# A New Inverse Optimal Control Method for Discrete-time Systems

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Abstract: This paper presents a new approach based on extended kalman filter (EKF) to construct a control lyapunov function (CLF). This function will be used in establishing the control law of inverse optimal control for discrete-time nonlinear systems. The main aim of the inverse optimal control is to avoid the solution of the difficult Hamilton-Jacobi-Bellman (HJB) equation which is resulted from the traditional solution of nonlinear optimal control problem. The relevance of the proposed scheme is illustrated through MATLAB simulation. The results show the effectiveness of the proposed method.



## 1 INTRODUCTION

The design of optimal controllers for nonlinear systems has been an area of intense research interest in control theory. Optimal nonlinear control deals with the problem of finding a stabilizing control law for a given nonlinear system and achieving a certain optimality criterion . In general, solving the nonlinear optimal control problem leads to the Hamilton-Jacobi-Bellman (HJB) equation. This equation has no exact analytical solution for the general nonlinear case (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011; Freeman and Kokotovic, 1996). The HJB equation is reduced to the Riccati equation in the case of linear quadratic regulator (LQR) (Kalman, 1964).

The inverse optimal control problem, which initially presented by Kalman for linear systems, deals with the question of whether a given state feedback can be the optimal control with respect to some useful performance index (Kalman, 1964). In nonlinear case, the inverse optimal control approach circumvents the task of solving a Hamilton-Jacobi-Bellman equation. The main idea behind the theory of inverse optimal control is that it is required to construct a stabilizing feedback control law based on a priori knowledge of a control lyapunov function (CLF) as a first step, then this control law will be used to optimize a meaningful cost functional (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011; Freeman and Kokotovic, 1996). This definition can be a bit confusing if it is compared to the definition of optimal control theory, where the cost function should be known before designing the control law.

In this paper, the inverse optimality approach depends on defining a control lyapunov function (CLF). Unfortunately, there are no systemic techniques to define a CLF for general nonlinear systems. In the literature, it is well known that the existence of a control lyapunov function leads to lyapunov stability in the system (Sanchez and Ornelas-Tellez, 2013; Khalil, 1996). Moreover, any CLF can be considered as meaningful cost function in optimal control problems (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011). In (Ornelas-Tellez et al., 2011) a quadratic CLF was proposed for the inverse optimal control problem ; this function depends on a time- variant parameter, where the speed-gradient (SG) algorithm was proposed to adjust this parameter.

In this research, the same quadratic control lyapunov function which proposed in (Ornelas et al., 2011; Ornelas-Tellez et al., 2011) is used, then the overall parameters of this function are adjusted in a recursive way by using the mean of extended kalman filter (EKF) Algorithm. The researchers in the field of nonlinear estimation problems used the EKF algorithm as an estimator in both the state of a nonlinear dynamic system and in parameters estimation process for many applications, such as induction motor control and Fuzzy modeling control problems (Simon, 2002; Yazid et al., 2011).

The novel contribution of this paper is that the EKF algorithm is used as on-line parameters identifier in order to construct the CLF within the control loop of the inverse optimal control.

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The remainder of this paper is organized as follows: Section 2 briefly describes the nonlinear optimal control and the discrete time HJB equation. Section 3 introduces some mathematical notations and definitions related to the inverse optimal control and control lyapunov function. In Section 4, the extended kalman filter algorithm is presented. In Section 5, an in-depth explanation on the proposed design method is given. Section 6 presents a nonlinear test example and the simulation results. Some conclusions are drawn in section 7.

## 2 DISCRETE TIME HAMILTON-JACOBI-BELLMAN EQUATION FOR NONLINEAR OPTIMAL CONTROL

Considering an affine-in-input nonlinear dynamical system of the form:

$$x_{k+1} = f(x_k) + g(x_k)u_k \tag{1}$$

Where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the control input. $f(x_k) \in \mathbb{R}^n$, $g(x_k) \in \mathbb{R}^{n\times m}$. Without loss of generality, It can be assumed that the origin $(x = 0)$ is the equilibrium point of the system (1), $f(0) = 0$ and $g(x_k) \neq 0$ for all $x_k \neq 0$. system (1) is assumed to be stabilizable on a predefined compact set $\Omega \in \mathbb{R}^n$.

- Definition 1: Stabilizable System; A nonlinear dynamical system is said to be a stabilizable system on a compact set $\Omega \in \mathbb{R}^n$ if there exists a control input $U \in \mathbb{R}^m$ such that, for all initial conditions $x_0 \in \Omega$, the state $x_k \to 0$ as $k \to \infty$ (Khalil, 1996).

It is desired to determine a control law $u_k$, which minimizes the following cost functional:

$$V(x_k) = \sum_{n=k}^{\infty} (L(x_n) + u_n^T E u_n) \tag{2}$$

Where $V : \mathbb{R}^n \to \mathbb{R}^+$ is the cost functional, $L : \mathbb{R}^n \to \mathbb{R}^+$ is positive semi-definite function, and $E : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a real symmetric positive definite weighting matrix which could be a function of the system's states. Equation (2) can be written as:

$$V(x_k) = L(x_k) + u_k^T E u_k + V(x_{k+1}) \tag{3}$$

It is assumed that the boundary condition of the function is equal to zero (i.e. $V(x = 0) = 0$) in order to use $V(x_k)$ as a lyapunov function in the next section. From Bellman's optimality principle, it is known that the $V^*(x_k)$ value function is time invariant and satisfies the discrete time (DT) bellman equation for the infinite horizon optimization case (Sanchez and Ornelas-Tellez, 2013; Nakamura et al., 2007):

$$V^*(x_k) = \min_{u_k}\{L(x_k) + u_k^T E u_k + V^*(x_{k+1})\} \tag{4}$$

The formula of the optimal control $u_k^*$ can be calculated by taking the gradient of the right-hand side of (4) with respect to $u_k$. Therefore, the optimal control $u_k^*$ will be:

$$u_k^* = -\frac{1}{2} E^{-1} g^T(x_k) \frac{\partial V^*(x_{k+1})}{\partial x_{k+1}} \tag{5}$$

By substituting the optimal control formula $u_k^*$ in $V^*(x_k)$ at (4), the DT HJB equation will be:

$$\begin{aligned} V^*(x_k) &= L(x_k) + V^*(x_{k+1}) \\ &+ \frac{1}{4} \frac{\partial V^{*T}(x_{k+1})}{\partial x_{k+1}} g(x_k) E^{-1} g^T(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}} \end{aligned} \tag{6}$$

## 3 FUNDAMENTAL OF INVERSE OPTIMAL CONTROL

The proposed inverse optimal control approach in this research depends on the control lyapunov (CLF). Due to this, some important properties for lyapunov function from the literature are stated here.

### 3.1 Control Lyapunov Function

- Definition 2: A positive definite function $M(x_k)$ satisfying the condition $M(x_k) \to \infty$ as $\|x_k\| \to \infty$ is said to be radially unbounded (Sanchez and Ornelas-Tellez, 2013; Khalil, 1996).
- Definition 3: Let $M(x_k)$ be radially unbounded, with $c > 0$, $\forall\, x_k \neq 0$, $M(0) = 0$.
  If for any $x_k \in \mathbb{R}^n$ , there exit real values $u_k$ such that $\triangle M(x_k, u_k) < 0$ where the difference $\triangle M(x_k, u_k)$ is defined as:

  $$M(x_{k+1}) - M(x_k) = M(f(x_k) + g(x_k)u_k) - M(x_k)$$

  Then M(.) is said to be "Discrete-time control lyapunov function for the system" (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011).
- Definition 4: The equilibrium point $x_k = 0$ is globally asymptotically stable if there exists a function $M : \mathbb{R}^n \to \mathbb{R}$ such that (i) M is a positive definite function, decrescent and radially unbounded, and (ii) $-\triangle M(x_k, u_k)$ is a positive definite function (Sanchez and Ornelas-Tellez, 2013; LaSalle, 1986).

- Definition 5: Suppose that there exists a positive definite function M and three constants as following:

  $$c1, c2, c3 > 0, \quad P > 1$$

  Such that:

  $$c1\|x\|^P \leq M(x_k) \leq c2\|x\|^P$$

  $$\triangle M(x_k) \leq c3\|x\|^P, \ \forall k \geq 0, \ \forall x_k \in \mathbb{R}^n$$

  Then $x_k = 0$ is an exponentially stable equilibrium for the system (Sanchez and Ornelas-Tellez, 2013; LaSalle, 1986).

### 3.2 Inverse Optimal Control

In the literature of inverse optimal control, a theorem for the discrete-time inverse optimal control has been published and the proof of this theorem is well-done in (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011). This research depends directly on this theorem. For that, the theorem is just stated here without any proof.

- Theorem 1: The control law $u_k^*$ in (5) can be assumed to be inverse optimal control if:

  a) It achieves a global exponential stability of the equilibrium point $x_k = 0$ for the system.

  b) It minimizes the cost functional in (2). For which $L(x_k) := -\overline{M}$ Where:

  $$\overline{M} := M(x_{k+1}) - M(x_k) + u_k^{*T} E u_k^* \leq 0 \quad (7)$$

The inverse optimal control is based on the knowledge of $M(x_k)$. Hence, a CLF $M(x_k)$ is proposed such that (a) and (b) are guaranteed. That is, instead of solving HJB in (6) for $V(x_k)$, a candidate quadratic control lyapunov function $M(x_k)$ is proposed with the form

$$M(x_k) = \frac{1}{2} x_k^T P x_k \qquad P = P^T > 0 \quad (8)$$

Hence, The CLF $M(x_k)$ is used instead of $V(x_k)$. It is required to select an appropriate matrix P in order to achieve stability. Moreover, the control law $u_k^*$ with the proposed quadratic control lyapunov function will optimize the cost functional. The state feedback control law can be rewritten as:

$$u_k^* = -\frac{1}{2}(E + \frac{1}{2} g^T(x_k) P g(x_k))^{-1} * g(x_k)^T P f(x_k) \quad (9)$$

The process of finding an appropriate P matrix to satisfy the required performance is still a hot research topic (Sanchez and Ornelas-Tellez, 2013; Ornelas et al., 2011; Ornelas-Tellez et al., 2011). Figure 1 illustrates the distinction between the traditional solution for the nonlinear optimal control problem and the inverse optimal control approach.

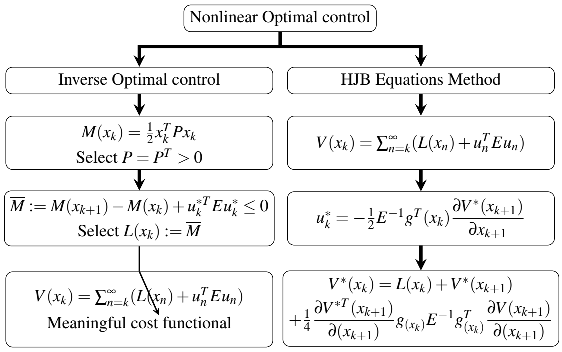


Figure 1: The inverse optimal control approach and the traditional solution for optimal control problem.

## 4 EXTENDED KALMAN FILTER

The Kalman filter (KF) has become a standard technique to be used as an optimal estimator and a quite easy method for estimating the un-measurable states of the linear systems. For nonlinear systems, the extended Kalman filter can be used if the nonlinearity of the system were sufficiently smooth (Simon, 2002).

### 4.1 Extended Kalman Filter Equations

The nonlinear process model is described as:

$$X_k = g(u_{k-1}, x_{k-1}) + w_{k-1} \quad (10)$$

$$Z_k = h(x_k) + v_k \quad (11)$$

State transition probability and measurement probability are the nonlinear functions $g$ and $h$, respectively. $w_k$ and $v_k$ are the process and observation noises. These noises are assumed to be zero mean multivariate Gaussian noises with covariance $Q_k$ and $R_k$, respectively. Here, $u_k$ is the input control vector. The discrete-time equations of extended kalman filter are illustrated in Figure 2. Where the matrix $G_k$ is the Jacobian of the state function and it is defined as the derivatives of each component of $g$ w.r.t. each component of $x_{k-1}$. Moreover, the matrix $H_k$ is the Jacobian of the measurement function and it is defined as the derivatives of each component of $h$ w.r.t. each component of $x_k$.

Recently, the Extended Kalman Filter has been utilized in parameters estimation for real-time control in nonlinear system (Simon, 2002; Yazid et al., 2011).

### 4.2 Stability Analysis in EKF Applications

Since the covariance matrices which used in EKF are approximations and the estimation is based on the linearization of nonlinear functions $g$ and $h$, there is no

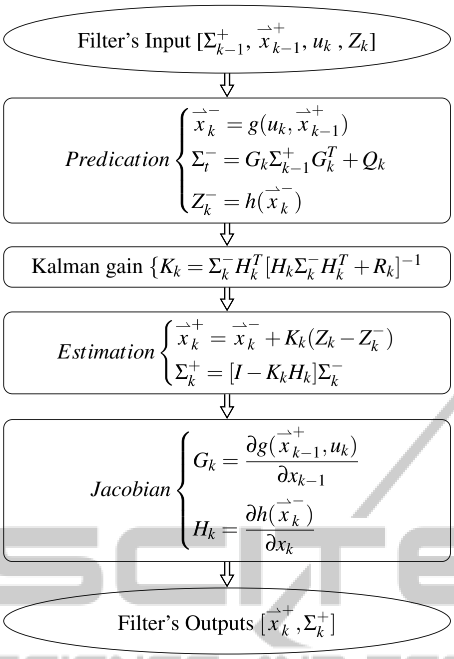


Figure 2: Extended Kalman Filter Equations.

guarantee of stability and performance for the system prior to experimental data analysis. Indeed, the approach seems to work well if the linearization is sufficiently smooth and a proper tuning for filter parameters is achieved (Raol et al., 2004). Section 5 shows how to modify the EKF equations in order to estimate the parameters in the proposed inverse optimal control law.

# 5 EKF FOR INVERSE OPTIMAL CONTROL

In the proposed approach, the EKF equations are used in order to estimate the parameters of the P matrix. This matrix will be used in establishing the quadratic control lyapunov function as following:

$$M(x_k) = \frac{1}{2} x_k^T P x_k \qquad P = P^T > 0 \qquad (12)$$

Where $\vec{x}_{k-1}^{+} = [P1 \quad P2 \quad ... \quad Pn]$, and $P1, \quad P2, \quad ...Pn$ are the elements of matrix P to be estimated. It can be defined the state function as one to one mapping of those parameters:

$$\vec{x}_k^{-} = g(u_k, \vec{x}_{k-1}^{+}) = \vec{x}_{k-1}^{+} = \begin{bmatrix} P1 \\ P2 \\ . \\ . \\ Pn \end{bmatrix} \qquad (13)$$

The state Jacobian matrix $G_k$ is equal to the identity matrix.

$$G_k = \begin{bmatrix} \frac{\partial P_1}{\partial P_1} & \frac{\partial P_1}{\partial P_2} & \cdots & \frac{\partial P_1}{\partial P_n} \\ \frac{\partial P_2}{\partial P_1} & \frac{\partial P_2}{\partial P_2} & \cdots & \frac{\partial P_2}{\partial P_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_n}{\partial P_1} & \frac{\partial P_n}{\partial P_2} & \cdots & \frac{\partial P_n}{\partial P_n} \end{bmatrix} = I \qquad (14)$$

For simplicity, it can be assumed that $Q_k$ is constant during the process:

$$Qk = q_0 \times I \qquad (15)$$

$$\vec{x}_k^{+} = S_0 \times I \qquad (16)$$

For the following estimator's equation in EKF:

$$\vec{x}_k^{+} = \vec{x}_k^{-} + K_k(Z_k - Z_k^{-}) \qquad (17)$$

The term $(Z_k - Z_k^{-})$ is used to calculate the difference between the measurement value and the estimated one. Hence, this term can be adapted in order to be suitable for the proposed research as following: $Z_k^{-}$ In the EKF equations can be used as error indicator, $Z_k$ can be set to be equal to zero in order to minimize the total error $(Z_k - Z_k^{-})$. Moreover, the Root_Mean_Square_Error (RMSE) of all states output will be used as error observer instead of measurement error $Z_k^{-}$. (i.e. $Z_k^{-}$= RMSE), which equal to $h(\vec{x}_k^{-})$ as shown in predication equation in figure 2.

$$Z_k^{-} = h(\vec{x}_k^{-}) = RMSE = \sqrt{\frac{(X_1 - X_{1ref})^2 + (X_2 - X_{2ref})^2 + \ldots + (X_n - X_{nref})^2}{n}} \qquad (18)$$

To calculate the Jacobian $Hk$ , it is required to define $h(\vec{x}_k^{-})$ as a function of the P parameters $[P1 \quad P2 \quad Pn]$. Then the Jacobian matrix can be found as the following equation:

$$Hk = \begin{bmatrix} \frac{\partial h(\vec{x}_k^{-})}{\partial P_1} & \frac{\partial h(\vec{x}_k^{-})}{\partial P_2} & \cdots & \frac{\partial h(\vec{x}_k^{-})}{\partial P_n} \end{bmatrix} \qquad (19)$$

Figure 3 shows the block diagram of the proposed method.

The steps of the proposed approach are illustrated here:

1. Find suitable initial values for the parameters of matrix P.
2. Choose suitable values for the covariance matrices: $Q_k$ and $R_k$.

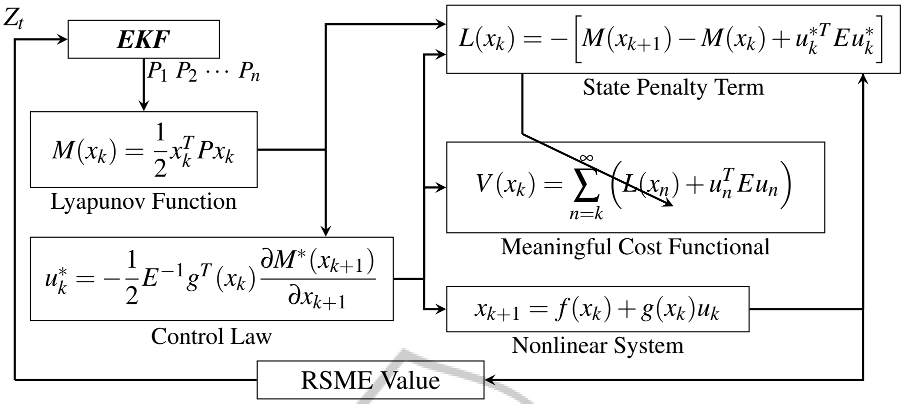


Figure 3: EKF_based Inverse Optimal Control for Discrete-Time Nonlinear System.

3. Calculate the error observer (RMSE) from the current states which equal to $h(\overrightarrow{\bar{x}_k})$.
4. Apply the proposed EKF equations to get the estimated values of the parameters.
5. Construct the control lyapunov function (CLF), and then establish the control law of the inverse optimal control.
6. Calculate the penalty term $L(x_k)$ for the meaningful cost functional.
7. Test the system's states and exit if the states arrived to your target.
8. Return to step 3 to calculate the new states.
9. The values of initial parameters and the covariance matrices can be changed in order to get better performance.

In summary, the EKF algorithm will estimate a new P matrix at each step. This new P matrix should minimize the RMSE value if the filter is well adjusted. As it is mentioned before in 4, the stability and the performance of the nonlinear system can be adjusted by tuning the filter's parameters.

# 6 EXAMPLE AND SIMULATION RESULTS

The performance of the proposed method is illustrated in the following nonlinear example:

$$f(x_k) = \begin{bmatrix} x_{1,k}x_{2,k} - 0.8x_{2,k} \\ x_{1,k}^2 + 1.8x_{2,k} \end{bmatrix} \; g(x_k) = \begin{bmatrix} 0 \\ -2 + cos(x_{2,k}) \end{bmatrix}$$

The stabilizing optimal control law can be calculated according to (9). Matrix P is estimated by the proposed method, as in section 5. Where $E = 0.5$ is the constant in the cost function equation. The initial condition for the states is $X_0 = [2 \quad -2]$. The EKF algorithm constants are selected to be: $Q_0 = 100$; $R_0 = 0.01$; $P_0 = 100$. The phase portrait for both unstable nonlinear system and the stabilized nonlinear system is illustrated in figure 4. The behavior of both states response and the control law for the stabilizing nonlinear system with respect to the time step is illustrated in figure 5. Figure 6 displays the evaluation of the cost functional $V(x_k)$.

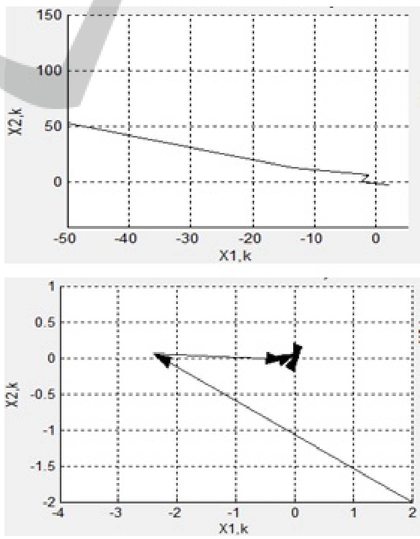


Figure 4: The phase portrait for the unstable system (a), and for stabilized nonlinear system (b).

The previous example is used by the authors in (Sanchez and Ornelas-Tellez, 2013) in order to test the effectiveness of the main theorem. This theorem is used in this research (theorem 1 in section 3). Moreover, the same example is used in (Ornelas-Tellez et al., 2011) to test the performance of speed gradient algorithm for inverse optimal control. In (Ornelas-Tellez et al., 2011) a quadratic function of the form $V(x_k) = \frac{1}{2} x_k^T P x_k$ was proposed as a CLF for the in-

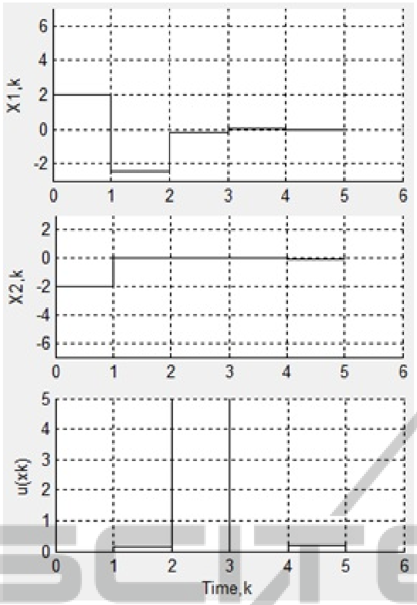


Figure 5: States response and the control law.

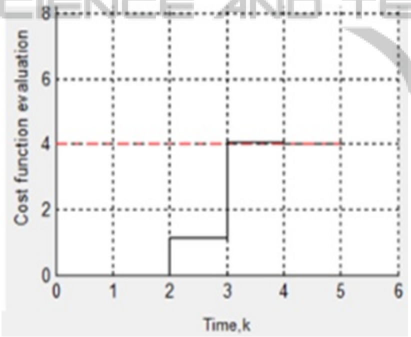


Figure 6: Cost function evaluation.

verse optimal control problem, this CLF depends only on one time-variant parameter $P_k$, where $P = P_k * P^{'}$ and $P^{'}$ is a predefined matrix. Then this parameter $P_k$ is adjusted by the mean of speed-gradient (SG) algorithm. In this research, the simulation results indicate that the proposed method has better performance compared to the existing method as shown in Table 1.

Table 1: A Comparison between EKF based Approach and Other approaches.

| Methods | $X_1 \to 0$ | $X_2 \to 0$ | Cost functional |
|---|---|---|---|
| Main theorem | 10 Steps | 8 Steps | 40 |
| Speed Gradient | 8 Steps | 7 Steps | 10 |
| EKF Based | 3 Steps | 2 Steps | 4 |

## 7 CONCLUSIONS

In this paper, a new approach related to inverse optimal control problem for discrete-time nonlinear systems is proposed. By using inverse optimal control technique, there is no need to solve the Hamilton-Jacobi-Bellman (HJB) equation which is resulted from the traditional solution of nonlinear optimal control. For this new approach, a discrete-time control lyapunov function (CLF) in a quadratic form is proposed, whose parameters is determined by using extended kalman filter (EKF) algorithm. This CLF will be used to establish the inverse optimal control law. The validation of the proposed method is made through MATLAB simulation. The results illustrate that the proposed controller ensures stabilization of nonlinear systems and minimizes a cost functional.

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