems fall into two general classes: indirect methods and direct methods. In an indirect method (Bryson and Ho, 1975; Kelley, 1962; Stryk and Burlich,1992; Keller, 1968), we rely on the Minimum Principle and other necessary conditions to obtain a two-point boundary-value problem (TPBVP), which is then numerically solved for optimal trajectories. The main advantages of indirect methods are their high solution accuracy and the guarantee that the solution satisfies the optimality conditions. However, indirect methods are frequently subject to severe convergence problems. Frequently, without a good guess for the missing initial conditions, and a priori knowledge of the constrained and unconstrained arcs, convergence may not be achieved at all, or may require some very long and tedious computational effort. In the direct methods (Stryk, 1993; Benson, 2004; Elnagar et. al., 1995; Fraroo and Ross, 2001; Rao, et. al. 2010) the continuous optimal control problem is parametrized as a finite dimensional problem. Well-developed algorithms for constrained parameter optimization - also called non-linear programming (NLP) solvers for historical reasons - then solve the resulting optimization problem numerically. There are several popular methods which transform the optimal control problem into a parameter optimization problem. In present time the most popular methods are the collocation method and pseudo-spectral methods. Numerical optimization of the constrained parameter optimization typically involves finding hundreds of unknown parameters subject to hundreds of constraints. The computation time, especially when the initial guess is far from the solution, may become quite significant. This fact may be critical for real-time applications or in applications where the solution is needed for a huge number of cases (say with various terminal conditions).

Gradient in function space (Kelley, 1962; Bryson and Denham 2010) is a well-known method which may be characterized as a hybrid method, merging direct and indirect methods. On the one hand necessary conditions for the adjoint system are met, whereas on the other hand the control function is directly sought by the method of gradients. The control function is iteratively updated based on the current state/adjoint solution by evaluating the corresponding Green's function and using gradient correction (steepest descent) in function space. No further NLP solvers are needed for the solution.

Guided rocket is a new field in rockets development, which offers several improvements such as extensions of existing rockets range, improved accuracy, trajectory shaping, etc. These improvements are achieved by providing the rockets maneuverability either by aerodynamics means or by using small pyrotechnical motors (pulsers). Some approximate methods have been employed in the past to this problem (e.g. Kelley et. al., 1982). However, finding the optimal trajectories for rockets

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is still an important challenge. The computation time in these applications is of a particular importance for two main reasons: (a) fast calculation of trajectories is needed just before launching a rocket to a new target; (b) real time corrections in flight might be required due to disturbances and/or target movement.

The main purpose of this work is to revisit the gradient method in function space in order to obtain easily implementable and fast, albeit less accurate, trajectories for maneuvering rockets. The method can be used either by itself or as an accelerating method for more accurate techniques.

2 GRADIENTS IN FUNCTION SPACE

For completeness will present here the methods of gradient based on (Kelley, 1962). Consider the following state-space representation of a dynamic system:

$$\dot{x}(t) = g(x(t), u(t), t) \tag{1}$$

Where *x(t)* and u(t) are n-dimensional and m-dimensional vectors, respectively. For a given initial condition, we want to minimize some terminal cost $P(x_f)$. For simplicity, let us assume that t_f is specified. Let *u(t)* and *x(t)* be some guess values for the state and control variables respectively, and consider a sufficiently small variation $\delta u(t)$ and the resulting $\delta x(t)$ determined by the linearized equation:

$$\delta\dot{x}(t) = g_x^T(t)\delta x(t) + g_u^T(t)\delta u(t);\ \delta x(t_0) = 0 \tag{2}$$

Where g_u and g_x are Jacobian matrices. Consider now the adjoint system, defined by the linear time-varying differential equations:

$$\dot{\lambda}(t) = -g_x(t)\lambda(t) \tag{3}$$

One can readily obtain, using (2) and (3), that:

$$\begin{aligned} &\frac{d(\lambda^T(t)\delta x(t))}{dt} = \\ &\dot{\lambda}^T(t)\delta x(t) + \lambda^T(t)\delta\dot{x}(t) = \\ &\lambda^T(t)g_u^T(t)\delta u(t) \end{aligned} \tag{4}$$

On the other hand, the cost variation can be written (to first order) as:

$$\delta P = P_{x_f}^T \delta x_f \tag{5}$$

where P_{xf} if the gradient of *P* with respect to x_f. Using Eq. (3), with the following terminal conditions,

$$\lambda(t_f) = P_{x_f} \tag{6}$$

we get, from Eq. (4) and Eq. (5), and the fact the initial condition is given, that

$$\begin{aligned} &\delta P = \int_{t_0}^{t_f} (g_u(t)\lambda(t))^T \delta u(t)dt \equiv \\ &\int_{t_0}^{t_f} \mu(t)^T \delta u(t)dt \end{aligned} \tag{7}$$

The term $\mu(t)$ is the gradient of the cost in the control function space (Courant and Hilbert, 1953). Under control iterations, the steepest descent will be in its negative direction. This fact can be easily derived from Schwarz's inequality, as follows:

$$\begin{aligned} &[\delta P]^2 = \left[\int_{t_0}^{t_f} \mu^T(t)\delta u(t)dt\right]^2 \le \\ &\int_{t_0}^{t_f} |\mu(t)|^2\, dt \cdot \int_{t_0}^{t_f} |\delta u(t)|^2\, dt \end{aligned} \tag{8}$$

For the case $\mu(t) \neq 0$ the upper limit on the left is obtained (under equality) for

$$\delta u(t) = k \cdot \mu(t) \tag{9}$$

k is any constant real number. For the minimization of *P,* this constant should have a sign opposite to the sign of function space.

Hence $\mu(t)$ is in the (current) direction of the gradient in function space. For a different derivation the reader is referred to (Kelley, 1962). Notice that for the case $\mu(t)$=0 the optimal control cannot be determined by this method!

3 MAXIMAL ROCKET RANGE

3.1 Problem Formulation

The dynamic modeling of a lifting rocket will be introduced first. For simplicity, post-boost dynamics in a flat earth 2-D scenario is assumed, governed by the following continuous dynamic equations:

$$\begin{aligned} &\dot{V} = -\frac{\rho V^2 S\left[C_{D0} + K(C_{L\alpha}\alpha)^2\right]}{2m} \\ &\quad - g\sin\gamma \\ &\dot{\gamma} = \frac{\rho V^2 S C_{L\alpha}\alpha}{2mV} - \frac{g\cos\gamma}{V} \\ &\dot{h} = V\sin\gamma \\ &\dot{R} = V\cos\gamma \end{aligned} \tag{10}$$

R is range, *h* is altitude, *V* is velocity, *γ* is the flight-path angle, *m* is mass, *S* is a reference area, *α* is the angle of attack, *ρ* is air density; *g* is gravity, C_{D0} and *K* are the parabolic drag coefficients, and finally $C_{L\alpha}$ is the lift slope coefficient.

The control in this problem is the angle-of-attack *α*. It assumed to be changed instantaneously without any time delay (point mass approximation). Increasing the angle-of-attack creates the required lift force, but it also increases the induced drag.

The maximal range problem is to minimize the following cost by the control for a given *x*(0):

$$P(t_f) = -R(t_f) \tag{11}$$

Remark: Notice that t_f is not specified; in practice we should find it by the terminal condition of reaching the ground (see below).

3.2 The Adjoint System and Green's Function

From (3) and (10) we readily obtain the following adjoint system:

$$\dot{\lambda}_v = -(\lambda_h \sin\gamma + \lambda_\gamma \left[\frac{\rho S C_{L\alpha}\alpha}{2m} + \frac{g\cos\gamma}{V^2}\right] - \lambda_v \frac{\rho V S C_D}{m} + \lambda_R \cos\gamma)$$

$$\dot{\lambda}_\gamma = -(\lambda_h V \cos\gamma + \lambda_\gamma \frac{g\sin\gamma}{V} - \lambda_v g\cos\gamma - \lambda_R V \sin\gamma) \tag{12}$$

$$\dot{\lambda}_h = -\frac{SV}{2m}\frac{d\rho}{dh}\left[\lambda_\gamma C_{L\alpha}\alpha - \lambda_v C_D V\right]$$

$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = 0$$

λ_i is the adjoint (co-state) associated with the state variable *i*. As already explained, in the gradient method we use a present guess for the control, the state and the adjoint variables, where the terminal conditions for the adjoints (in this maximal range problem) are all zeros except for λ_R which is 1 (from Eq. 6). The Green's function for this problem becomes:

$$\mu(t) = \frac{\rho V^2}{m} S K C_{L\alpha}^2 \alpha \lambda_V(t) + \frac{1}{2m}\rho V S C_{L\alpha}\lambda_\gamma(t) \tag{13}$$

The control function is updated, as follows:

$$\alpha_{new}(t) = \alpha_{old}(t) - k\mu(t) \tag{14}$$

The scalar *k* is some positive fixed number which determines the step size.

As the terminal time is unknown, we also update its value in order to obtain $h(t_f)=0$, thus:

$$t_{f\,new} = t_{f\,old} - b \cdot h(t_f) \cdot \dot{h}(t_f) \tag{15}$$

where *b* is some positive fixed number. We iterate on the problem by resolving Equations (10) – (15) until some convergence condition is satisfied.

3.3 Computational Results

A fictitious 140 kg rocket with 94 km non-lifting range is considered. The initial end-of-boost angle is fixed to 53 deg. and the rocket flies a non-lifting trajectory up to its apogee. The maximal angle-of-attack is set to 15°. The lift coefficient is $C_{L\alpha}=8$ with the reference area of 0.0405 m^2; and the drag coefficients C_{D0} and *K* are 0.14 and 0.127, respectively. It is required to extend the range to its maximum (with the initial conditions set at the apogee). Fig. 1 presents in blue the optimal trajectories obtained by two direct approaches: GPOPS (Rao et. al. 2010) and the cubic-spline based collocation method (Stryk, 1993). Also shown in red is the trajectory obtained by the gradient-in-function-space method. The first two solutions overlap and their maximal range is identical 171km. The gradient-in-function-space solution reaches somewhat shorter range (169 km) hence it should be considered sub-optimal. The optimal flight time is 313 sec.

Remark: The NLP solver for GPOPS was NPOPT or IPOPT, whichever runs faster. The NLP solver for the collocation method was IPOPT. As already said, the gradient method does not use any NLP solver.

The CPU computation times for this example were as follows: 27.4 sec for GPOPS; 35 sec for the collocation method; and only 2.4 sec for the gradient method. Note that these computation times are based on MATLAB implementations on an INTEL CORE i7vPro, hence are far from being minimal (efficient coding can reduce it by orders of magnitude). Evidently one can consider them only on a comparative basis. There is at least one order of magnitude saving in the computation time for the gradient method. This result has been obtained in numerous other examples. At the very least it can be used as an accelerating method for the other approaches (Bryson, 1999). Trials based on this idea have reduced the computation time for the collocation method by a factor of three.

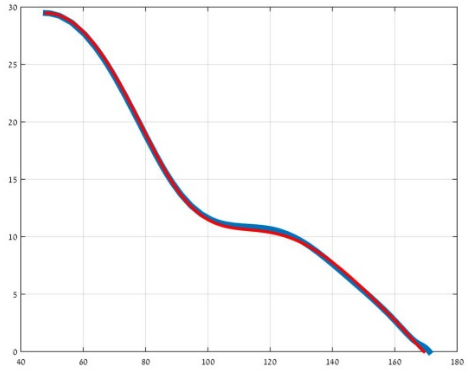

Figure 1: Maximal range trajectories.

4 FIXED ROCKET RANGE

4.1 Problem Formulation

In most practical applications the range of the rocket is fixed. The optimization problem is therefore aimed at a different cost function. One plausible candidate is the control effort. The reasons are threefold:

a. The domain of static stability is typically small for rockets and they may stall at even medium angle-of-attack values.

b. Wind gusts may increase the effective angle-of-attack causing even earlier stall conditions.

c. It will also minimize the requirements from the servos activating the control surfaces.

Hence the following cost will be considered:

$$J = \int_0^{t_f} \alpha^2(t)dt \tag{16}$$

To obtain a Meyer's formulation, a 5[th] state representing the accumulated control effort is introduced. Thus the system becomes:

$$\begin{aligned}
\dot{V} &= -\frac{\rho V^2 S\left[C_{D0} + K(C_{L\alpha}\alpha)^2\right]}{2m} \\
&\quad - g\sin\gamma \\
\dot{\gamma} &= \frac{\rho V^2 S C_{L\alpha}\alpha}{2mV} - \frac{g\cos\gamma}{V} \\
\dot{h} &= V\sin\gamma \\
\dot{R} &= V\cos\gamma \\
\dot{P} &= 0.5\alpha^2
\end{aligned} \tag{17}$$

And the cost is simply the terminal 5[th] state value:

$$J = P(t_f) \tag{18}$$

In order to obtain the required range, the problem can be simplified by changing the independent state from time to range. This is advisable due to fact that it behaves monotonically with time and has fixed initial and terminal values (Kelley, 1962). Dividing (17) through by $\dot{R}$ the systems equations are reduced to:

$$\begin{aligned}
\frac{dV}{dR} &= -\frac{D}{mV\cos\gamma} - \frac{g}{V}\tan\gamma \\
\frac{d\gamma}{dR} &= \frac{L}{mV^2\cos\gamma} - \frac{g}{V^2} \\
\frac{dh}{dR} &= \tan\gamma \\
\frac{dP}{dR} &= \frac{\alpha^2}{V\cos\gamma}
\end{aligned} \tag{19}$$

The terminal altitude needs to be zero. To this end we introduce a penalty function (Kelley, 1962)

$$\tilde{P}(R_f) = P(R_f) + W \cdot h(R_f)^2 \tag{20}$$

for some large positive scalar W.

4.2 The Adjoint System and Green's Function

Similarly to the previous section, the adjoint system is calculated by (3) with R being the independent variable. For each iteration we first integrate (19) forward, and then integrate the associated adjoint equations backward with two sets of terminal values: (0 0 0 1) yielding - from (5) - the control-effort influence function $\mu_1(R)$; and (0 0 1 0) yielding the terminal altitude influence function $\mu_2(R)$. We then combine them to obtain a single influence function for the total cost (21), as follows

$$\mu(R) = \mu_1(R) + 2W \cdot h(R_f) \cdot \mu_2(R) \tag{21}$$

We proceed as before

$$\alpha_{new}(R) = \alpha_{old}(R) - k \cdot \mu(R) \tag{22}$$

The scalar k is some positive fixed number which determines the step size.

4.3 Computational Results

Fig. 2 presents the trajectories obtained by the gradient method (red) and by GPOPS (blue), for a rocket flying to the fixed range of 140 km. The flight time is about 190 sec. As seen, the trajectories are fairly close to each other but the gradient method results are again sub-optimal, with some intermediate higher maneuver. This maneuver entails a total cost of 0.6769 sec, compared with merely 0.6242 sec of GPOPS. However, the CPU computation time for the latter was 28 sec, as opposed to 3 sec for the former.

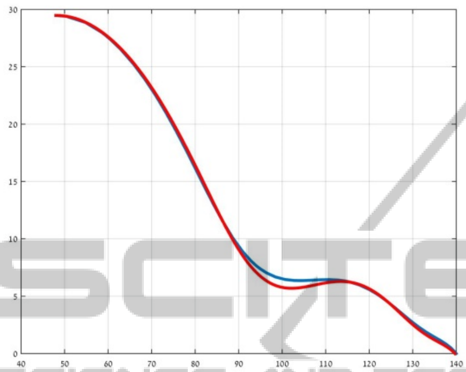

Figure 2: Fixed range trajectories.

5 CONCLUSIONS

Present day computational methods, in particular direct methods such as pseudo-spectral and collocation methods, are widely and successfully in use. Bryson's and Kelley's old but powerful ideas of Gradients in Function Space are much less used today, perhaps under the impression that the current methods are superior and therefore these techniques belong to the past.

The purpose of this position paper was to somewhat rectify this impression by claiming that, at least for fast computations and very simple implementations, Gradients in Function Space can still be an invaluable method. The computation time is, typically, one order of magnitude lower than for the direct methods, and the implementation (e.g. the number of code lines needed to perform the calculations, the required memory size, etc.) is also much less demanding as no NLP solver is required. Consequently, the algorithm fits very well with on-board computations. Optimal rocket trajectory is a problem where such advantages are important.

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REFERENCES

Bryson A. E. and Ho Y-C, *Applied Optimal Control*, Hemisphere P. C, 1975, pp. 50-120.

Kelley H. J., "Methods of Gradients", Chapter 6 in *Optimization Techniques* ed. G. Leitman, Academic Press 1962, pp. 218-222.

Stryk O. and Bulirsch R." Direct and Indirect Methods For Trajectory Optimization", Annals of Operations Research 37(1992) 357-373.

Keller H. B., *Numerical Methods for Two-Point Boundary Value Problems*, Blaisdell, New York, 1968. pp.1-150.

Stryk O., "Numerical Solution of Optimal Control Problems by Direct Collocation, *Optimal Control*, ed. Bulirsch R, Miele A. and Stoer J., Birkhauser Verlag, Basel, 1993, pp. 129–143..

Benson D. A, Gauss Pseudospectral Transcription for Optimal Control, MIT Ph. D. Thesis, Department of Aeronautics and Astronautics, November 2004.

Elnagar J., Kazemi M. A., and Razzaghi M., "The Pseudospectral Legendre Method for Discretizing Optimal Control Problems," IEEE Transactions on Automatic Control, Vol. 40, No. 10, October 1995, pp. 1793-1796.

Fahroo F. and Ross I. M., "Costate Estimation by a Legendre Pseudospectral Method," Journal of Guidance, Control, and Dynamics, Vol. 24, No. 2, 2001, pp. 270–277.

Rao A. V. et.al., "Algorithm 902: GPOPS, A MATLAB Software for Solving Multiple-Phase Optimal Control Problems Using the Gauss Pseudospectral Method", ACM Transactions on Mathematical Software, Vol. 37, No. 2, Article 22, April 2010.

Bryson A. E. and Denham W. F. "A Steepest-Ascent Method for Solving Optimum Programming Problems," J. Appl. Mech. 29(2), 247-257, 2011.

Kelley H. J., Cliff E. M. and Lutze, F. H, "Boost–glide Range-Optimal Guidance," Optimal Control Applications and Methods, Volume 3, Issue 3, pages 293–298, July-September 1982.

Courant R. and Hilbert D., *Methods of Mathematical Physics* vol.X , Interscience Publishers Inc. 1953, pp. 222-224.

Bryson A. E., *Dynamic Optimization,* Addison Wesley Longman, 1999.