# A Modified Fuzzy Lee-Carter Method for Modeling Human Mortality

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Abstract: Human mortality modeling and forecasting are important study fields since mortality rates are essential in financial and social policy making. Among many others, Lee Carter (LC) model is one of the most popular stochastic method in mortality forecasting. Koissi and Shapiro fuzzified the standard LC model and eliminated the assumptions of homoscedasticity and the ambiguity on the size of the error term variances. In this study, a modified version of fuzzy LC model incorporating singular value decomposition (SVD) technique is proposed. Utilizing SVD instead of ordinary least squares in the fuzzy LC model allows the model to capture existing fluctuations in mortality rates and yields a better fit. The proposed method is applied to Finland mortality data for years 1925 to 2009. The results are compared with Koissi and Shapiro's fuzzy LC method and the standard LC method. Numerical findings show that proposed method gives statistically better results in generating small spreads and in estimating mortality rates when compared with Koissi and Shapiro's method.

## 1 INTRODUCTION

Human mortality modeling and forecasting are two important factors for development planning and decision making in various disciplines. Projecting and estimating issues such as unemployment rates, income levels, household consumptions, composition of labour force, and school enrolment are among mortality modeling application areas. In fact, mortality rates together with fertility and migration rates are the vital demographic indicators of population dynamics (Keyfitz, 1977). Mortality projections generate a basis for public financing, productivity growth, and monetary policy decisions (Lindh, 2003) and public and private retirement systems (Danesi, Haberman and Millossovich, 2015), life insurance schemes (Ahmadi and Li 2014), social security and healthcare planning (French, 2014), and etc.

Stochastic mortality modeling methods have a significant area in demographic estimation studies since they come up with stochastic estimations for the mortality rates, and provide forecast intervals for them via considering their deviations (Booth, 2006). Time series methods are major extrapolative stochastic methods used for mortality forecasting based solely on historic data (Lee and Carter, 1992; Lee and Tuljapurkar, 1994; Li and Chan, 2005; de Jong and Tickle, 2006). Time series methods do not permit the inclusion of exogenous variables, that is, they do not involve the effects of technological developments and etc. in estimating the future population.

Among the existing studies, Lee-Carter (LC) model is one the most extensively studied stochastic method in mortality forecasting. It simply takes age and sex into account together with matrix decomposition to obtain single time varying mortality indices. According to Lee and Carter (1992), mortality can be modeled as:

$$\ln(\mathbf{m}_{x,t}) = \mathbf{a}_x + \mathbf{b}_x \mathbf{k}_t + \varepsilon_{x,t} \qquad (1)$$

where $\mathbf{m}_{x,t}$ is the central death rate for age $x$ at time $t$, $\mathbf{a}_x$ and $\mathbf{b}_x$ are age-specific constants and $\mathbf{k}_t$ is time-variant mortality index. The error term $\varepsilon_{x,t}$ is normally distributed with mean 0 and variance $\sigma_\varepsilon^2$, and stands for the past effects that are not reflected by the model.

Lee and Carter use singular value decomposition method (SVD) to estimate mortality index $\mathbf{k}_t$ and age-specific constants $\mathbf{a}_x$, and $\mathbf{b}_x$. Then, they use the



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estimated $\mathbf{k}_t$ to forecast the future $\mathbf{k}_t$ values and their standard deviations.

In literature, many improvements to the LC model have been suggested. Renshaw and Haberman (2003) add a double bilinear predictor structure to the model to include the effects of age differences, whereas Brouhns, Denuit and Vermunt (2002) fit the mortality rates at each age group via a Poisson regression model. The problems related with outliers in historic data are tried to be overcome by several parametric and nonparametric smoothing techniques (Currie, Durban and Eilers, 2004; de Jong and Tickle, 2006; Hyndman and Ullah, 2007; Lazar and Denuit, 2009; Hatzapoulos and Haberman, 2011). Further developments in LC model are accomplished by Giacometti et al (2012), Ahmadi and Li (2014).

## 1.1 Fuzzy LC Model

LC model is a very popular method in mortality forecasting since it is a simple model that can be used for capturing the mortality trends in most of the developed countries (Christiansen, Niemeyer and Teigiszerová, 2015). However, in some cases the application of LC model has limited results. The outputs may not reflect a reasonable trend due to lack of relevant data for whole age and sex groups or in case of random fluctuations due to small sample size or exogenous effects (Ahcan et al., 2014). Standard LC model uses SVD method and assumes that error terms are normally distributed with constant variance, $\sigma_\varepsilon^2$. This is a strict homoscedasticity assumption which is difficult to satisfy especially in cases where precise and enough historic data are not available. The magnitude of this variance is assumed to be small for acceptable forecasts but there is an obvious ambiguity in how small it should be (Lee, 2000). The ambiguity problem about homoscedasticity is studied by Koissi and Shapiro (2006). They reformulated the standard Lee-Carter model with incorporating fuzziness into the model. In their approach, minimum fuzziness criterion derived by Tanaka, Ueijima and Asai (1982) and Chang and Ayyub (2001) in a fuzzy least-squares regression method is used for estimating the mortality.

The fuzzy formulation of the LC model is:

$$\tilde{Y}_{x,t} = \tilde{A}_x \underset{T_W}{\oplus} \tilde{B}_x \underset{T_W}{\otimes} \tilde{K}_t \quad \text{for } x = x_1,...x_N,\ t = t_1, t_1+1, ..., t_1+T-1 \tag{2}$$

where $\tilde{Y}_{x,t}$ are known fuzzy log-mortality rate of age group $x$ at time $t$, $\tilde{A}_x$ and $\tilde{B}_x$ are the unknown fuzzy age-specific parameters, and $\tilde{K}_t$ is the unknown fuzzy time-variant mortality index. Here, $\tilde{A}_x$, $\tilde{B}_x$, and $\tilde{K}_t$ can be defined as fuzzy symmetric triangular numbers as $\tilde{A}_x = (a_x, \alpha_x)$, $\tilde{B}_x = (b_x, \beta_x)$, and $\tilde{K}_t = (k_t, \delta_t)$, where $a_x$, $b_x$, and $k_t$ are the centers and $\alpha_x$, $\beta_x$, and $\delta_t$ are the spreads of the corresponding fuzzy numbers, and log-mortality rate refers to natural logarithm of a mortality rate. Equation (2) treats the log-mortality rate for age cohort $x$ at time $t$ as a confidence interval by fuzzifying it instead of considering it as a crisp number. Koissi and Shapiro argue that this sounds realistic as exact values of mortality rates are seldom known.

## 1.2 Motivation for a Modified Fuzzy LC Model

The fuzzy formulation of LC model requires the fuzzification of crisp $Y_{x,t}$ values. Koissi and Shapiro use fuzzy least squares regression based on minimum fuzziness criterion developed by Tanaka et al., (1982) and Chang and Ayyub (2001). They try to find $\tilde{A}_0 = (c_{0x}, s_{0x})$, $\tilde{A}_1 = (c_{1x}, s_{1x})$, and $\tilde{Y}_{x,t} = (y_{x,t}, e_{x,t})$ with centers $c_{0x}$, $c_{1x}$, and $y_{x,t}$, and spreads $s_{0x}$, $s_{1x}$, and $e_{x,t}$, so that:

$$(y_{x,t}, e_{x,t}) = (c_{0x}, s_{0x}) + (c_{1x}, s_{1x}) \times t \tag{3}$$

for each age group $x$.

Koissi and Shapiro first apply ordinary least squares regression (OLS) to obtain center values such that

$$Y_{x,t} = c_{0x} + c_{1x} \times t, \tag{4}$$

Then, the spreads are determined by solving a linear programming (LP) problem based on minimum fuzziness criterion suggested by and Chang and Ayyub (2001).

Equation (4) treats time $t$ as an independent variable. Although in most of the mortality modeling techniques mortality rates are treated as time series, it may not be proper to use time directly as the only explanatory variable in the model. In fact, t, the independent variable in equation (3), is a monotonically increasing variable, hence the center and spread of log-mortality rate (dependent variables in equation (3)) take a linear form. In this paper, to overcome this issue, a modified version of the fuzzification of crisp $Y_{x,t}$ values based on singular value decomposition (SVD) technique is proposed. Thus the fluctuations in log-mortality rates can be captured by the model. The modified fuzzy LC model proposed in this study also aims to eliminate the homoscedasticity assumptions and assumptions related to the magnitude of error term variances. Moreover, the modified model can be used in cases where there are concerns about the ambiguity of data and when the number of data prohibits the usage of standard LC or other stochastic methods.

## 2 METHODOLOGY

The modified fuzzy LC method can be analyzed in two parts: fuzzification of observed $Y_{x,t}$ values, and finding the fuzzy model parameters for estimating log-mortality rates. The proposed modifications are about the first part, while second part is dealt with the same approach as Koissi and Shapiro's except the solution approach.

### 2.1 Part I: Fuzzification of $Y_{x,t}$ Values

A modified version of Koissi and Shapiro's method that fuzzifies $Y_{x,t}$ values on SVD technique is proposed in this study. That is given the log-mortality rates $Y_{x,t}$, the task is to find $\tilde{A}_0 = (c_{0x}, s_{0x})$, $\tilde{A}_1 = (c_{1x}, s_{1x})$, and $\tilde{Y}_{x,t} = (y_{x,t}, e_{x,t})$ with centers $c_{0x}$, $c_{1x}$, and $y_{x,t}$, and spreads $s_{0x}$, $s_{1x}$, and $e_{x,t}$, such that:

$$(y_{x,t}, e_{x,t}) = (c_{0x}, s_{0x}) + (c_{1x}, s_{1x}) \times f_t \quad (5)$$

for each age group $x$, where $f_t$ is an unknown fuzzification index varying with time $t$. $f_t$ can be expressed as $f_t = g_t(\vec{m}_{xt})$, where $g_t$ is a function mapping $\vec{m}_{xt}$ to fuzzification index $f_t$ for each time $t$, and $\vec{m}_{xt}$ is a vector composed of mortality rates $m_{x_1 t}, m_{x_2 t}, ..., m_{x_N t}$ for each time $t$ and age group $x_i = x_1, ...x_N$. $f_t$ can be viewed as the unknown regressor of equation (5) which is capable of capturing the fluctuations in log-mortality rates. $t$, the independent variable in equation (3) is a monotonically increasing variable, hence the center and spread of log-mortality rate (dependent variables in equation (2)) take a linear form. However, the proposed fuzzification index $f_t$ which is based on the aggregated age group mortality rates, does not necessarily show a linear trend. Consequently equation (5) generates a better fitting model.

In equation (5), since the value of the independent variable $f_t$ is unknown, OLS cannot be used. Substituting $f_t$ in equation (4) yields the following equation (6) as:

$$y_{x,t} = c_{0x} + c_{1x} \times f_t \quad (6)$$

and the independent variable $f_t$ is obtained by using SVD method. SVD is a dimension reduction method in which the original data points are approximated in a lower dimensional space by highlighting the underlying trend of the original data (Mandel, 1982). In general, the method is based on the linear algebra theorem asserting that it is possible to decompose an $m \times n$ rectangular matrix $A$ into the product of three matrices: $A = USV^T$, where $U$ is an $m \times m$ orthogonal matrix whose columns are orthonormal eigenvectors of $AA^T$, $V$ is an $n \times n$ orthogonal matrix whose columns are orthonormal eigenvectors of $A^T A$, and $S$ is an $m \times n$ diagonal matrix containing the square roots of eigenvalues from $U$ or $V$ in descending order. In fact the diagonal matrix $S$ captures the characteristics of matrix $A$, because of the fact that it is composed of eigenvalues of its left and right eigenvectors.

By expressing the matrix $A$ with the eigenvalues in matrix $S$, new coordinate axes composed of the orthogonal vectors defined by the columns of matrices $U$ or $V$ can be generated. Then the projections of the original data points in matrix $A$ to the new coordinate space can be defined with the help of the corresponding eigenvalues in matrix $S$.

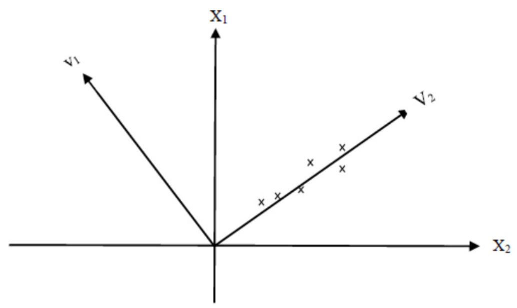


Figure 1: Geometric interpretation of SVD method for a matrix *A*.

That is, SVD method aims to reorient the coordinate axes in such a way that these axes follow a more similar pattern to the points of matrix $A$. Figure 1 shows the geometric interpretation of the method as an example. Assuming that matrix A is a $6\times 2$ matrix composed of six data points, In Figure 1 these six data points, defined in the coordinate plane of $x_1$-$x_2$, can also be expressed in the coordinate plane of $v_1$-$v_2$.

To utilize SVD method in equation (6) for estimating the unknown parameters $c_{0x}$, $c_{1x}$, and $f_t$; the following procedure is applied. First, $f_t$ 's are normalized to sum to 0 and $c_{1x}$'s to sum to 1. Then, $c_{0x}$ must equal the average over time of $y_{x,t}$ (this follows from the fact that the average value of $f_t$ 's is set to 0). Moreover, each $f_t$ must equal to the sum over age of $(y_{x,t} - c_{0x})$, since the sum of $c_{1x}$'s is set to unity. Then, $c_{1x}$'s are estimated by regressing $(y_{x,t} - c_{0x})$ on $f_t$ without a constant term separately for each age group $x$ (Lee and Carter, 1992). The spread optimization part of Koissi and Shapiro's model which is rewritten as:

$$\text{minimize } Ts_{0x} + s_{1x} \sum_{t=t_0}^{t_0+T-1} |f_t| \tag{7}$$

subject to:

$$c_{0x} + c_{1x} f_t + (1-h)[s_{0x} + s_{1x}|f_t|] \geq y_{x,t}, \quad \text{for } \forall t = t_0, t_0+1, ..., t_0+T-1 \tag{8}$$

$$c_{0x} + c_{1x} f_t - (1-h)[s_{0x} + s_{1x}|f_t|] \leq y_{x,t}, \quad \text{for } \forall t = t_0, t_0+1, ..., t_0+T-1 \tag{9}$$

$$s_{0x}, s_{1x} \geq 0 \tag{10}$$

Here, the objective is to minimize the total spreads. Equations (8) and (9) guarantee that each log-mortality rate $\tilde{Y}_{x,t}$ falls within the estimated $\hat{\tilde{Y}}_{xt}$ at a level $h$, which is a predetermined small parameter (Koissi and Shapiro prefer using $h = 0$).

## 2.2 Part II: Finding the Model Parameters

Once the log-mortality rates are fuzzified, the next step in Koissi and Shapiro's method is to find appropriate parameters $\tilde{A}_x$, $\tilde{B}_x$ and $\tilde{K}_t$ for equation (2). At this point it is worth mentioning that with multiplication of triangular fuzzy numbers, the characteristics of the numbers are not preserved although addition of triangular fuzzy numbers also results in a triangular fuzzy number. Mesiar (1997) shows that with weakest triangular norm ($T_W$) based multiplication and addition the shape of the membership function is preserved for LR-type fuzzy numbers.

For two symmetric triangular fuzzy numbers $\tilde{A} = (a, l_A)$ and $\tilde{B} = (b, l_B)$, the shape preserving $T_W$ -based multiplication and addition are (Koissi and Shapiro, 2006):

$$\tilde{A} \underset{T_W}{\oplus} \tilde{B} = (a+b, \max(l_A, l_B)) \tag{11}$$

$$\tilde{A} \underset{T_W}{\otimes} \tilde{B} = (ab, \max(l_A|b|, l_B|a|)) \tag{12}$$

Using equations (11) and (12), equation (2) can be rewritten as:

$$\tilde{Y}_{x,t} = (a_x + b_x k_t,\ \max(\alpha_x, |b_x|\delta_x, \beta_x|k_t|)) \tag{13}$$

To find the unknown parameters $a_x$, $b_x$, $k_t$, $\alpha_x$, $\beta_x$, and $\delta_t$; Koissi and Shapiro suggest a solution to equation (2) by minimizing the total squared distance between $\tilde{A}_x \underset{T_W}{\oplus} \tilde{B}_x \underset{T_W}{\otimes} \tilde{K}_t$ and $\tilde{Y}_{x,t}$. Here, they make use of Diamond distance as the fuzzy distance measure. Diamond distance (Diamond, 1988) between two symmetric triangular fuzzy numbers $\tilde{A}_1 = (a_1, \alpha_1)$ and $\tilde{A}_2 = (a_2, \alpha_2)$ is defined as:

$$D_{LR}(\tilde{A}_1, \tilde{A}_2) = (a_1 - a_2)^2 + [(a_1 - \alpha_1) - (a_2 - \alpha_2)]^2 + [(a_1 + \alpha_1) - (a_2 + \alpha_2)]^2 \tag{14}$$

Minimizing total Diamond distance leads to following optimization problem for each age cohort $x$ and time $t$:

$$\text{Minimize } \sum_x \sum_t D_{LR}[\tilde{A}_x \underset{T_W}{\oplus} (\tilde{B}_x \underset{T_W}{\otimes} \tilde{K}_t), \tilde{Y}_{x,t}]^2 \tag{15}$$

where

$$D_{LR}[\tilde{A}_x \underset{T_W}{\oplus} (\tilde{B}_x \underset{T_W}{\otimes} \tilde{K}_t), \tilde{Y}_{x,t}]^2 = (a_x + b_x k_t - y_{x,t})^2 + [a_x + b_x k_t - \max\{\alpha_x, |b_x|\delta_t, \beta_x|k_t|\} - (y_{x,t} - e_{x,t})]^2 + [a_x + b_x k_t + \max\{\alpha_x, |b_x|\delta_t, \beta_x|k_t|\} - (y_{x,t} + e_{x,t})]^2 \tag{16}$$

This is an unconstrained nonlinear problem as finding the optimal values of parameters $a_x$, $b_x$, $k_t$, $\alpha_x$, $\beta_x$, and $\delta_t$ require dealing with a maximum function. Applying SVD, $a_x$ can be obtained as:

$$a_x = \frac{1}{T}\sum_t y_{x,t} \tag{17}$$

Finding the parameters $b_x$, $k_t$, $\alpha_x$, $\beta_x$, and $\delta_t$ is less straightforward, because, the structure of equation (15) does not allow using a derivative based solution algorithm. Hence, *fminsearch* tool of MATLAB optimization application for unconstrained optimization problems can be utilized to find the unknown parameters. *fminsearch* is a derivative free method for unconstrained nonlinear optimization problems based on Nelder-Mead simplex algorithm (Nelder and Mead, 1965).

# 3 NUMERICAL FINDINGS

The proposed method is applied to mortality data for Finland. The reason why Finland dataset is selected for application is that the mortality rates in Finland show some fluctuations due to some exogenous effects such as World War II. Furthermore, Koissi and Shapiro also apply their method on Finland dataset. In this study, standard LC and the fuzzy LC models are also applied to the same dataset and the outcomes are compared with the results obtained from the proposed method. The data is obtained freely from "Human Mortality Database" at www.mortality.org. In all computations total mortality rates (for both sexes) of seventeen consecutive five-year-periods 1925-1929, 1930-1934 …, 2005-2009, and twenty two age cohorts of [0, 1), [1-5), [5, 10), …, [100, 105) are used (making 374 data points in total).

To demonstrate the results, three example periods are selected and given in Table 1, 2, and 3. These tables display the spreads of fuzzified values of Finland for selected five-year-periods of 1925-1929 (the first time period), 1965-1969 (the mid-time period in dataset) and 2005-2009 (the last time period) respectively. The results in these tables are calculated via Koissi and Shapiro's fuzzified LC model (spread$_{OLS}$) and the modified fuzzy LC model (spread$_{SVD}$).

Tables 1 to 3 illustrate that proposed method give smaller spreads compared to Koissi and Shapiro's method for ten age groups in 1925-1929 period, for sixteen age groups in 1965-1969 period, and for twenty age groups in 2005-2009. This shows that the number of smaller spreads generated during fuzzification of $Y_{x,t}$ by the proposed method are increasing by time. This trend can be explained by the advances in accurate data approaches which result in vagueness reduction, thus smaller spreads. When the whole dataset is considered, paired t-test results show that the proposed method is superior to Koissi and Shapiro's method in terms of smaller spread generation (t-value=13.53, p-value=0.000), smaller absolute distances between observed $Y_{x,t}$ and center values of fuzzified $Y_{x,t}$ (t-value=5.07, p-value=0.000) and smaller squared distances between observed $Y_{x,t}$ and center values of fuzzified $Y_{x,t}$ (t-value=3.88, p-value=0.000) during the fuzzification of log-mortality rates.

The two methods are also compared in terms of their $\tilde{Y}_{x,t}$ estimations based on the model parameters obtained from the second parts of the methods. Figure 2 and 3 illustrate the observed $Y_{x,t}$, and estimated centers of $\tilde{Y}_{x,t}$ with Koissi and

Table 1: Spreads of fuzzified log-mortality values for Finland, 1925-1929.

| Age group | Spread$_{OLS}$ | Spread$_{SVD}$ | Age group | Spread$_{OLS}$ | Spread$_{SVD}$ |
|---|---|---|---|---|---|
| [0, 1) | 0.3220 | 0.4419 | [50, 55) | 0.0750 | 0.1560 |
| [1, 5) | 0.4920 | 0.3556 | [55, 60) | 0.1250 | 0.1533 |
| [5, 10) | 0.5300 | 0.1723 | [60, 65) | 0.1540 | 0.1837 |
| [10, 15) | 0.4890 | 0.2333 | [65, 70) | 0.1860 | 0.2686 |
| [15, 20) | 0.9170 | 0.4520 | [70, 75) | 0.1920 | 0.3123 |
| [20, 25) | 1.6470 | 0.9371 | [75, 80) | 0.2137 | 0.2960 |
| [25, 30) | 1.3170 | 0.6741 | [80, 85) | 0.1970 | 0.2603 |
| [30, 35) | 1.0320 | 0.4829 | [85, 90) | 0.2110 | 0.2300 |
| [35, 40) | 0.7380 | 0.3062 | [90, 95) | 0.2340 | 0.2473 |
| [40, 45) | 0.3860 | 0.1300 | [95, 100) | 0.2340 | 0.2556 |
| [45, 50) | 0.1380 | 0.1173 | [100, 105) | 0.3750 | 0.4252 |

Table 2: Spreads of fuzzified log-mortality values for Finland, 1965-1969.

| Age group | $Spread_{OLS}$ | $Spread_{SVD}$ | Age group | $Spread_{OLS}$ | $Spread_{SVD}$ |
|---|---|---|---|---|---|
| [0, 1) | 0.3700 | 0.3161 | [50, 55) | 0.0750 | 0.0931 |
| [1, 5) | 0.4923 | 0.2088 | [55, 60) | 0.1250 | 0.1498 |
| [5, 10) | 0.5300 | 0.1304 | [60, 65) | 0.1540 | 0.1627 |
| [10, 15) | 0.4890 | 0.1914 | [65, 70) | 0.1860 | 0.1847 |
| [15, 20) | 0.9170 | 0.4520 | [70, 75) | 0.2080 | 0.2074 |
| [20, 25) | 1.6470 | 0.4548 | [75, 80) | 0.2194 | 0.2331 |
| [25, 30) | 1.3170 | 0.3177 | [80, 85) | 0.1970 | 0.2184 |
| [30, 35) | 1.0320 | 0.2312 | [85, 90) | 0.2110 | 0.2300 |
| [35, 40) | 0.7380 | 0.1385 | [90, 95) | 0.2340 | 0.1424 |
| [40, 45) | 0.3860 | 0.1300 | [95, 100) | 0.2340 | 0.1717 |
| [45, 50) | 0.1380 | 0.0754 | [100, 105) | 0.3750 | 0.1945 |

Table 3: Spreads of fuzzified log-mortality values for Finland, 2005-2009.

| Age group | $Spread_{OLS}$ | $Spread_{SVD}$ | Age group | $Spread_{OLS}$ | $Spread_{SVD}$ |
|---|---|---|---|---|---|
| [0, 1) | 0.4180 | 0.2102 | [50, 55) | 0.0750 | 0.0401 |
| [1, 5) | 0.4926 | 0.0852 | [55, 60) | 0.1250 | 0.1468 |
| [5, 10) | 0.5300 | 0.0951 | [60, 65) | 0.1540 | 0.1450 |
| [10, 15) | 0.4890 | 0.1561 | [65, 70) | 0.1860 | 0.1141 |
| [15, 20) | 0.9170 | 0.4520 | [70, 75) | 0.2240 | 0.1192 |
| [20, 25) | 1.6470 | 0.0487 | [75, 80) | 0.2251 | 0.1801 |
| [25, 30) | 1.3170 | 0.0175 | [80, 85) | 0.1970 | 0.1831 |
| [30, 35) | 1.0320 | 0.0194 | [85, 90) | 0.2110 | 0.2300 |
| [35, 40) | 0.7380 | 0.0028 | [90, 95) | 0.2340 | 0.0542 |
| [40, 45) | 0.3860 | 0.1300 | [95, 100) | 0.2340 | 0.1011 |
| [45, 50) | 0.1380 | 0.0401 | [100, 105) | 0.3750 | 0.0003 |

Shapiro's and modified methods for age groups [5, 10) and [40-45) respectively. These two age groups are selected randomly as examples. In both figures, the horizontal axis stand for time periods (1=1925-1929, …, 17=2005-2009), whereas the vertical axis depicts the log-mortality rates. The numerical findings show that the proposed method displays better similarity between observed and estimated log-mortality rates compared to Koissi and Shapiro's method. In fact, paired t-test results show that the modified method is superior to Koissi and Shapiro's method in terms of smaller spread generation (t-value=13.97, p-value=0.000), smaller absolute distances between observed $Y_{x,t}$ and center values of fuzzified $Y_{x,t}$ (t-value=2.69, p-value=0.004) and smaller squared distances between observed $Y_{x,t}$ and center values of fuzzified $Y_{x,t}$ (t-value=4.19, p-value=0.000) in estimating the log-mortality rates.

As depicted in Figure 2 and 3, the proposed method gives better fits mainly due to the utilization of SVD in fuzzifying $Y_{x,t}$ values. On the other hand, Koissi and Shapiro make use of OLS, therefore, the resulting centers of $\tilde{Y}_{x,t}$ follows a linear trend which is incapable of capturing the fluctuations in data.

However, in Finland mortality rates during World War II are higher compared to the other periods, thus the data show fluctuations and even outlier points for some age groups. In contrast to Koissi and Shapiro's method, the proposed method has the ability to reflect data pattern, thus it gives better fits as the estimation of model parameters phase utilizes the better fitted $\tilde{Y}_{x,t}$ values.

Finally, the standard LC method is applied to the same dataset as well (although the homoscedasticity assumption is violated). When the proposed method is compared with the standard LC method, paired t-tests on absolute and squared distances between the observed $Y_{x,t}$ and the estimated centers of $\tilde{Y}_{x,t}$ show that standard LC method gives better results than the modified one (t-value=6.20, p-value=0.004; t-value=4.09, p-value=0.000 respectively). However, as mentioned before, standard LC model cannot be applied in cases where there is vagueness in assumptions related with the homoscedasticity and the magnitude of variance of error terms. In fact, the standard LC method cannot be used in this data set as it requires the data in each age group to be normally distributed with mean 0 and a small

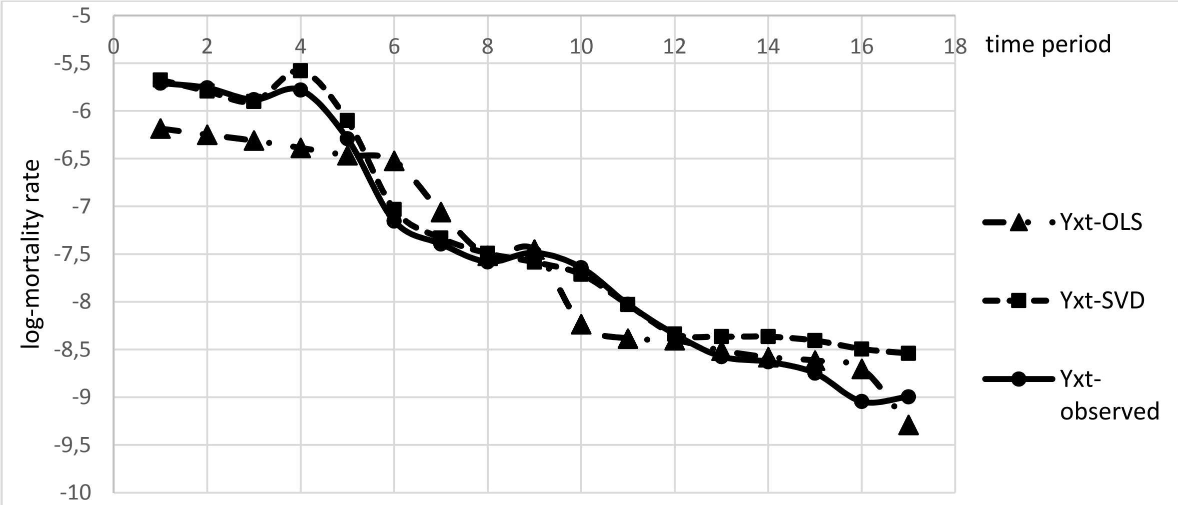


Figure 2: Comparison of observed $Y_{x,t}$ and estimated centers of $\tilde{Y}_{x,t}$ with Koissi and Shapiro's and modified methods for age group [5, 10).

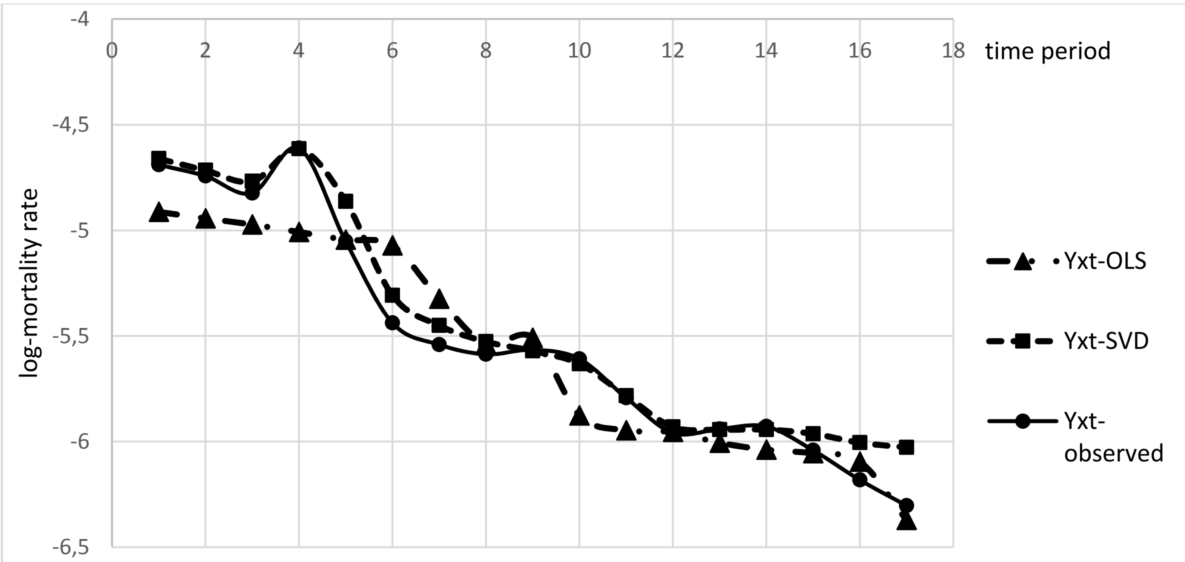


Figure 3: Comparison of observed $Y_{x,t}$ and estimated centers of $\tilde{Y}_{x,t}$ with Koissi and Shapiro's and modified methods for age group [40, 45).

variance $\sigma_{\varepsilon}^{2}$. In Finland data set, there are seventeen data points for each age group separately which do not a normality test to be performed to see whether the homoscedasticity assumption is met. Thus, the better results obtained by standard LC method do not make sense as the basic assumption of standard LC approach is violated.

# 4 CONCLUSIONS

In this paper, a modified version of Koissi and Shapiro's fuzzified LC method is proposed. The proposed method makes use of SVD in fuzzification of observed log-mortality rates instead of taking time as the independent variable. Numerical findings show that proposed method is better in smaller spread generation and mortality rate estimation even the utilized dataset reveal some fluctuations within time.

The proposed method can be used in cases of heteroscedasticity and other violations where standard LC method cannot be applied. In fact the method gives reasonable estimations when the number or the quality of data do not permit standard LC or similar stochastic methods to be used.

The future mortality rates can be forecasted via estimating future $\tilde{K}_t$ values with some suitable fuzzy time series analysis based on the $\tilde{K}_t$ values obtained from the modified model. As well as this, the modified fuzzy LC method for estimating mortality rates can be extended to model fertility and migration rates. Once the three vital rates (mortality, fertility, and migration rates) are known it may be possible to develop a fuzzy population forecasting model, which may be a research topic of a future work.

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