An Order Hyperresolution Calculus for Gödel Logic with Truth Constants and Equality, Strict Order, Delta

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Abstract: In (Guller, 2014), we have generalised the well-known hyperresolution principle to the first-order Gödel logic with truth constants. This paper is a continuation of our work. We propose a hyperresolution calculus suitable for automated deduction in a useful expansion of Gödel logic by intermediate truth constants and the equality, $\eqcirc$, strict order, $\prec$, projection, Δ, operators. We solve the deduction problem of a formula from a countable theory in this expansion. We expand Gödel logic by a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$. Our approach is based on translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We shall investigate the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard G-algebra and truth constants are interpreted by 'themselves'. The hyperresolution calculus is refutation sound and complete for a countable order clausal theory under a certain condition for the set of truth constants occurring in the theory. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants and the equality, $\eqcirc$, strict order, $\prec$, projection, Δ, operators.

1 INTRODUCTION

Current research in many-valued logics is mainly concerned with left-continuous t-norm based logics including the fundamental fuzzy logics: Gödel, Łukasiewicz, and Product ones. Most explorations of t-norm based logics are focused on tautologies and deduction calculi with the only distinguished truth degree 1, (Hájek, 2001). However, in many real-world applications, one may be interested in representation and inference with explicit partial truth; besides the truth constants *0*, *1*, intermediate truth constants are involved in. In the literature, two main approaches to expansions with truth constants, are described. Historically, the first one has been introduced in (Pavelka, 1979), where the propositional Łukasiewicz logic is augmented by truth constants $\bar{r}$, $r \in [0,1]$, Pavelka's logic (*PL*). A formula of the form $\bar{r} \to \phi$ evaluated to 1 expresses that the truth value of ϕ is greater than or equal to r. In (Novák et al., 1999), further development of evaluated formulae, and in (Hájek, 2001), Rational Pavelka's logic (*RPL*) - a simplification of *PL*, are described. Another approach relies on traditional algebraic semantics. Various completeness results for expansions of t-norm based logics with countably many truth constants are investigated, among others, in (Esteva et al., 2001; Savický et al., 2006; Esteva et al., 2007b; Esteva et al., 2007a; Esteva et al., 2009; Esteva et al., 2010a; Esteva et al., 2010b).

In (Guller, 2012; Guller, 2015), we have generalised the well-known hyperresolution principle to the first-order Gödel logic for the general case. Our approach is based on translation of a formula of Gödel logic to an equivalent satisfiable finite order clausal theory, consisting of order clauses. We have introduced a notion of quantified atom: a formula a is a quantified atom if $a = Qx\, p(t_0,\dots,t_\tau)$ where Q is a quantifier ($\forall$, $\exists$); $p(t_0,\dots,t_\tau)$ is an atom; x is a variable occurring in $p(t_0,\dots,t_\tau)$; for all $i \leq \tau$, either $t_i = x$ or x does not occur in t_i (t_i is a free term in the quantified atom). The notion of quantified atom is all important. It permits us to extend classical unification to quantified atoms without any additional computational cost. Two quantified atoms $Qx\, p(t_0,\dots,t_\tau)$ and $Q'x'\, p'(t'_0,\dots,t'_\tau)$ are unifiable if

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$Q = Q'$, $x = x'$, $p = p'$, and the left-right sequence of free terms of $Qx\,p(t_0,\dots,t_\tau)$ is unifiable with the left-right sequence of free terms of $Q'x'\,p'(t'_0,\dots,t'_\tau)$ in the standard manner. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. On the basis of the hyperresolution principle, a calculus operating over order clausal theories, has been devised. The calculus is proved to be refutation sound and complete for the countable case with respect to the standard G-algebra $G = ([0,1], \leq, \vee, \wedge, \Rightarrow, \overline{}, \boldsymbol{\eqcirc}, \boldsymbol{\prec}, 0, 1)$ augmented by binary operators $\boldsymbol{\eqcirc}$ and $\boldsymbol{\prec}$ for $\eqcirc$ and $\prec$, respectively. As another step, one may incorporate a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$, to get a modification of the hyperresolution calculus suitable for automated deduction with explicit partial truth (Guller, 2014). We shall investigate the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard G-algebra G and truth constants are interpreted by 'themselves'. We say that a set $\{0,1\} \subseteq X$ of truth constants is admissible with respect to suprema and infima if, for all $\emptyset \neq Y_1, Y_2 \subseteq X$ and $\bigvee Y_1 = \bigwedge Y_2$, $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 \in Y_2$. Then the hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of all truth constants occurring in the theory, is admissible with respect to suprema and infima. This condition obviously covers the case of finite order clausal theories. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants and the operators $\boldsymbol{\eqcirc}, \boldsymbol{\prec}, \boldsymbol{\Delta}$ on $[0,1]$:

$$a \boldsymbol{\eqcirc} b = \begin{cases} 1 & \textit{if } a = b, \\ 0 & \textit{else}; \end{cases} \qquad a \boldsymbol{\prec} b = \begin{cases} 1 & \textit{if } a < b, \\ 0 & \textit{else}; \end{cases}$$

$$\boldsymbol{\Delta} a = \begin{cases} 1 & \textit{if } a = 1, \\ 0 & \textit{else}; \end{cases}$$

which strengthens a similar result for prenex formulae of G^{Δ}_{∞} stated in Conclusion of (Baaz et al., 2012).

Some applications of our hyperresolution calculus may lead to computational linguistics, to design and analysis of scientific (natural) language processing systems (Mandelíková, 2012; Mandelíková, 2014).

The paper is organised as follows. Section 2 gives the basic notions and notation concerning the first-order Gödel logic. Section 3 deals with clause form translation. In Section 4, we propose a hyperresolution calculus with truth constants and prove its refutational soundness, completeness. Section 5 brings conclusions.

2 FIRST-ORDER GÖDEL LOGIC

Throughout the paper, we shall use the common notions and notation of first-order logic. $\mathbb{N} \mid \mathbb{Z}$ designates the set of natural | integer numbers and $\leq \mid <$ the standard order | strict order on $\mathbb{N} \mid \mathbb{Z}$. By $\mathcal{L}$ we denote a first-order language. $\mathit{Var}_{\mathcal{L}} \mid \mathit{Func}_{\mathcal{L}} \mid \mathit{Pred}_{\mathcal{L}} \mid \mathit{Term}_{\mathcal{L}} \mid \mathit{GTerm}_{\mathcal{L}} \mid \mathit{Atom}_{\mathcal{L}} \mid \mathit{GAtom}_{\mathcal{L}}$ denotes the set of all variables | function symbols | predicate symbols | terms | ground terms | atoms | ground atoms of $\mathcal{L}$. $\mathit{ar}_{\mathcal{L}} : \mathit{Func}_{\mathcal{L}} \cup \mathit{Pred}_{\mathcal{L}} \longrightarrow \mathbb{N}$ denotes the mapping assigning an arity to every function and predicate symbol of $\mathcal{L}$. We assume truth constants - nullary predicate symbols $\mathit{0}, \mathit{1} \in \mathit{Pred}_{\mathcal{L}}$, $\mathit{ar}_{\mathcal{L}}(\mathit{0}) = \mathit{ar}_{\mathcal{L}}(\mathit{1}) = 0$; $\mathit{0}$ denotes the false and $\mathit{1}$ the true in $\mathcal{L}$. Let $\mathbb{C}_{\mathcal{L}} \subseteq (0,1)$ be countable. In addition, we assume a countable set of nullary predicate symbols $\overline{C}_{\mathcal{L}} = \{\bar{c} \mid \bar{c} \in \mathit{Pred}_{\mathcal{L}}, \mathit{ar}_{\mathcal{L}}(\bar{c}) = 0, c \in \mathbb{C}_{\mathcal{L}}\} \subseteq \mathit{Pred}_{\mathcal{L}}$; $\{\mathit{0}\}$, $\{\mathit{1}\}$, $\overline{C}_{\mathcal{L}}$ are pairwise disjoint. $\mathit{0}$, $\mathit{1}$, $\bar{c} \in \overline{C}_{\mathcal{L}}$ are called truth constants. We denote $\mathit{Tcons}_{\mathcal{L}} = \{\mathit{0}, \mathit{1}\} \cup \overline{C}_{\mathcal{L}} \subseteq \mathit{Pred}_{\mathcal{L}}$. Let $X \subseteq \mathit{Tcons}_{\mathcal{L}}$. We denote $\overline{X} = \{0 \mid \mathit{0} \in X\} \cup \{1 \mid \mathit{1} \in X\} \cup \{c \mid \bar{c} \in X \cap \overline{C}_{\mathcal{L}}\} \subseteq [0,1]$. We introduce a new unary connective Δ, Delta, and binary connectives $\eqcirc$, equality, $\prec$, strict order. By $\mathit{OrdForm}_{\mathcal{L}}$ we designate the set of all so-called order formulae of $\mathcal{L}$ built up from $\mathit{Atom}_{\mathcal{L}}$ and $\mathit{Var}_{\mathcal{L}}$ using the connectives: $\neg$, negation, Δ, $\wedge$, conjunction, $\vee$, disjunction, $\rightarrow$, implication, $\leftrightarrow$, equivalence, $\eqcirc$, $\prec$, and the quantifiers: $\forall$, the universal one, $\exists$, the existential one.[1] In the paper, we shall assume that $\mathcal{L}$ is a countable first-order language; hence, all the above mentioned sets of symbols and expressions are countable. Let $\varepsilon \mid \varepsilon_i$, $1 \leq i \leq m \mid \upsilon_i$, $1 \leq i \leq n$, be either an expression or a set of expressions or a set of sets of expressions of $\mathcal{L}$, in general. By $\mathit{vars}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Var}_{\mathcal{L}} \mid \mathit{freevars}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Var}_{\mathcal{L}} \mid \mathit{boundvars}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Var}_{\mathcal{L}} \mid \mathit{funcs}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Func}_{\mathcal{L}} \mid \mathit{preds}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Pred}_{\mathcal{L}} \mid \mathit{atoms}(\varepsilon_1,\dots,\varepsilon_m) \subseteq \mathit{Atom}_{\mathcal{L}}$ we denote the set of all variables | free variables | bound variables | function symbols | predicate symbols | atoms of $\mathcal{L}$ occurring in $\varepsilon_1,\dots,\varepsilon_m$. ε is closed iff $\mathit{freevars}(\varepsilon) = \emptyset$. By ℓ we denote the empty sequence. By $|\varepsilon_1,\dots,\varepsilon_m| = m$ we denote the length of the sequence $\varepsilon_1,\dots,\varepsilon_m$. We define the concatenation of the sequences $\varepsilon_1,\dots,\varepsilon_m$ and $\upsilon_1,\dots,\upsilon_n$ as $(\varepsilon_1,\dots,\varepsilon_m),(\upsilon_1,\dots,\upsilon_n) = \varepsilon_1,\dots,\varepsilon_m,\upsilon_1,\dots,\upsilon_n$. Note that concatenation of sequences is associative.

Let X, Y, Z be sets, $Z \subseteq X$; $f : X \longrightarrow Y$ be a mapping. By $\|X\|$ we denote the set-theoretic cardinality of X. X being a finite subset of Y is denoted as

[1] We assume a decreasing connective and quantifier precedence: $\forall$, $\exists$, $\neg$, Δ, $\eqcirc$, $\prec$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.

$X \subseteq_{\mathcal{F}} Y$. We designate $\mathcal{P}(X) = \{x \mid x \subseteq X\}$; $\mathcal{P}(X)$ is the power set of X; $\mathcal{P}_{\mathcal{F}}(X) = \{x \mid x \subseteq_{\mathcal{F}} X\}$; $\mathcal{P}_{\mathcal{F}}(X)$ is the set of all finite subsets of X; $f[Z] = \{f(z) \mid z \in Z\}$; $f[Z]$ is the image of Z under f; $f|_Z = \{(z, f(z)) \mid z \in Z\}$; $f|_Z$ is the restriction of f onto Z. Let $\gamma \leq \omega$. A sequence δ of X is a bijection $\delta : \gamma \longrightarrow X$. Recall that X is countable if and only if there exists a sequence of X. Let I be a set and $S_i \neq \emptyset$, $i \in I$, be sets. A selector $\mathcal{S}$ over $\{S_i \mid i \in I\}$ is a mapping $\mathcal{S} : I \longrightarrow \bigcup\{S_i \mid i \in I\}$ such that for all $i \in I$, $\mathcal{S}(i) \in S_i$. We denote $\mathcal{S}el(\{S_i \mid i \in I\}) = \{\mathcal{S} \mid \mathcal{S} \textit{ is a selector over } \{S_i \mid i \in I\}\}$. $\mathbb{R}$ designates the set of real numbers and $\leq \mid <$ the standard order | strict order on $\mathbb{R}$. We denote $\mathbb{R}_0^+ = \{c \mid 0 \leq c \in \mathbb{R}\}$, $\mathbb{R}^+ = \{c \mid 0 < c \in \mathbb{R}\}$; $[0,1] = \{c \mid 0 \leq c \leq 1, c \in \mathbb{R}\}$; $[0,1]$ is the unit interval. Let $c \in \mathbb{R}^+$. $\log c$ denotes the binary logarithm of c. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_0^+$. f is of the order of g, in symbols $f \in O(g)$, iff there exist $n_0 \in \mathbb{N}$ and $c^* \in \mathbb{R}_0^+$ such that for all $n \geq n_0$, $f(n) \leq c^* \cdot g(n)$.

Let $t \in Term_{\mathcal{L}}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$. The size of $t \mid \phi$, in symbols $|t| \mid |\phi|$, is defined as the number of nodes of its standard tree representation. We define the size of T as $|T| = \sum_{\phi \in T} |\phi|$. By $varseq(\phi)$, $vars(varseq(\phi)) \subseteq Var_{\mathcal{L}}$, we denote the sequence of all variables of $\mathcal{L}$ occurring in ϕ which is built up via the left-right preorder traversal of ϕ. For example, $varseq(\exists w\,(\forall x\, p(x,x,z) \vee \exists y\, q(x,y,z))) = w,x,x,x,z,y,x,y,z$ and $|w,x,x,x,z,y,x,y,z| = 9$. A sequence of variables will often be denoted as $\bar{x}$, $\bar{y}$, $\bar{z}$, etc. Let $Q \in \{\forall, \exists\}$ and $\bar{x} = x_1, \ldots, x_n$ be a sequence of variables of $\mathcal{L}$. By $Q\bar{x}\phi$ we denote $Qx_1 \ldots Qx_n\,\phi$.

Gödel logic is interpreted by the standard G-algebra augmented by the operators $\overset{\centerdot}{=}$, $\overset{\centerdot}{\prec}$, $\boldsymbol{\Delta}$ for the connectives $\overset{\centerdot}{=}$, $\prec$, Δ, respectively.

$$G = ([0,1], \leq, \vee, \wedge, \Rightarrow, \overline{}, \overset{\centerdot}{=}, \overset{\centerdot}{\prec}, \boldsymbol{\Delta}, 0, 1)$$

where $\vee \mid \wedge$ denotes the supremum | infimum operator on $[0,1]$;

$$a \Rightarrow b = \begin{cases} 1 & \textit{if } a \leq b, \\ b & \textit{else}; \end{cases} \qquad \overline{a} = \begin{cases} 1 & \textit{if } a = 0, \\ 0 & \textit{else}; \end{cases}$$

$$a \overset{\centerdot}{=} b = \begin{cases} 1 & \textit{if } a = b, \\ 0 & \textit{else}; \end{cases} \qquad a \overset{\centerdot}{\prec} b = \begin{cases} 1 & \textit{if } a < b, \\ 0 & \textit{else}; \end{cases}$$

$$\boldsymbol{\Delta} a = \begin{cases} 1 & \textit{if } a = 1, \\ 0 & \textit{else}. \end{cases}$$

Recall that G is a complete linearly ordered lattice algebra; $\vee \mid \wedge$ is commutative, associative, idempotent, monotone; $0 \mid 1$ is its neutral element; the residuum operator $\Rightarrow$ of $\wedge$ satisfies the condition of residuation:

$$\text{for all } a,b,c \in G,\ a \wedge b \leq c \Longleftrightarrow a \leq b \Rightarrow c; \qquad (1)$$

Gödel negation $\overline{}$ satisfies the condition:

$$\text{for all } a \in G,\ \overline{a} = a \Rightarrow 0; \qquad (2)$$

the following properties, which will be exploited later, hold:[2]

for all $a,b,c \in G$,

$$a \vee b \wedge c = (a \vee b) \wedge (a \vee c), \quad \text{(distributivity of } \vee \text{ over } \wedge) \quad (3)$$

$$a \wedge (b \vee c) = a \wedge b \vee a \wedge c, \quad \text{(distributivity of } \wedge \text{ over } \vee) \quad (4)$$

$$a \Rightarrow b \vee c = (a \Rightarrow b) \vee (a \Rightarrow c), \qquad (5)$$

$$a \Rightarrow b \wedge c = (a \Rightarrow b) \wedge (a \Rightarrow c), \qquad (6)$$

$$a \vee b \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c), \qquad (7)$$

$$a \wedge b \Rightarrow c = (a \Rightarrow c) \vee (b \Rightarrow c), \qquad (8)$$

$$a \Rightarrow (b \Rightarrow c) = a \wedge b \Rightarrow c, \qquad (9)$$

$$((a \Rightarrow b) \Rightarrow b) \Rightarrow b = a \Rightarrow b, \qquad (10)$$

$$(a \Rightarrow b) \Rightarrow c = ((a \Rightarrow b) \Rightarrow b) \wedge (b \Rightarrow c) \vee c, \qquad (11)$$

$$(a \Rightarrow b) \Rightarrow 0 = ((a \Rightarrow 0) \Rightarrow 0) \wedge (b \Rightarrow 0), \qquad (12)$$

$$\boldsymbol{\Delta} a = a \overset{\centerdot}{=} 1. \qquad (13)$$

An interpretation I for $\mathcal{L}$ is a triple $\big(\mathcal{U}_I, \{f^I \mid f \in Func_{\mathcal{L}}\}, \{p^I \mid p \in Pred_{\mathcal{L}}\}\big)$ defined as follows: $\mathcal{U}_I \neq \emptyset$ is the universum of I; every $f \in Func_{\mathcal{L}}$ is interpreted as a function $f^I : \mathcal{U}_I^{ar_{\mathcal{L}}(f)} \longrightarrow \mathcal{U}_I$; every $p \in Pred_{\mathcal{L}}$ is interpreted as a $[0,1]$-relation $p^I : \mathcal{U}_I^{ar_{\mathcal{L}}(p)} \longrightarrow [0,1]$. A variable assignment in I is a mapping $Var_{\mathcal{L}} \longrightarrow \mathcal{U}_I$. We denote the set of all variable assignments in I as $\mathcal{S}_I$. Let $e \in \mathcal{S}_I$ and $u \in \mathcal{U}_I$. A variant $e[x/u] \in \mathcal{S}_I$ of e with respect to x and u is defined as

$$e[x/u](z) = \begin{cases} u & \textit{if } z = x, \\ e(z) & \textit{else}. \end{cases}$$

Let $t \in Term_{\mathcal{L}}$, $\bar{x}$ be a sequence of variables of $\mathcal{L}$, $\phi \in OrdForm_{\mathcal{L}}$. In I with respect to e, we define the value $\|t\|_e^I \in \mathcal{U}_I$ of t by recursion on the structure of t, the value $\|\bar{x}\|_e^I \in \mathcal{U}_I^{|\bar{x}|}$ of $\bar{x}$, the truth value $\|\phi\|_e^I \in [0,1]$ of ϕ by recursion on the structure of ϕ, as usual. Notice that $0_e^I = 0$, $1_e^I = 1$, for all $\bar{c} \in \overline{C}_{\mathcal{L}}$, $\bar{c}_e^I = c$, $\|\phi_1 \leftrightarrow \phi_2\|_e^I = (\|\phi_1\|_e^I \Rightarrow \|\phi_2\|_e^I) \wedge (\|\phi_2\|_e^I \Rightarrow \|\phi_1\|_e^I)$. Let ϕ be closed. Then, for all $e, e' \in \mathcal{S}_I$, $\|\phi\|_e^I = \|\phi\|_{e'}^I$. Let $e \in \mathcal{S}_I \neq \emptyset$. We denote $\|\phi\|^I = \|\phi\|_e^I$.

Let $\mathcal{L} \mid \mathcal{L}'$ be a first-order language and $I \mid I'$ be an interpretation for $\mathcal{L} \mid \mathcal{L}'$. $\mathcal{L}'$ is an expansion of $\mathcal{L}$ iff $Func_{\mathcal{L}'} \supseteq Func_{\mathcal{L}}$ and $Pred_{\mathcal{L}'} \supseteq Pred_{\mathcal{L}}$; on the other side, we say $\mathcal{L}$ is a reduct of $\mathcal{L}'$. I' is an expansion of I to $\mathcal{L}'$ iff $\mathcal{L}'$ is an expansion of $\mathcal{L}$, $\mathcal{U}_{I'} = \mathcal{U}_I$, for all $f \in Func_{\mathcal{L}}$, $f^{I'} = f^I$, for all $p \in Pred_{\mathcal{L}}$, $p^{I'} = p^I$;

[2] We assume a decreasing operator precedence: $\overline{}$, $\boldsymbol{\Delta}$, $\overset{\centerdot}{=}$, $\overset{\centerdot}{\prec}$, $\wedge$, $\vee$, $\Rightarrow$.

on the other side, we say I is a reduct of I' to $\mathcal{L}$, in symbols $I = I'|_{\mathcal{L}}$.

An order theory of $\mathcal{L}$ is a set of order formulae of $\mathcal{L}$. Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$, $e \in \mathcal{S}_I$. ϕ is true in I with respect to e, written as $I \models_e \phi$, iff $\|\phi\|_e^I = 1$. I is a model of ϕ, in symbols $I \models \phi$, iff, for all $e \in \mathcal{S}_I$, $I \models_e \phi$. I is a model of T, in symbols $I \models T$, iff, for all $\phi \in T$, $I \models \phi$. ϕ is a logically valid formula iff, for every interpretation I for $\mathcal{L}$, $I \models \phi$. ϕ is equivalent to ϕ', in symbols $\phi \equiv \phi'$, iff, for every interpretation I for $\mathcal{L}$ and $e \in \mathcal{S}_I$, $\|\phi\|_e^I = \|\phi'\|_e^I$. We denote $tcons(\phi) = \{0,1\} \cup (preds(\phi) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$ and $tcons(T) = \{0,1\} \cup (preds(T) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$.

3 TRANSLATION TO CLAUSAL FORM

In the propositional case (Guller, 2010), we have proposed some translation of a formula to an equivalent *CNF* containing literals of the form either a or $a \rightarrow b$ or $(a \rightarrow b) \rightarrow b$ where a is a propositional atom and b is either a propositional atom or the propositional constant 0. An output equivalent *CNF* may be of exponential size with respect to the input formula; we had laid no restrictions on use of the distributivity law (3) during translation to conjunctive normal form. To avoid this disadvantage, we have devised translation to *CNF* via interpolation using new atoms, which produces an output *CNF* of linear size at the cost of being only equisatisfiable to the input formula. A similar approach exploiting the renaming subformulae technique can be found in (Plaisted and Greenbaum, 1986; de la Tour, 1992; Hähnle, 1994; Nonnengart et al., 1998; Sheridan, 2004). A *CNF* is further translated to a finite set of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is either a propositional atom or a propositional constant, 0, 1, and $\diamond \in \{\eqcirc, \prec\}$.

We now describe some generalisation of the mentioned translation to the first-order case. At first, we introduce a notion of quantified atom. Let $a \in Form_{\mathcal{L}}$. a is a quantified atom of $\mathcal{L}$ iff $a = Qx\, p(t_0, \dots, t_\tau)$ where $p(t_0, \dots, t_\tau) \in Atom_{\mathcal{L}}$, $x \in vars(p(t_0, \dots, t_\tau))$, either $t_i = x$ or $x \notin vars(t_i)$. $QAtom_{\mathcal{L}} \subseteq Form_{\mathcal{L}}$ denotes the set of all quantified atoms of $\mathcal{L}$. $QAtom_{\mathcal{L}}^Q \subseteq QAtom_{\mathcal{L}}$, $Q \in \{\forall, \exists\}$, denotes the set of all quantified atoms of $\mathcal{L}$ of the form Qxa. Let $\varepsilon \mid \varepsilon_i$, $1 \le i \le m \mid \upsilon_i$, $1 \le i \le n$, be either an expression or a set of expressions or a set of sets of expressions of $\mathcal{L}$, in general. By $qatoms(\varepsilon_1, \dots, \varepsilon_m) \subseteq QAtom_{\mathcal{L}}$ we denote the set of all quantified atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. We denote $qatoms^Q(\varepsilon_1, \dots, \varepsilon_m) = qatoms(\varepsilon_1, \dots, \varepsilon_m) \cap QAtom_{\mathcal{L}}^Q$, $Q \in \{\forall, \exists\}$. Let $Qx\, p(t_0, \dots, t_\tau) \in QAtom_{\mathcal{L}}$ and $p(t'_0, \dots, t'_\tau) \in Atom_{\mathcal{L}}$. We denote

$$boundindset(Qx\, p(t_0, \dots, t_\tau)) = \{i \mid i \le \tau, t_i = x\} \neq \emptyset.$$

Let $I = \{i \mid i \le \tau, x \notin vars(t_i)\}$ and $r_1, \dots, r_k$, $r_i \le \tau$, $k \le \tau$, for all $1 \le i < i' \le k$, $r_i < r_{i'}$, be a sequence such that $\{r_i \mid 1 \le i \le k\} = I$. We denote

$$freetermseq(Qx\, p(t_0, \dots, t_\tau)) = t_{r_1}, \dots, t_{r_k},$$
$$freetermseq(p(t'_0, \dots, t'_\tau)) = t'_0, \dots, t'_\tau.$$

We further introduce order clauses in Gödel logic. Let $l \in OrdForm_{\mathcal{L}}$. l is an order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\varepsilon_i \in Atom_{\mathcal{L}} \cup QAtom_{\mathcal{L}}$, $\diamond \in \{\eqcirc, \prec\}$. The set of all order literals of $\mathcal{L}$ is designated as $OrdLit_{\mathcal{L}} \subseteq OrdForm_{\mathcal{L}}$. An order clause of $\mathcal{L}$ is a finite set of order literals of $\mathcal{L}$; since $\eqcirc$ is commutative, for all $\varepsilon_1 \eqcirc \varepsilon_2 \in OrdLit_{\mathcal{L}}$, we identify $\varepsilon_1 \eqcirc \varepsilon_2$ and $\varepsilon_2 \eqcirc \varepsilon_1 \in OrdLit_{\mathcal{L}}$ with respect to order clauses. An order clause $\{l_1, \dots, l_n\}$ is written in the form $l_1 \vee \dots \vee l_n$. The order clause $\emptyset$ is called the empty order clause and denoted as $\square$. An order clause $\{l\}$ is called a unit order clause and denoted as l; if it does not cause the ambiguity with the denotation of the single order literal l in given context. We designate the set of all order clauses of $\mathcal{L}$ as $OrdCl_{\mathcal{L}}$. Let $l, l_0, \dots, l_n \in OrdLit_{\mathcal{L}}$ and $C, C' \in OrdCl_{\mathcal{L}}$. We define the size of C as $|C| = \sum_{l \in C} |l|$. By $l \vee C$ we denote $\{l\} \cup C$ where $l \notin C$. Analogously, by $l_0 \vee \dots \vee l_n \vee C$ we denote $\{l_0\} \cup \dots \cup \{l_n\} \cup C$ where, for all $i, i' \le n$, $i \ne i'$, $l_i \notin C$ and $l_i \ne l_{i'}$. By $C \vee C'$ we denote $C \cup C'$. C is a subclause of C', in symbols $C \sqsubseteq C'$, iff $C \subseteq C'$. An order clausal theory of $\mathcal{L}$ is a set of order clauses of $\mathcal{L}$. A unit order clausal theory is a set of unit order clauses.

Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$, $T, T' \subseteq OrdForm_{\mathcal{L}}$, $S, S' \subseteq OrdCl_{\mathcal{L}}$, I be an interpretation for $\mathcal{L}$, $e \in \mathcal{S}_I$. Note that $I \models_e l$ if and only if either $l = \varepsilon_1 \eqcirc \varepsilon_2$, $\|\varepsilon_1 \eqcirc \varepsilon_2\|_e^I = 1$, $\|\varepsilon_1\|_e^I = \|\varepsilon_2\|_e^I$; or $l = \varepsilon_1 \prec \varepsilon_2$, $\|\varepsilon_1 \prec \varepsilon_2\|_e^I = 1$, $\|\varepsilon_1\|_e^I < \|\varepsilon_2\|_e^I$. C is true in I with respect to e, written as $I \models_e C$, iff there exists $l^* \in C$ such that $I \models_e l^*$. I is a model of C, in symbols $I \models C$, iff, for all $e \in \mathcal{S}_I$, $I \models_e C$. I is a model of S, in symbols $I \models S$, iff, for all $C \in S$, $I \models C$. $\phi' \mid T' \mid C' \mid S'$ is a logical consequence of $\phi \mid T \mid C \mid S$, in symbols $\phi \mid T \mid C \mid S \models \phi' \mid T' \mid C' \mid S'$, iff, for every model I of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$, $I \models \phi' \mid T' \mid C' \mid S'$. $\phi \mid T \mid C \mid S$ is satisfiable iff there exists a model of $\phi \mid T \mid C \mid S$ for $\mathcal{L}$. Note that both $\square$ and $\square \in S$ are unsatisfiable. $\phi \mid T \mid C \mid S$ is equisatisfiable to $\phi' \mid T' \mid C' \mid S'$ iff $\phi \mid T \mid C \mid S$ is satisfiable if and only if $\phi' \mid T' \mid C' \mid S'$ is satisfiable. We denote $tcons(S) = \{0,1\} \cup (preds(S) \cap \overline{C}_{\mathcal{L}}) \subseteq Tcons_{\mathcal{L}}$. Let $S \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}}$. We define the size of S as $|S| = \sum_{C \in S} |C|$. l is a simplified order literal of $\mathcal{L}$ iff $l = \varepsilon_1 \diamond \varepsilon_2$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq QAtom_{\mathcal{L}}$. The set of all simplified order literals of $\mathcal{L}$

is designated as $SimOrdLit_{\mathcal{L}} \subseteq OrdLit_{\mathcal{L}}$. We denote $SimOrdCl_{\mathcal{L}} = \{C \,|\, C \in OrdCl_{\mathcal{L}}, C \subseteq SimOrdLit_{\mathcal{L}}\} \subseteq OrdCl_{\mathcal{L}}$. Let $\tilde{f}_0 \notin Func_{\mathcal{L}}$; $\tilde{f}_0$ is a new function symbol. Let $\mathbb{I} = \mathbb{N} \times \mathbb{N}$; $\mathbb{I}$ is an infinite countable set of indices. Let $\tilde{\mathbb{P}} = \{\tilde{p}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{P}} \cap Pred_{\mathcal{L}} = \emptyset$; $\tilde{\mathbb{P}}$ is an infinite countable set of new predicate symbols.

From a computational point of view, the worst case time and space complexity will be estimated using the logarithmic cost measurement. Let $\mathcal{A}$ be an algorithm. $\#O_{\mathcal{A}}(In) \geq 1$ denotes the number of all elementary operations executed by $\mathcal{A}$ on an input In.

3.1 Substitutions

We assume the reader to be familiar with the standard notions and notation of substitutions. We introduce a few definitions and denotations; some of them are slightly different from the standard ones, but found to be more convenient. Let $X = \{x_i \,|\, 1 \leq i \leq n\} \subseteq Var_{\mathcal{L}}$. A substitution ϑ of $\mathcal{L}$ is a mapping $\vartheta : X \longrightarrow Term_{\mathcal{L}}$. ϑ may be written in the form $x_1/\vartheta(x_1), \dots, x_n/\vartheta(x_n)$. We denote $dom(\vartheta) = X \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$ and $range(\vartheta) = \bigcup_{x \in X} vars(\vartheta(x)) \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. The set of all substitutions of $\mathcal{L}$ is designated as $Subst_{\mathcal{L}}$. Let $\vartheta, \vartheta' \in Subst_{\mathcal{L}}$. ϑ is a variable renaming of $\mathcal{L}$ iff $\vartheta : dom(\vartheta) \longrightarrow Var_{\mathcal{L}}$, for all $x, x' \in dom(\vartheta)$, $x \neq x'$, $\vartheta(x) \neq \vartheta(x')$. We define $id_{\mathcal{L}} : Var_{\mathcal{L}} \longrightarrow Var_{\mathcal{L}}$, $id_{\mathcal{L}}(x) = x$. Let $t \in Term_{\mathcal{L}}$. ϑ is applicable to t iff $dom(\vartheta) \supseteq vars(t) = freevars(t)$. Let ϑ be applicable to t. We define the application $t\vartheta \in Term_{\mathcal{L}}$ of ϑ to t by recursion on the structure of t in the standard manner. Let $range(\vartheta) \subseteq dom(\vartheta')$. We define the composition of ϑ and ϑ' as $\vartheta \circ \vartheta' : dom(\vartheta) \longrightarrow Term_{\mathcal{L}}$, $\vartheta \circ \vartheta'(x) = \vartheta(x)\vartheta'$, $\vartheta \circ \vartheta' \in Subst_{\mathcal{L}}$, $dom(\vartheta \circ \vartheta') = dom(\vartheta)$, $range(\vartheta \circ \vartheta') = range(\vartheta'|_{range(\vartheta)})$. Note that composition of substitutions is associative. ϑ' is a regular extension of ϑ iff $dom(\vartheta') \supseteq dom(\vartheta)$, $\vartheta'|_{dom(\vartheta)} = \vartheta$, $\vartheta'|_{dom(\vartheta') - dom(\vartheta)}$ is a variable renaming such that $range(\vartheta'|_{dom(\vartheta') - dom(\vartheta)}) \cap range(\vartheta) = \emptyset$. Let $a \in Atom_{\mathcal{L}}$. ϑ is applicable to a iff $dom(\vartheta) \supseteq vars(a) = freevars(a)$. Let ϑ be applicable to a and $a = p(t_1, \dots, t_\tau)$. We define the application of ϑ to a as $a\vartheta = p(t_1\vartheta, \dots, t_\tau\vartheta) \in Atom_{\mathcal{L}}$. Let $Qx\,a \in QAtom_{\mathcal{L}}$. ϑ is applicable to $Qx\,a$ iff $dom(\vartheta) \supseteq freevars(Qx\,a)$ and $x \notin range(\vartheta|_{freevars(Qx\,a)})$. Let ϑ be applicable to $Qx\,a$. We define the application of ϑ to $Qx\,a$ as $(Qx\,a)\vartheta = Qx\,a(\vartheta|_{freevars(Qx\,a)} \cup x/x) \in QAtom_{\mathcal{L}}$. Let $\varepsilon_1 \diamond \varepsilon_2 \in OrdLit_{\mathcal{L}}$. ϑ is applicable to $\varepsilon_1 \diamond \varepsilon_2$ iff, for both i, ϑ is applicable to ε_i. Let ϑ be applicable to $\varepsilon_1 \diamond \varepsilon_2$. Then, for both i, ϑ is applicable to ε_i, $dom(\vartheta) \supseteq freevars(\varepsilon_i)$, $dom(\vartheta) \supseteq freevars(\varepsilon_1) \cup freevars(\varepsilon_2) = freevars(\varepsilon_1 \diamond \varepsilon_2)$. We define the application of ϑ to $\varepsilon_1 \diamond \varepsilon_2$ as $(\varepsilon_1 \diamond \varepsilon_2)\vartheta = \varepsilon_1\vartheta \diamond \varepsilon_2\vartheta \in OrdLit_{\mathcal{L}}$. Let $E \subseteq \mathsf{A}$, $\mathsf{A} = Term_{\mathcal{L}} \mid \mathsf{A} = Atom_{\mathcal{L}} \mid \mathsf{A} = QAtom_{\mathcal{L}} \mid \mathsf{A} = OrdLit_{\mathcal{L}}$. ϑ is applicable to E iff, for all $\varepsilon \in E$, ϑ is applicable to ε. Let ϑ be applicable to E. Then, for all $\varepsilon \in E$, ϑ is applicable to ε, $dom(\vartheta) \supseteq freevars(\varepsilon)$, $dom(\vartheta) \supseteq \bigcup_{\varepsilon \in E} freevars(\varepsilon) = freevars(E)$. We define the application of ϑ to E as $E\vartheta = \{\varepsilon\vartheta \,|\, \varepsilon \in E\} \subseteq \mathsf{A}$. Let $\varepsilon, \varepsilon' \in \mathsf{A} \mid \varepsilon, \varepsilon' \in OrdCl_{\mathcal{L}}$. ε' is an instance of ε of $\mathcal{L}$ iff there exists $\vartheta^* \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\vartheta^*$. ε' is a variant of ε of $\mathcal{L}$ iff there exists a variable renaming $\rho^* \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\rho^*$. Let $C \in OrdCl_{\mathcal{L}}$ and $S \subseteq OrdCl_{\mathcal{L}}$. C is an instance $\mid$ a variant of S of $\mathcal{L}$ iff there exists $C^* \in S$ such that C is an instance $\mid$ a variant of C^* of $\mathcal{L}$. We denote $Inst_{\mathcal{L}}(S) = \{C \,|\, C \textit{ is an instance of } S \textit{ of } \mathcal{L}\} \subseteq OrdCl_{\mathcal{L}}$ and $Vrnt_{\mathcal{L}}(S) = \{C \,|\, C \textit{ is a variant of } S \textit{ of } \mathcal{L}\} \subseteq OrdCl_{\mathcal{L}}$.

ϑ is a unifier of $\mathcal{L}$ for E iff $E\vartheta$ is a singleton set. Note that there does not exist a unifier for $\emptyset$. Let $\theta \in Subst_{\mathcal{L}}$. θ is a most general unifier of $\mathcal{L}$ for E iff θ is a unifier of $\mathcal{L}$ for E, and for every unifier ϑ of $\mathcal{L}$ for E, there exists $\gamma^* \in Subst_{\mathcal{L}}$ such that $\vartheta|_{freevars(E)} = \theta|_{freevars(E)} \circ \gamma^*$. By $mgu_{\mathcal{L}}(E) \subseteq Subst_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for E. Let $\overline{E} = E_0, \dots, E_n$, $E_i \subseteq \mathsf{A}_i$, either $\mathsf{A}_i = Term_{\mathcal{L}}$ or $\mathsf{A}_i = Atom_{\mathcal{L}}$ or $\mathsf{A}_i = QAtom_{\mathcal{L}}$ or $\mathsf{A}_i = OrdLit_{\mathcal{L}}$. ϑ is applicable to $\overline{E}$ iff, for all $i \leq n$, ϑ is applicable to E_i. Let ϑ be applicable to $\overline{E}$. Then, for all $i \leq n$, ϑ is applicable to E_i, $dom(\vartheta) \supseteq freevars(E_i)$, $dom(\vartheta) \supseteq \bigcup_{i \leq n} freevars(E_i) = freevars(\overline{E})$. We define the application of ϑ to $\overline{E}$ as $\overline{E}\vartheta = E_0\vartheta, \dots, E_n\vartheta$, $E_i\vartheta \subseteq \mathsf{A}_i$. ϑ is a unifier of $\mathcal{L}$ for $\overline{E}$ iff, for all $i \leq n$, ϑ is a unifier of $\mathcal{L}$ for E_i. Note that if there exists $i^* \leq n$ and $E_{i^*} = \emptyset$, then there does not exist a unifier for $\overline{E}$. θ is a most general unifier of $\mathcal{L}$ for $\overline{E}$ iff θ is a unifier of $\mathcal{L}$ for $\overline{E}$, and for every unifier ϑ of $\mathcal{L}$ for $\overline{E}$, there exists $\gamma^* \in Subst_{\mathcal{L}}$ such that $\vartheta|_{freevars(\overline{E})} = \theta|_{freevars(\overline{E})} \circ \gamma^*$. By $mgu_{\mathcal{L}}(\overline{E}) \subseteq Subst_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for $\overline{E}$.

Theorem 3.1 (Unification Theorem)**.** *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} Atom_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in mgu_{\mathcal{L}}(\overline{E})$ such that $range(\theta^*|_{vars(\overline{E})}) \subseteq vars(\overline{E})$.*

Proof. By induction on $\|vars(\overline{E})\|$; a modification of the proof of Theorem 2.3 (Unification Theorem) in (Apt, 1988), Section 2.4, pp. 5–6. □

Theorem 3.2 (Extended Unification Theorem)**.** *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} Atom_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} QAtom_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} OrdLit_{\mathcal{L}}$, and $boundvars(\overline{E}) \subseteq V \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists $\theta^* \in mgu_{\mathcal{L}}(\overline{E})$ such that $range(\theta^*|_{freevars(\overline{E})}) \cap V = \emptyset$.*

Proof. A straightforward consequence of Theorem 3.1. □

3.2 A Formal Treatment

Translation of an order formula or a theory to clausal form, is based on the following lemma:

Lemma 3.3. *Let* $n_\phi, n_0 \in \mathbb{N}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$.

(I) *There exist either* $J_\phi = \emptyset$ *or* $J_\phi = \{(n_\phi, j) \mid j \le n_{J_\phi}\}$, $J_\phi \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, *and* $S_\phi \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_\phi\}}$ *such that*

- (a) $\|J_\phi\| \le 2 \cdot |\phi|$;
- (b) *either* $J_\phi = \emptyset$, $S_\phi = \{\square\}$ *or* $J_\phi = S_\phi = \emptyset$ *or* $J_\phi \ne \emptyset$, $\square \notin S_\phi \ne \emptyset$;
- (c) *there exists an interpretation* $\mathfrak{A}$ *for* $\mathcal{L}$ *and* $\mathfrak{A} \models \phi$ *if and only if there exists an interpretation* $\mathfrak{A}'$ *for* $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$ *and* $\mathfrak{A}' \models S_\phi$, *satisfying* $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;
- (d) $|S_\phi| \in O(|\phi|^2)$; *the number of all elementary operations of the translation of* ϕ *to* S_ϕ, *is in* $O(|\phi|^2)$; *the time and space complexity of the translation of* ϕ *to* S_ϕ, *is in* $O(|\phi|^2 \cdot (\log(1+n_\phi) + \log|\phi|))$;
- (e) *if* $S_\phi \ne \emptyset, \{\square\}$, *then* $J_\phi \ne \emptyset$, *for all* $C \in S_\phi$, $\emptyset \ne preds(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$;
- (f) *for all* $a \in qatoms(S_\phi)$, *there exists* $\mathbb{j}^* \in J_\phi$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;
- (g) *for all* $\mathbb{j} \in J_\phi$, *there exists a sequence* $\bar{x}$ *of variables of* $\mathcal{L}$ *and* $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S_\phi)$ *satisfying, for all* $a \in atoms(S_\phi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; *if there exists* $a^* \in qatoms(S_\phi)$ *and* $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, *then there exists* $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S_\phi)$ *satisfying, for all* $a \in qatoms(S_\phi)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;
- (h) $tcons(S_\phi) \subseteq tcons(\phi)$.

(II) *There exist* $J_T \subseteq \{(i,j) \mid i \ge n_0\}$ *and* $S_T \subseteq SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T\}}$ *such that*

- (a) *either* $J_T = \emptyset$, $S_T = \{\square\}$ *or* $J_T = S_T = \emptyset$ *or* $J_T \ne \emptyset$, $\square \notin S_T \ne \emptyset$;
- (b) *there exists an interpretation* $\mathfrak{A}$ *for* $\mathcal{L}$ *and* $\mathfrak{A} \models T$ *if and only if there exists an interpretation* $\mathfrak{A}'$ *for* $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$ *and* $\mathfrak{A}' \models S_T$, *satisfying* $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;
- (c) *if* $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$, *then* $J_T \subseteq_{\mathcal{F}} \{(i,j) \mid i \ge n_0\}$, $\|J_T\| \le 2 \cdot |T|$, $S_T \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J_T\}}$, $|S_T| \in O(|T|^2)$; *the number of all elementary operations of the translation of* T *to* S_T, *is in* $O(|T|^2)$; *the time and space complexity of the translation of* T *to* S_T, *is in* $O(|T|^2 \cdot \log(1 + n_0 + |T|))$;
- (d) *if* $S_T \ne \emptyset, \{\square\}$, *then* $J_T \ne \emptyset$, *for all* $C \in S_T$, $\emptyset \ne preds(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$;
- (e) *for all* $a \in qatoms(S_T)$, *there exists* $\mathbb{j}^* \in J_T$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;
- (f) *for all* $\mathbb{j} \in J_T$, *there exists a sequence* $\bar{x}$ *of variables of* $\mathcal{L}$ *and* $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S_T)$ *satisfying, for all* $a \in atoms(S_T)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; *if there exists* $a^* \in qatoms(S_T)$ *and* $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, *then there exists* $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S_T)$ *satisfying, for all* $a \in qatoms(S_T)$ *and* $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;
- (g) $tcons(S_T) \subseteq tcons(T)$.

Proof. Technical, using interpolation. It is straightforward to prove the following statements:

Let $n_\theta \in \mathbb{N}$ and $\theta \in OrdForm_{\mathcal{L}}$. There exists $\theta' \in OrdForm_{\mathcal{L}}$ such that (14)

- (a) $\theta' \equiv \theta$;
- (b) $|\theta'| \le 2 \cdot |\theta|$; θ' can be built up from θ via a postorder traversal of θ with $\#O(\theta) \in O(|\theta|)$ and the time, space complexity in $O(|\theta| \cdot (\log(1+n_\theta) + \log|\theta|))$;
- (c) θ' does not contain $\neg$ and Δ;
- (d) $\theta' \in Tcons_{\mathcal{L}}$; or for every subformula of θ' of the form $\varepsilon_1 \diamond \varepsilon_2$, $\diamond \in \{\wedge, \vee, \leftrightarrow\}$, $\varepsilon_i \ne 0, 1$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$; for every subformula of θ' of the form $\varepsilon_1 \rightarrow \varepsilon_2$, $\varepsilon_1 \ne 0, 1$, $\varepsilon_2 \ne 1$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$; for every subformula of θ' of the form $\varepsilon_1 \eqcirc \varepsilon_2$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$; for every subformula of θ' of the form $\varepsilon_1 \prec \varepsilon_2$, $\varepsilon_1 \ne 1$, $\varepsilon_2 \ne 0$, $\{\varepsilon_1, \varepsilon_2\} \not\subseteq Tcons_{\mathcal{L}}$; for every subformula of θ' of the form $Qx\varepsilon_1$, $Q \in \{\forall, \exists\}$, $\varepsilon_1 \notin Tcons_{\mathcal{L}}$;
- (e) $tcons(\theta') \subseteq tcons(\theta)$.

The proof is by induction on the structure of θ.

Let $n_\theta \in \mathbb{N}$, $\theta \in OrdForm_{\mathcal{L}} - \{0, 1\}$, (14c,d) hold for θ; $\bar{x}$ be a sequence of variables, $vars(\theta) \subseteq vars(\bar{x}) \subseteq Var_{\mathcal{L}}$; $\mathbb{i} = (n_\theta, j_{\mathbb{i}}) \in \{(n_\theta, j) \mid j \in \mathbb{N}\}$, $\tilde{p}_{\mathbb{i}} \in \tilde{\mathbb{P}}$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. There exist $J = \{(n_\theta, j) \mid j_{\mathbb{i}} + 1 \le j \le n_J\} \subseteq \{(n_\theta, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \le n_J$, $\mathbb{i} \notin J$, and $S \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{j}} \mid \mathbb{j}\in J\}}$ such that (15)

- (a) $\|J\| \le |\theta| - 1$;
- (b) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow \theta \in OrdForm_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}}$;
- (c) $|S| \le 27 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S can be built up from θ and $\tilde{f}_0(\bar{x})$ via a preorder traversal of θ with $\#O(\theta, \tilde{f}_0(\bar{x})) \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(d) for all $C \in S$, $\emptyset \neq preds(C) \cap \tilde{\mathbb{P}} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}$, $\tilde{p}_{\mathbb{i}}(\bar{x}) = 1, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1 \notin S$;

(e) for all $a \in qatoms(S)$, there exists $\mathbb{j}^* \in J$ and $preds(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(f) for all $\mathbb{j} \in \{\mathbb{i}\} \cup J$, $\tilde{p}_{\mathbb{j}}(\bar{x}) \in atoms(S)$ satisfying, for all $a \in atoms(S)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = \tilde{p}_{\mathbb{j}}(\bar{x})$; $\tilde{p}_{\mathbb{i}} \notin preds(qatoms(S))$, for all $\mathbb{j} \in J$, if there exists $a^* \in qatoms(S)$ and $preds(a^*) = \{\tilde{p}_{\mathbb{j}}\}$, then there exists $Qx\,\tilde{p}_{\mathbb{j}}(\bar{x}) \in qatoms(S)$ satisfying, for all $a \in qatoms(S)$ and $preds(a) = \{\tilde{p}_{\mathbb{j}}\}$, $a = Qx\,\tilde{p}_{\mathbb{j}}(\bar{x})$;

(g) $tcons(S) = tcons(\theta)$.

The proof is by induction on the structure of θ using the interpolation rules in Table 1.

(I) By (14) for n_ϕ, ϕ, there exists $\phi' \in OrdForm_{\mathcal{L}}$ such that (14a–e) hold for n_ϕ, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in Tcons_{\mathcal{L}} - \{1\}$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$ and $S_\phi = \{\Box\} \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L}}$. Case 2: $\phi' = 1$. We put $J_\phi = \emptyset \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$ and $S_\phi = \emptyset \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L}}$. Case 3: $\phi' \notin Tcons_{\mathcal{L}}$. We put $\bar{x} = varseq(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_\phi, j_{\mathbb{i}})$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (15) for n_ϕ, ϕ', $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_\phi, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, $S \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (15a–g) hold for ϕ', $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, S. We put $n_{J_\phi} = n_J$, $J_\phi = \{(n_\phi, j) \mid j \leq n_{J_\phi}\} \subseteq \{(n_\phi, j) \mid j \in \mathbb{N}\}$, $S_\phi = \{\tilde{p}_{\mathbb{i}}(\bar{x}) = 1\} \cup S \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$. (II) straightforwardly follows from (I). The lemma is proved. □

The described translation produces order clausal theories in some restrictive form, which will be utilised in inference using our order hyperresolution calculus to get shorter deductions in average case, cf. Section 4. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. S is admissible iff

(a) for all $a \in qatoms(S)$, $preds(a) \subseteq P$;

(b) for all $\tilde{p} \in P$, there exists a sequence $\bar{x}$ of variables of $\mathcal{L}$ and $\tilde{p}(\bar{x}) \in atoms(S)$ satisfying, for all $a \in atoms(S)$ and $preds(a) = \{\tilde{p}\}$, a is an instance of $\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$; if there exists $a^* \in qatoms(S)$ and $preds(a^*) = \{\tilde{p}\}$, then there exists $Qx\,\tilde{p}(\bar{x}) \in qatoms(S)$ satisfying, for all $a \in qatoms(S)$ and $preds(a) = \{\tilde{p}\}$, a is an instance of $Qx\,\tilde{p}(\bar{x})$ of $\mathcal{L} \cup P$.

(a) and (b) imply that for all $Qx\,a, Q'x'\,a' \in qatoms(S)$, if $preds(a) = preds(a')$, then $Q = Q'$, $x = x'$, $boundindset(Qx\,a) = boundindset(Q'x'\,a')$.

Theorem 3.4. *Let $n_0 \in \mathbb{N}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$. There exist $J_T^\phi \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$ such that*

(i) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}$ and $\mathfrak{A}' \models S_T^\phi$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*

(ii) *if $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$, then $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^\phi\| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2 + |\phi|^2)$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1 + n_0) + \log|\phi|))$;*

(iii) *S_T^ϕ is admissible;*

(iv) *$tcons(S_T^\phi) \subseteq tcons(\phi) \cup tcons(T)$.*

Proof. Similar to that of Lemma 3.3(I). We get by Lemma 3.3(II) for $n_0 + 1$, T that there exist $J_T \subseteq \{(i,j) \mid i \geq n_0 + 1\}$, $S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$, and Lemma 3.3(II a–g) hold for $n_0 + 1$, T, J_T, S_T. By (14) for n_0, ϕ, there exists $\phi' \in OrdForm_{\mathcal{L}}$ such that (14a–e) hold for n_0, ϕ, ϕ'. We distinguish three cases for ϕ'. Case 1: $\phi' \in Tcons_{\mathcal{L}} - \{1\}$. We put $J_T^\phi = J_T \subseteq \{(i,j) \mid i \geq n_0 + 1\} \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = S_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$. Case 2: $\phi' = 1$. We put $J_T^\phi = \emptyset \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = \{\Box\} \subseteq SimOrdCl_{\mathcal{L}}$. Case 3: $\phi' \notin Tcons_{\mathcal{L}}$. We put $\bar{x} = varseq(\phi')$, $j_{\mathbb{i}} = 0$, $\mathbb{i} = (n_0, j_{\mathbb{i}})$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (15) for n_0, $\forall \bar{x} \phi'$, $\bar{x}$, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist $J = \{(n_0, j) \mid 1 \leq j \leq n_J\} \subseteq \{(n_0, j) \mid j \in \mathbb{N}\}$, $j_{\mathbb{i}} \leq n_J$, $\mathbb{i} \notin J$, $S \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J\}}$, and (15a–g) hold for $\forall \bar{x} \phi'$, $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J, S. We put $J_T^\phi = J_T \cup \{\mathbb{i}\} \cup J \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi = S_T \cup \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$. The theorem is proved. □

Corollary 3.5. *Let $n_0 \in \mathbb{N}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$. There exist $J_T^\phi \subseteq \{(i,j) \mid i \geq n_0\}$ and $S_T^\phi \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$ such that*

(i) *$T \models \phi$ if and only if S_T^ϕ is unsatisfiable;*

(ii) *if $T \subseteq_{\mathcal{F}} OrdForm_{\mathcal{L}}$, then $J_T^\phi \subseteq_{\mathcal{F}} \{(i,j) \mid i \geq n_0\}$, $\|J_T^\phi\| \in O(|T| + |\phi|)$, $S_T^\phi \subseteq_{\mathcal{F}} SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T^\phi\}}$, $|S_T^\phi| \in O(|T|^2 + |\phi|^2)$; the number of all elementary operations of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 + |\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^ϕ, is in $O(|T|^2 \cdot \log(1 + n_0 + |T|) + |\phi|^2 \cdot (\log(1 + n_0) + \log|\phi|))$;*

(iii) *S_T^ϕ is admissible;*

(iv) *$tcons(S_T^\phi) \subseteq tcons(\phi) \cup tcons(T)$.*

Table 1: Interpolation rules.

Case	Laws	
$\theta = \theta_1 \wedge \theta_2$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow \theta_1 \wedge \theta_2}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\end{array}\right\}}$$	(6), (8)	(16)
$\lvert\text{Consequent}\rvert = 24 + 16 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \theta_1 \vee \theta_2$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \vee \theta_2)}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \\ \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\end{array}\right\}}$$	(5), (7)	(17)
$\lvert\text{Consequent}\rvert = 24 + 16 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \theta_1 \rightarrow \theta_2, \theta_2 \neq \mathit{0}$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \rightarrow \theta_2)}{\{\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\}}$$		(18)
$\lvert\text{Consequent}\rvert = 15 + 9 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \theta_1 \rightarrow \mathit{0}$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \rightarrow \mathit{0})}{\{\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \mathit{0} \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{0}, \mathit{0} \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\}}$$		(19)
$\lvert\text{Consequent}\rvert = 12 + 4 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert$		
$\theta = \theta_1 \leftrightarrow \theta_2$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \leftrightarrow \theta_2)}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}), \\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\end{array}\right\}}$$		(20)
$\lvert\text{Consequent}\rvert = 27 + 17 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \theta_1 \eqcirc \theta_2$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \eqcirc \theta_2)}{\{\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{0}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\}}$$		(21)
$\lvert\text{Consequent}\rvert = 15 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \theta_1 \prec \theta_2$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow (\theta_1 \prec \theta_2)}{\{\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{0}, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\}}$$		(22)
$\lvert\text{Consequent}\rvert = 15 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \leftrightarrow \theta_2\rvert$		
$\theta = \forall x \theta_1$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow \forall x \theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \forall x \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\}}$$		(23)
$\lvert\text{Consequent}\rvert = 5 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert$		
$\theta = \exists x \theta_1$		
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \leftrightarrow \exists x \theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \exists x \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\}}$$		(24)
$\lvert\text{Consequent}\rvert = 5 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert \le 27 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \leftrightarrow \theta_1\rvert$		

Proof. Let $T \models \phi$. Then, for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models T$ or $\mathfrak{A} \models \phi$; by Theorem 3.4(i), there does not exist an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}$ and $\mathfrak{A}' \models S_T^\phi$; S_T^ϕ is unsatisfiable.

Let S_T^ϕ is unsatisfiable. Then, for every interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^\phi\}$, $\mathfrak{A}' \not\models S_T^\phi$; by Theorem 3.4(i), there does not exist an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T$, $\mathfrak{A} \not\models \phi$; for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, $\mathfrak{A} \not\models T$ or $\mathfrak{A} \models \phi$; $T \models \phi$; (i) holds.

(ii–iv) are the same as Theorem 3.4(ii–iv); (ii–iv) hold. The corollary is proved. □

4 HYPERRESOLUTION OVER ORDER CLAUSES

In this section, we propose an order hyperresolution calculus with truth constants operating over order clausal theories, and prove its refutational soundness, completeness.

4.1 Order Hyperresolution Rules

At first, we introduce some basic notions and notation concerning chains of order literals. A chain Ξ of $\mathcal{L}$ is a sequence $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$, $\varepsilon_i \diamond_i \upsilon_i \in OrdLit_{\mathcal{L}}$, such that for all $i < n$, $\upsilon_i = \varepsilon_{i+1}$. ε_0 is the beginning element of Ξ and υ_n the ending element of Ξ. $\varepsilon_0 \,\Xi\, \upsilon_n$ denotes Ξ together with its respective beginning and ending element. Let $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain of $\mathcal{L}$. Ξ is an equality chain of $\mathcal{L}$ iff, for all $i \le n$, $\diamond_i = \eqcirc$. Ξ is an increasing chain of $\mathcal{L}$ iff there exists $i^* \le n$ such that $\diamond_{i^*} = \prec$. Ξ is a contradiction of $\mathcal{L}$ iff Ξ is an increasing chain of $\mathcal{L}$ of the form $\varepsilon_0 \,\Xi\, \mathit{0}$ or $\mathit{1} \,\Xi\, \upsilon_n$ or $\varepsilon_0 \,\Xi\, \varepsilon_0$. Let $S \subseteq OrdCl_{\mathcal{L}}$ be unit and $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain | an equality chain | an increasing chain | a contradiction of $\mathcal{L}$. Ξ is a chain | an equality chain | an increasing chain | a contradiction of S iff, for all $i \le n$, $\varepsilon_i \diamond_i \upsilon_i \in S$.

Let $\tilde{\mathbb{W}} = \{\tilde{w}_{\mathbb{i}} \,|\, \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{\mathbb{W}} \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$; $\tilde{\mathbb{W}}$ is an infinite countable set of new function symbols. Let $\mathcal{L}$ contain a constant (nullary function) symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. We denote $GOrdCl_{\mathcal{L}} = \{C \,|\, C \in OrdCl_{\mathcal{L}} \text{ is closed}\} \subseteq OrdCl_{\mathcal{L}}$, $GInst_{\mathcal{L}}(S) = \{C \,|\, C \in GOrdCl_{\mathcal{L}} \text{ is an instance of } S \text{ of } \mathcal{L}\} \subseteq GOrdCl_{\mathcal{L}}$, $ordtcons(S) = \{\mathit{0} \prec \mathit{1}\} \cup \{\mathit{0} \prec \bar{c} \,|\, \bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}\} \cup \{\bar{c} \prec \mathit{1} \,|\, \bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}\} \cup \{\bar{c}_1 \prec \bar{c}_2 \,|\, \bar{c}_1, \bar{c}_2 \in tcons(S) \cap \overline{C}_{\mathcal{L}}, c_1 < c_2\} \subseteq GOrdCl_{\mathcal{L}}$. A basic order hyperresolution calculus is defined in Table 2.

The basic order hyperresolution calculus can be generalised to an order hyperresolution one in Table 3. Let $\mathcal{L}_0 = \mathcal{L} \cup P$, a reduct of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, and $S_0 = \emptyset \subseteq GOrdCl_{\mathcal{L}_0} \mid OrdCl_{\mathcal{L}_0}$. Let $\mathcal{D} = C_1, \dots, C_n$, $C_\kappa \in GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P} \mid OrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$, $n \ge 1$. $\mathcal{D}$ is a deduction of C_n from S by basic order hyperresolution iff, for all $1 \le \kappa \le n$, $C_\kappa \in ordtcons(S) \cup GInst_{\mathcal{L}_{\kappa-1}}(S)$, or there exist $1 \le j_k^* \le \kappa - 1$, $k = 1, \dots, m$, such that C_κ is a basic order resolvent of $C_{j_1^*}, \dots, C_{j_m^*} \in S_{\kappa-1}$ using Rule (25)–(31) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$; $\mathcal{D}$ is a deduction of C_n from S by order hyperresolution iff, for all $1 \le \kappa \le n$, $C_\kappa \in ordtcons(S) \cup S$, or there exist $1 \le j_k^* \le \kappa - 1$, $k = 1, \dots, m$, such that C_κ is an order resolvent of $C'_{j_1^*}, \dots, C'_{j_m^*} \in S_{\kappa-1}^{Vr}$ using Rule (32)–(38) with respect to $\mathcal{L}_{\kappa-1}$ and $S_{\kappa-1}$ where $C'_{j_k^*}$ is a variant of $C_{j_k^*} \in S_{\kappa-1}$ of $\mathcal{L}_{\kappa-1}$; $\mathcal{L}_\kappa$ and S_κ are defined by recursion on $1 \le \kappa \le n$ as follows:

$$\mathcal{L}_\kappa = \begin{cases} \mathcal{L}_{\kappa-1} \cup \{\tilde{w}\} & \textit{in case of Rule } (30), (31) \mid (37), (38), \\ \mathcal{L}_{\kappa-1} & \textit{else}; \end{cases}$$

$$S_\kappa = S_{\kappa-1} \cup \{C_\kappa\} \subseteq GOrdCl_{\mathcal{L}_\kappa} \mid OrdCl_{\mathcal{L}_\kappa},$$

$$S_\kappa^{Vr} = Vrnt_{\mathcal{L}_\kappa}(S_\kappa) \subseteq OrdCl_{\mathcal{L}_\kappa}.$$

$\mathcal{D}$ is a refutation of S iff $C_n = \square$. We denote

$$clo^{\mathcal{BH}}(S) = \{C \,|\, \textit{there exists a deduction of } C \textit{ from } S \textit{ by basic order hyperresolution}\} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P},$$

$$clo^{\mathcal{H}}(S) = \{C \,|\, \textit{there exists a deduction of } C \textit{ from } S \textit{ by order hyperresolution}\} \subseteq OrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}.$$

4.2 Refutational Soundness and Completeness

We are in position to prove the refutational soundness and completeness of the order hyperresolution calculus. At first, we list some auxiliary lemmata.

Lemma 4.1 (Lifting Lemma)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. Let $C \in clo^{\mathcal{BH}}(S)$. There exists $C^* \in clo^{\mathcal{H}}(S)$ such that C is an instance of C^* of $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$.*

Proof. Technical, analogous to the standard one. □

Lemma 4.2 (Reduction Lemma)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$ and $S \subseteq OrdCl_{\mathcal{L} \cup P}$. Let $\{\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee C_i \,|\, i \le n\} \subseteq clo^{\mathcal{BH}}(S)$ such that for all $\mathcal{S} \in \mathcal{S}el(\{\{j \,|\, j \le k_i\}_i \,|\, i \le n\})$, there exists a contradiction of $\{\varepsilon_{\mathcal{S}(i)}^i \diamond_{\mathcal{S}(i)}^i \upsilon_{\mathcal{S}(i)}^i \,|\, i \le n\} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$. There exists $\emptyset \ne I^* \subseteq \{i \,|\, i \le n\}$ such that $\bigvee_{i \in I^*} C_i \in clo^{\mathcal{BH}}(S)$.*

Table 2: Basic order hyperresolution rules.

(*Basic order hyperresolution rule*) (25)

$$\frac{l_0 \vee C_0, \dots, l_n \vee C_n \in S_{\kappa-1}}{\bigvee_{i=0}^{n} C_i \in S_\kappa};$$

$l_0, \dots, l_n$ *is a contradiction of* $\mathcal{L}_{\kappa-1}$.

$\bigvee_{i=0}^{n} C_i$ is a basic order hyperresolvent of $l_0 \vee C_0, \dots, l_n \vee C_n$.

(*Basic order trichotomy rule*) (26)

$$\frac{a, b \in atoms(S_{\kappa-1}), a \in \overline{C}_{\mathcal{L}}, b \notin Tcons_{\mathcal{L}}, qatoms(S) = \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

(*Basic order trichotomy rule*) (27)

$$\frac{a, b \in atoms(S_{\kappa-1}) - \{0, 1\}, \{a, b\} \not\subseteq Tcons_{\mathcal{L}}, qatoms(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

$a \prec b \vee a \eqcirc b \vee b \prec a$ is a basic order trichotomy resolvent of a and b.

(*Basic order* $\forall$*-quantification rule*) (28)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma \in S_\kappa};$$

$t \in GTerm_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in Subst_{\mathcal{L}_{\kappa-1}}, dom(\gamma) = \{x\} = vars(a).$

$\forall x\, a \prec a\gamma \vee \forall x\, a \eqcirc a\gamma$ is a basic order $\forall$-quantification resolvent of $\forall x\, a$.

(*Basic order* $\exists$*-quantification rule*) (29)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1})}{a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a \in S_\kappa};$$

$t \in GTerm_{\mathcal{L}_{\kappa-1}}, \gamma = x/t \in Subst_{\mathcal{L}_{\kappa-1}}, dom(\gamma) = \{x\} = vars(a).$

$a\gamma \prec \exists x\, a \vee a\gamma \eqcirc \exists x\, a$ is a basic order $\exists$-quantification resolvent of $\exists x\, a$.

(*Basic order* $\forall$*-witnessing rule*) (30)

$$\frac{\forall x\, a \in qatoms^{\forall}(S_{\kappa-1}), b \in atoms(S_{\kappa-1}) \cup qatoms(S_{\kappa-1})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\forall x\, a), freetermseq(b)|,$
$\gamma = x/\tilde{w}(freetermseq(\forall x\, a), freetermseq(b)) \in Subst_{\mathcal{L}_\kappa}, dom(\gamma) = \{x\} = vars(a).$

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is a basic order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Basic order* $\exists$*-witnessing rule*) (31)

$$\frac{\exists x\, a \in qatoms^{\exists}(S_{\kappa-1}), b \in atoms(S_{\kappa-1}) \cup qatoms(S_{\kappa-1})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$\tilde{w} \in \tilde{\mathbb{W}} - Func_{\mathcal{L}_{\kappa-1}}, ar(\tilde{w}) = |freetermseq(\exists x\, a), freetermseq(b)|,$
$\gamma = x/\tilde{w}(freetermseq(\exists x\, a), freetermseq(b)) \in Subst_{\mathcal{L}_\kappa}, dom(\gamma) = \{x\} = vars(a).$

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is a basic order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

Proof. Technical, analogous to the one of Proposition 2, (Guller, 2009). □

Lemma 4.3 (Unit Lemma)**.** *Let* $\mathcal{L}$ *contain a constant symbol. Let* $P \subseteq \tilde{\mathbb{P}}$ *and* $S \subseteq OrdCl_{\mathcal{L} \cup P}$. *Let* $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \,|\, \iota < \gamma\}$, $\gamma \leq \omega$. *There exists* $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma\})$ *such that there does not exist a contradiction of* $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma\} \subseteq GOrdCl_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}$.

Table 3: Order hyperresolution rules.

(*Order hyperresolution rule*) (32)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_{\kappa-1}^{Vr}}{\Big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\Big)\theta \in S_\kappa};$$

$\textit{for all } i < i' \le n,$
$\quad \mathit{freevars}(\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \bigvee_{j=1}^{m_i} l_j^i) \cap \mathit{freevars}(\bigvee_{j=0}^{k_{i'}} \varepsilon_j^{i'} \diamond_j^{i'} \upsilon_j^{i'} \vee \bigvee_{j=1}^{m_{i'}} l_j^{i'}) = \emptyset,$
$\theta \in \mathit{mgu}_{\mathcal{L}_{\kappa-1}}\Big(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n, \{\upsilon_0^0, \varepsilon_0^1\}, \dots, \{\upsilon_0^{n-1}, \varepsilon_0^n\}, \{a,b\}\Big),$
$\mathit{dom}(\theta) = \mathit{freevars}\big(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \,|\, j \le k_i, i \le n\}, \{l_j^i \,|\, 1 \le j \le m_i, i \le n\}\big),$
$a = \varepsilon_0^0, b = \mathit{1} \textit{ or } a = \upsilon_0^n, b = \mathit{0} \textit{ or } a = \varepsilon_0^0, b = \upsilon_0^n, \textit{ there exists } i^* \le n \textit{ such that } \diamond_0^{i^*} = \prec .$

$\big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\big)\theta$ is an order hyperresolvent of $\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n$.

(*Order trichotomy rule*) (33)

$$\frac{a,b \in \mathit{atoms}(S_{\kappa-1}), a \in \overline{C}_{\mathcal{L}}, b \notin \mathit{Tcons}_{\mathcal{L}}, \mathit{qatoms}(S) = \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa}.$$

(*Order trichotomy rule*) (34)

$$\frac{a,b \in \mathit{atoms}(S_{\kappa-1}^{Vr}) - \{\mathit{0},\mathit{1}\}, \{a,b\} \not\subseteq \mathit{Tcons}_{\mathcal{L}}, \mathit{qatoms}(S) \neq \emptyset}{a \prec b \vee a \eqcirc b \vee b \prec a \in S_\kappa};$$

$\mathit{vars}(a) \cap \mathit{vars}(b) = \emptyset.$

$a \prec b \vee a \eqcirc b \vee b \prec a$ is an order trichotomy resolvent of a and b.

(*Order* $\forall$*-quantification rule*) (35)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1})}{\forall x\, a \prec a \vee \forall x\, a \eqcirc a \in S_\kappa}.$$

$\forall x\, a \prec a \vee \forall x\, a \eqcirc a$ is an order $\forall$-quantification resolvent of $\forall x\, a$.

(*Order* $\exists$*-quantification rule*) (36)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1})}{a \prec \exists x\, a \vee a \eqcirc \exists x\, a \in S_\kappa}.$$

$a \prec \exists x\, a \vee a \eqcirc \exists x\, a$ is an order $\exists$-quantification resolvent of $\exists x\, a$.

(*Order* $\forall$*-witnessing rule*) (37)

$$\frac{\forall x\, a \in \mathit{qatoms}^{\forall}(S_{\kappa-1}^{Vr}), b \in \mathit{atoms}(S_{\kappa-1}^{Vr}) \cup \mathit{qatoms}(S_{\kappa-1}^{Vr})}{a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a \in S_\kappa};$$

$\mathit{freevars}(\forall x\, a) \cap \mathit{freevars}(b) = \emptyset,$
$\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)|,$
$\gamma = x/\tilde{w}(\mathit{freetermseq}(\forall x\, a), \mathit{freetermseq}(b)) \cup \mathit{id}|_{\mathit{vars}(a)-\{x\}} \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} \cup (\mathit{vars}(a) - \{x\}) = \mathit{vars}(a).$

$a\gamma \prec b \vee b \eqcirc \forall x\, a \vee b \prec \forall x\, a$ is an order $\forall$-witnessing resolvent of $\forall x\, a$ and b.

(*Order* $\exists$*-witnessing rule*) (38)

$$\frac{\exists x\, a \in \mathit{qatoms}^{\exists}(S_{\kappa-1}^{Vr}), b \in \mathit{atoms}(S_{\kappa-1}^{Vr}) \cup \mathit{qatoms}(S_{\kappa-1}^{Vr})}{b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b \in S_\kappa};$$

$\mathit{freevars}(\exists x\, a) \cap \mathit{freevars}(b) = \emptyset,$
$\tilde{w} \in \tilde{\mathbb{W}} - \mathit{Func}_{\mathcal{L}_{\kappa-1}}, \mathit{ar}(\tilde{w}) = |\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)|,$
$\gamma = x/\tilde{w}(\mathit{freetermseq}(\exists x\, a), \mathit{freetermseq}(b)) \cup \mathit{id}|_{\mathit{vars}(a)-\{x\}} \in \mathit{Subst}_{\mathcal{L}_\kappa}, \mathit{dom}(\gamma) = \{x\} \cup (\mathit{vars}(a) - \{x\}) = \mathit{vars}(a).$

$b \prec a\gamma \vee \exists x\, a \eqcirc b \vee \exists x\, a \prec b$ is an order $\exists$-witnessing resolvent of $\exists x\, a$ and b.

Proof. Technical, a straightforward consequence of König's Lemma and Lemma 4.2. $\square$

Let $\{0,1\} \subseteq X \subseteq [0,1]$. X is admissible with respect to suprema and infima iff, for all $\emptyset \neq Y_1, Y_2 \subseteq X$ and $\bigvee Y_1 = \bigwedge Y_2$, $\bigvee Y_1 \in Y_1$, $\bigwedge Y_2 \in Y_2$. Let $\{\mathit{0},\mathit{1}\} \subseteq Tc \subseteq Tcons_{\mathcal{L}}$. Tc is admissible with respect to suprema and infima iff $\{0,1\} \subseteq \overline{Tc} \subseteq [0,1]$ is admissible with respect to suprema and infima.

Theorem 4.4 (Refutational Soundness and Completeness)**.** *Let $\mathcal{L}$ contain a constant symbol. Let $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $tcons(S)$ be admissible with respect to suprema and infima. $\square \in clo^{\mathcal{H}}(S)$ if and only if S is unsatisfiable.*

Proof. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$ and $C \in clo^{\mathcal{H}}(S) \subseteq OrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in clo^{\mathcal{H}}(S)$ and $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup P$. Hence, there exists an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$ such that $\mathfrak{A}' \models \square$, which is a contradiction; S is unsatisfiable.

($\Longleftarrow$) Let $\square \notin clo^{\mathcal{H}}(S)$. Then, by Lemma 4.1 for S, $\square$, $\square \notin clo^{\mathcal{BH}}(S)$; we have $\mathcal{L}$, $\tilde{\mathbb{P}}$, $\tilde{\mathbb{W}}$ are countable, $P \subseteq \tilde{\mathbb{P}}$, $S \subseteq OrdCl_{\mathcal{L}\cup P}$, $clo^{\mathcal{BH}}(S) \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$; P, $\mathcal{L} \cup P$, $OrdCl_{\mathcal{L}\cup P}$, S, $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$, $GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$, $clo^{\mathcal{BH}}(S)$ are countable; there exists $\gamma_1 \leq \omega$ and $\square \notin clo^{\mathcal{BH}}(S) = \{\bigvee_{j=0}^{k_\iota} \varepsilon_j^\iota \diamond_j^\iota \upsilon_j^\iota \,|\, \iota < \gamma_1\}$; by Lemma 4.3 for S, there exists $\mathcal{S}^* \in \mathcal{S}el(\{\{j \,|\, j \leq k_\iota\}_\iota \,|\, \iota < \gamma_1\})$ and there does not exist a contradiction of $\{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. We put $\mathbb{S} = \{\varepsilon^\iota_{\mathcal{S}^*(\iota)} \diamond^\iota_{\mathcal{S}^*(\iota)} \upsilon^\iota_{\mathcal{S}^*(\iota)} \,|\, \iota < \gamma_1\} \subseteq GOrdCl_{\mathcal{L}\cup\tilde{\mathbb{W}}\cup P}$. Then $ordtcons(S) \subseteq clo^{\mathcal{BH}}(S)$, $\mathbb{S} \supseteq ordtcons(S)$ is countable, unit, $(q)atoms(\mathbb{S}) \subseteq (q)atoms(clo^{\mathcal{BH}}(S))$; there does not exist a contradiction of $\mathbb{S}$. We have $\mathcal{L}$ contains a constant symbol. Hence, there exists $cn^* \in Func_{\mathcal{L}}$, $ar_{\mathcal{L}}(cn^*) = 0$. We put $\tilde{\mathbb{W}}^* = funcs(\mathbb{S}) \cap \tilde{\mathbb{W}} \subseteq \tilde{\mathbb{W}}$, $\tilde{\mathbb{W}}^* \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) \subseteq \tilde{\mathbb{W}} \cap (Func_{\mathcal{L}} \cup \{\tilde{f}_0\}) = \emptyset$,

$$\begin{aligned} \mathcal{U}_{\mathfrak{A}} &= GTerm_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}, cn^* \in \mathcal{U}_{\mathfrak{A}} \neq \emptyset, \\ \mathcal{B} &= atoms(\mathbb{S}) \cup qatoms(\mathbb{S}) \subseteq \\ &\qquad GAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P} \cup QAtom_{\mathcal{L}\cup\tilde{\mathbb{W}}^*\cup P}. \end{aligned}$$

We have $\mathbb{S}$ is countable. Then $tcons(S) = atoms(ordtcons(S)) \subseteq atoms(\mathbb{S}) \subseteq \mathcal{B}$, $\mathcal{B} = tcons(S) \cup (\mathcal{B} - tcons(S))$, $tcons(S) \cap (\mathcal{B} - tcons(S)) = \emptyset$, $atoms(\mathbb{S})$, $qatoms(\mathbb{S})$, $\mathcal{B}$, $tcons(S)$, $\mathcal{B} - tcons(S)$ are countable; there exist $\gamma_2 \leq \omega$ and a sequence $\delta_2 : \gamma_2 \longrightarrow \mathcal{B} - tcons(S)$ of $\mathcal{B} - tcons(S)$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$. $\varepsilon_1 \triangleq \varepsilon_2$ iff there exists an equality chain $\varepsilon_1 \Xi \varepsilon_2$ of $\mathbb{S}$. Note that $\triangleq$ is a binary symmetric transitive relation on $\mathcal{B}$. $\varepsilon_1 \lhd \varepsilon_2$ iff there exists an increasing chain $\varepsilon_1 \Xi \varepsilon_2$ of $\mathbb{S}$. Note that $\lhd$ is a binary transitive relation on $\mathcal{B}$.

$$\begin{aligned} &\mathit{0} \not\triangleq \mathit{1}, \mathit{1} \not\triangleq \mathit{0}, \mathit{0} \lhd \mathit{1}, \mathit{1} \not\lhd \mathit{0}, \\ &\text{for all } \varepsilon \in \mathcal{B},\ \varepsilon \not\lhd \mathit{0}, \mathit{1} \not\lhd \varepsilon, \varepsilon \not\lhd \varepsilon. \end{aligned} \tag{39}$$

The proof is straightforward; we have that there does not exist a contradiction of $\mathbb{S}$. Note that $\lhd$ is also irreflexive and a partial strict order on $\mathcal{B}$.

Let $tcons(S) \subseteq X \subseteq \mathcal{B}$. A partial valuation $\mathcal{V}$ is a mapping $\mathcal{V} : X \longrightarrow [0,1]$ such that $\mathcal{V}(\mathit{0}) = 0$, $\mathcal{V}(\mathit{1}) = 1$, for all $\bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $\mathcal{V}(\bar{c}) = c$. We denote $dom(\mathcal{V}) = X$, $tcons(S) \subseteq dom(\mathcal{V}) \subseteq \mathcal{B}$. We define a partial valuation $\mathcal{V}_\alpha$ by recursion on $\alpha \leq \gamma_2$ as follows:

$$\mathcal{V}_0 = \{(\mathit{0},0), (\mathit{1},1)\} \cup \{(\bar{c}, c) \,|\, \bar{c} \in tcons(S) \cap \overline{C}_{\mathcal{L}}\};$$

$$\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta_2(\alpha-1), \lambda_{\alpha-1})\} \qquad (1 \leq \alpha \leq \gamma_2 \textit{ is a successor ordinal}),$$

$$\mathbb{E}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \triangleq \delta_2(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{D}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \lhd \delta_2(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{U}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, \delta_2(\alpha-1) \lhd a, a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\lambda_{\alpha-1} = \begin{cases} \dfrac{\bigvee \mathbb{D}_{\alpha-1} + \bigwedge \mathbb{U}_{\alpha-1}}{2} & \textit{if } \mathbb{E}_{\alpha-1} = \emptyset, \\ \bigvee \mathbb{E}_{\alpha-1} & \textit{else}; \end{cases}$$

$$\mathcal{V}_{\gamma_2} = \bigcup_{\alpha < \gamma_2} \mathcal{V}_\alpha \quad (\gamma_2 \textit{ is a limit ordinal}).$$

For all $\alpha \leq \alpha' \leq \gamma_2$, $\mathcal{V}_\alpha$ is a partial valuation, $dom(\mathcal{V}_\alpha) = tcons(S) \cup \delta_2[\alpha]$, $\mathcal{V}_\alpha \subseteq \mathcal{V}_{\alpha'}$. (40)

The proof is by induction on $\alpha \leq \gamma_2$.

We list some auxiliary statements without proofs:

If $qatoms(S) = \emptyset$, then $qatoms(clo^{\mathcal{BH}}(S)) = \emptyset$. (41)

$$tcons(S) = tcons(clo^{\mathcal{BH}}(S)). \tag{42}$$

For all $a, b \in atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, such that $a, b \in atoms(S_n) \cup qatoms(S_n)$. (43)

For all $\emptyset \neq A \subseteq_{\mathcal{F}} atoms(clo^{\mathcal{BH}}(S)) \cup qatoms(clo^{\mathcal{BH}}(S))$, there exist a deduction $C_1, \dots, C_n$, $n \geq 1$, from S by basic order hyperresolution, associated $\mathcal{L}_n$, S_n, $S_n \subseteq GOrdCl_{\mathcal{L}_n}$, such that $A \subseteq atoms(S_n) \cup qatoms(S_n)$. (44)

For all $a \in tcons(S) \cap \overline{C}_{\mathcal{L}}$, $b \in \mathcal{B} - tcons(S)$, either $a \lhd b$ or $a \triangleq b$ or $b \lhd a$. (45)

Let $qatoms(S) \neq \emptyset$. For all $a, b \in \mathcal{B} - \{0, 1\}$, either $a \lhd b$ or ($a = b$ or $a \triangleq b$) or $b \lhd a$. (46)

$$\begin{aligned}&\text{For all } \alpha \leq \gamma_2, \text{ for all } a, b \in dom(\mathcal{V}_\alpha), \\ &\quad \text{if } a \triangleq b, \text{ then } \mathcal{V}_\alpha(a) = \mathcal{V}_\alpha(b); \\ &\quad \text{if } a \lhd b, \text{ then } \mathcal{V}_\alpha(a) < \mathcal{V}_\alpha(b); \\ &\quad \text{if } \mathcal{V}_\alpha(a) = 0, \text{ then } a = 0 \text{ or } a \triangleq 0; \\ &\quad \text{if } \mathcal{V}_\alpha(a) = 1, \text{ then } a = 1 \text{ or } a \triangleq 1; \\ &\text{for all } \alpha < \gamma_2, \\ &\quad \mathcal{V}_\alpha[dom(\mathcal{V}_\alpha)] \text{ is admissible with respect to} \\ &\quad \text{suprema and infima.}\end{aligned} \tag{47}$$

The proof is by induction on $\alpha \leq \gamma_2$ using the assumption that $\overline{tcons(S)}$ is admissible with respect to suprema and infima.

We put $\mathcal{V} = \mathcal{V}_{\gamma_2}$, $dom(\mathcal{V}) \overset{(40)}{=} tcons(S) \cup \delta[\gamma_2] = tcons(S) \cup (\mathcal{B} - tcons(S)) = \mathcal{B}$. We further list some other auxiliary statements without proofs:

$$\begin{aligned}&\text{For all } a, b \in \mathcal{B}, \\ &\quad \text{if } a \triangleq b, \text{ then } \mathcal{V}(a) = \mathcal{V}(b); \\ &\quad \text{if } a \lhd b, \text{ then } \mathcal{V}(a) < \mathcal{V}(b).\end{aligned} \tag{48}$$

For all $Qx\,a \in qatoms(clo^{\mathcal{BH}}(S))$ and $u \in \mathcal{U}_{\mathfrak{A}}$, $a(x/u) \in atoms(clo^{\mathcal{BH}}(S))$. (49)

$$\begin{aligned}&\text{For all } a \in \mathcal{B}, \\ &\quad \text{if } a = \forall x b, \text{ then } \mathcal{V}(a) = \textstyle\bigwedge_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)); \\ &\quad \text{if } a = \exists x b, \text{ then } \mathcal{V}(a) = \textstyle\bigvee_{u \in \mathcal{U}_{\mathfrak{A}}} \mathcal{V}(b(x/u)).\end{aligned} \tag{50}$$

We put

$$f^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} f(u_1, \dots, u_\tau) \\ \qquad \textit{if } f \in Func_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}, \\ cn^* \quad \textit{else}, \end{cases}$$
$$f \in Func_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$p^{\mathfrak{A}}(u_1, \dots, u_\tau) = \begin{cases} \mathcal{V}(p(u_1, \dots, u_\tau)) \\ \qquad \textit{if } p(u_1, \dots, u_\tau) \in \mathcal{B}, \\ 0 \quad \textit{else}, \end{cases}$$
$$p \in Pred_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$\mathfrak{A} = \big(\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \mid f \in Func_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}, \{p^{\mathfrak{A}} \mid p \in Pred_{\mathcal{L} \cup \tilde{\mathbb{W}} \cup P}\}\big),$$
an interpretation for $\mathcal{L} \cup \tilde{\mathbb{W}} \cup P$.

For all $C \in S$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $C(e|_{freevars(C)}) \in clo^{\mathcal{BH}}(S)$. (51)

It is straightforward to prove that for all $a \in \mathcal{B}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, $\|a\|^{\mathfrak{A}}_e = \mathcal{V}(a)$. Let $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \triangleq \varepsilon_2$, by (48) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) = \mathcal{V}(\varepsilon_2)$, $\|l\|^{\mathfrak{A}}_e = \|\varepsilon_1 \eqcirc \varepsilon_2\|^{\mathfrak{A}}_e = \|\varepsilon_1\|^{\mathfrak{A}}_e \eqcirc \|\varepsilon_2\|^{\mathfrak{A}}_e = \mathcal{V}(\varepsilon_1) \eqcirc \mathcal{V}(\varepsilon_2) = 1$. Let $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$, $\varepsilon_1 \lhd \varepsilon_2$, by (48) for ε_1, ε_2, $\mathcal{V}(\varepsilon_1) < \mathcal{V}(\varepsilon_2)$, $\|l\|^{\mathfrak{A}}_e = \|\varepsilon_1 \prec \varepsilon_2\|^{\mathfrak{A}}_e = \|\varepsilon_1\|^{\mathfrak{A}}_e \prec \|\varepsilon_2\|^{\mathfrak{A}}_e = \mathcal{V}(\varepsilon_1) \prec \mathcal{V}(\varepsilon_2) = 1$. So, for all $l \in \mathbb{S}$ and $e \in \mathcal{S}_{\mathfrak{A}}$, for both the cases $l = \varepsilon_1 \eqcirc \varepsilon_2 \in \mathbb{S}$ and $l = \varepsilon_1 \prec \varepsilon_2 \in \mathbb{S}$, $\|l\|^{\mathfrak{A}}_e = 1$; $\|l\|^{\mathfrak{A}}_e = 1$. Let $C \in S \subseteq OrdCl_{\mathcal{L} \cup P}$ and $e \in \mathcal{S}_{\mathfrak{A}}$. Then $e : Var_{\mathcal{L}} \longrightarrow \mathcal{U}_{\mathfrak{A}}$, $freevars(C) \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$, $e|_{freevars(C)} \in Subst_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $dom(e|_{freevars(C)}) = freevars(C)$, $range(e|_{freevars(C)}) = \emptyset$; $e|_{freevars(C)}$ is applicable to C; by (51) for C, e, $C(e|_{freevars(C)}) \in clo^{\mathcal{BH}}(S)$, there exists $l^* \in C(e|_{freevars(C)})$ and $l^* \in \mathbb{S}$, $\|l^*\|^{\mathfrak{A}}_e = 1$; there exists $l^{**} \in C \in OrdCl_{\mathcal{L} \cup P}$ and $l^{**} \in OrdLit_{\mathcal{L} \cup P} \subseteq OrdLit_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $freevars(l^{**}) \subseteq freevars(C)$; $e|_{freevars(l^{**})}$ is applicable to l^{**}, $l^{**}(e|_{freevars(l^{**})}) = l^*$; for all $t \in Term_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $a \in Atom_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P} \cup QAtom_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $l \in OrdLit_{\mathcal{L} \cup \tilde{\mathbb{W}}^* \cup P}$, $\|t\|^{\mathfrak{A}}_e = t(e|_{vars(t)}) = \|t(e|_{vars(t)})\|^{\mathfrak{A}}_e$, $\|a\|^{\mathfrak{A}}_e = \|a(e|_{freevars(a)})\|^{\mathfrak{A}}_e$, $\|l\|^{\mathfrak{A}}_e = \|l(e|_{freevars(l)})\|^{\mathfrak{A}}_e$; the proof is by induction on t and by definition; $\|l^{**}\|^{\mathfrak{A}}_e = \|l^{**}(e|_{freevars(l^{**})})\|^{\mathfrak{A}}_e = \|l^*\|^{\mathfrak{A}}_e = 1$; $\mathfrak{A} \models_e C$; $\mathfrak{A} \models S$, $\mathfrak{A}|_{\mathcal{L} \cup P} \models S$; S is satisfiable. The theorem is proved. □

Consider $S = \{0 \prec a\} \cup \{a \prec \overline{\tfrac{1}{n}} \mid n \geq 2\} \subseteq OrdCl_{\mathcal{L}}$, $a \in Pred_{\mathcal{L}} - Tcons_{\mathcal{L}}$, $ar_{\mathcal{L}}(a) = 0$. $tcons(S)$ is not admissible with respect to suprema and infima; for $\{0\}$ and $\{\overline{\tfrac{1}{n}} \mid n \geq 2\}$, $\bigvee\{0\} = \bigwedge\{\tfrac{1}{n} \mid n \geq 2\} = 0$, $0 \notin \{\tfrac{1}{n} \mid n \geq 2\}$. S is unsatisfiable; both the cases $\|a\|^{\mathfrak{A}} = 0$ and $\|a\|^{\mathfrak{A}} > 0$ lead to $\mathfrak{A} \not\models S$ for every interpretation $\mathfrak{A}$ for $\mathcal{L}$. However, $\square \notin clo^{\mathcal{H}}(S) = S \cup \{0 \prec 1\} \cup \{0 \prec \overline{\tfrac{1}{n}} \mid n \geq 2\} \cup \{\overline{\tfrac{1}{n}} \prec 1 \mid n \geq 2\} \cup \{\overline{\tfrac{1}{n_1}} \prec \overline{\tfrac{1}{n_2}} \mid n_1 > n_2 \geq 2\} \cup \{\overline{\tfrac{1}{n}} \prec a \vee \overline{\tfrac{1}{n}} \eqcirc a \vee a \prec \overline{\tfrac{1}{n}} \mid n \geq 2\} \cup \{\overline{\tfrac{1}{n}} \eqcirc a \vee a \prec \overline{\tfrac{1}{n}} \mid n \geq 2\} \cup \{\overline{\tfrac{1}{n}} \prec a \vee a \prec \overline{\tfrac{1}{n}} \mid n \geq 2\}$, using Rules (33) and (32); $clo^{\mathcal{H}}(S)$ contains the order clauses from S, from $ordtcons(S)$, and some superclauses of them. So, the condition on $tcons(S)$ being admissible with respect to suprema and infima, is necessary.

The deduction problem of a formula from a theory can be solved as follows:

Corollary 4.5. *Let $\mathcal{L}$ contain a constant symbol. Let $n_0 \in \mathbb{N}$, $\phi \in OrdForm_{\mathcal{L}}$, $T \subseteq OrdForm_{\mathcal{L}}$, $tcons(T)$ be admissible with respect to suprema and infima. There exist $J^{\phi}_T \subseteq \{(i,j) \mid i \geq n_0\}$ and $S^{\phi}_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J^{\phi}_T\}}$ such that $tcons(S^{\phi}_T)$ is admissible with respect to suprema and infima; $T \models \phi$ if and only if $\square \in clo^{\mathcal{H}}(S^{\phi}_T)$.*

Proof. By Corollary 3.5 for n_0, ϕ, T, there exist

$$J^{\phi}_T \subseteq \{(i,j) \mid i \geq n_0\}, S^{\phi}_T \subseteq SimOrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J^{\phi}_T\}}$$

Table 4: An example: $\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\,q_1(x) \rightarrow \overline{0.5})$.

$$\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\,q_1(x) \rightarrow \overline{0.5})$$

$$\Big\{\tilde{p}_0(x) \prec 1, \tilde{p}_0(x) \leftrightarrow \big(\underbrace{\forall x\,(q_1(x) \rightarrow \overline{0.3})}_{\tilde{p}_1(x)} \rightarrow \underbrace{(\exists x\,q_1(x) \rightarrow \overline{0.5})}_{\tilde{p}_2(x)}\big)\Big\} \qquad (18)$$

$$\Big\{\tilde{p}_0(x) \prec 1, \tilde{p}_1(x) \prec \tilde{p}_2(x) \vee \tilde{p}_1(x) \eqcirc \tilde{p}_2(x) \vee \tilde{p}_0(x) \eqcirc \tilde{p}_2(x), \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_0(x) \eqcirc 1, \tilde{p}_1(x) \leftrightarrow \forall x\,(\underbrace{q_1(x) \rightarrow \overline{0.3}}_{\tilde{p}_3(x)}), \tilde{p}_2(x) \leftrightarrow (\underbrace{\exists x\,q_1(x)}_{\tilde{p}_4(x)} \rightarrow \underbrace{\overline{0.5}}_{\tilde{p}_5(x)})\Big\} \qquad (23), (18)$$

$$\Big\{\tilde{p}_0(x) \prec 1, \tilde{p}_1(x) \prec \tilde{p}_2(x) \vee \tilde{p}_1(x) \eqcirc \tilde{p}_2(x) \vee \tilde{p}_0(x) \eqcirc \tilde{p}_2(x), \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_0(x) \eqcirc 1, \tilde{p}_1(x) \eqcirc \forall x\,\tilde{p}_3(x), \tilde{p}_3(x) \leftrightarrow (\underbrace{q_1(x)}_{\tilde{p}_6(x)} \rightarrow \underbrace{\overline{0.3}}_{\tilde{p}_7(x)}),$$

$$\tilde{p}_4(x) \prec \tilde{p}_5(x) \vee \tilde{p}_4(x) \eqcirc \tilde{p}_5(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_5(x), \tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_2(x) \eqcirc 1, \tilde{p}_4(x) \leftrightarrow \exists x\,\underbrace{q_1(x)}_{\tilde{p}_8(x)}, \tilde{p}_5(x) \eqcirc \overline{0.5}\Big\} \qquad (18), (24)$$

$$\Big\{\tilde{p}_0(x) \prec 1, \tilde{p}_1(x) \prec \tilde{p}_2(x) \vee \tilde{p}_1(x) \eqcirc \tilde{p}_2(x) \vee \tilde{p}_0(x) \eqcirc \tilde{p}_2(x), \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_0(x) \eqcirc 1, \tilde{p}_1(x) \eqcirc \forall x\,\tilde{p}_3(x),$$

$$\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x) \vee \tilde{p}_3(x) \eqcirc \tilde{p}_7(x), \tilde{p}_7(x) \prec \tilde{p}_6(x) \vee \tilde{p}_3(x) \eqcirc 1, \tilde{p}_6(x) \eqcirc q_1(x), \tilde{p}_7(x) \eqcirc \overline{0.3},$$

$$\tilde{p}_4(x) \prec \tilde{p}_5(x) \vee \tilde{p}_4(x) \eqcirc \tilde{p}_5(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_5(x), \tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_2(x) \eqcirc 1, \tilde{p}_4(x) \eqcirc \exists x\,\tilde{p}_8(x), \tilde{p}_8(x) \eqcirc q_1(x), \tilde{p}_5(x) \eqcirc \overline{0.5}\Big\}$$

$S^{\phi} = \Big\{$

$\boxed{\tilde{p}_0(x) \prec 1}$ [1]

$\tilde{p}_1(x) \prec \tilde{p}_2(x) \vee \tilde{p}_1(x) \eqcirc \tilde{p}_2(x) \vee \tilde{p}_0(x) \eqcirc \tilde{p}_2(x)$ [2]

$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_0(x) \eqcirc 1}$ [3]

$\boxed{\tilde{p}_1(x) \eqcirc \forall x\,\tilde{p}_3(x)}$ [4]

$\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x) \vee \boxed{\tilde{p}_3(x) \eqcirc \tilde{p}_7(x)}$ [5]

$\tilde{p}_7(x) \prec \tilde{p}_6(x) \vee \tilde{p}_3(x) \eqcirc 1$ [6]

$\boxed{\tilde{p}_6(x) \eqcirc q_1(x)}$ [7]

$\boxed{\tilde{p}_7(x) \eqcirc \overline{0.3}}$ [8]

$\boxed{\tilde{p}_4(x) \prec \tilde{p}_5(x) \vee \tilde{p}_4(x) \eqcirc \tilde{p}_5(x)} \vee \tilde{p}_2(x) \eqcirc \tilde{p}_5(x)$ [9]

$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_2(x) \eqcirc 1}$ [10]

$\boxed{\tilde{p}_4(x) \eqcirc \exists x\,\tilde{p}_8(x)}$ [11]

$\boxed{\tilde{p}_8(x) \eqcirc q_1(x)}$ [12]

$\boxed{\tilde{p}_5(x) \eqcirc \overline{0.5}}\Big\}$ [13]

Rule (32) : [1][3] :

$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)}$ [14]

Rule (32) : [10][14] :

$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)}$ [15]

repeatedly **Rule (32)** : [9][15] :

$\boxed{\tilde{p}_2(x) \eqcirc \tilde{p}_5(x)}$ [16]

Rule (35) : $\forall x\,\tilde{p}_3(x)$:

$\boxed{\forall x\,\tilde{p}_3(x) \prec \tilde{p}_3(x) \vee \forall x\,\tilde{p}_3(x) \eqcirc \tilde{p}_3(x)}$ [17]

$\overline{\mathbf{0.3}} \prec \overline{\mathbf{0.5}} \in \mathit{ordtcons}(S^{\phi})$

$\boxed{\overline{0.3} \prec \overline{0.5}}$ [18]

repeatedly **Rule (32)** : [4][5][8][13][14][16][17][18] :

$\boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x)}$ [19]

Rule (38) : $\exists x\,\tilde{p}_8(x), \overline{0.5}$:

$\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)}) \vee \boxed{\exists x\,\tilde{p}_8(x) \prec \overline{0.5} \vee \exists x\,\tilde{p}_8(x) \eqcirc \overline{0.5}}$ [20]

repeatedly **Rule (32)** : [11][13][15][20] :

$\boxed{\overline{0.5} \prec \tilde{p}_8(\tilde{w}_{(0,0)})}$ [21]

repeatedly **Rule (32)** : [7][8][12][19]; $\tilde{w}_{(0,0)}$: [18][21] :

$\square$ [22]

and Corollary 3.5(i,iv) hold for ϕ, T, S^{ϕ}_{T}; we have $\mathit{tcons}(T)$ is admissible with respect to suprema and infima, $\mathit{tcons}(S^{\phi}_{T}) \subseteq \mathit{tcons}(\phi) \cup \mathit{tcons}(T)$; $\mathit{tcons}(\phi) \subseteq_{\mathcal{F}} \mathit{Tcons}_{\mathcal{L}}$, $\mathit{tcons}(S^{\phi}_{T})$ is admissible with respect to suprema and infima; we have $T \models \phi$ if and only if S^{ϕ}_{T} is unsatisfiable; by Theorem 4.4 for $\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J^{\phi}_{T}\}$, S^{ϕ}_{T}, S^{ϕ}_{T} is unsatisfiable if and only if $\square \in \mathit{clo}^{\mathcal{H}}(S^{\phi}_{T})$; $T \models \phi$ if and only if $\square \in \mathit{clo}^{\mathcal{H}}(S^{\phi}_{T})$. The corollary is proved. $\square$

In Table 4, we show that $\phi = \forall x\,(q_1(x) \rightarrow \overline{0.3}) \rightarrow (\exists x\,q_1(x) \rightarrow \overline{0.5}) \in \mathit{OrdForm}_{\mathcal{L}}$ is logically valid using the translation to order clausal form and the order hyperresolution calculus.

5 CONCLUSIONS

In the paper, we have proposed a modification of the hyperresolution calculus from (Guller, 2012; Guller,

2015) which is suitable for automated deduction in the first-order Gödel logic with explicit partial truth. Gödel logic is expanded by a countable set of intermediate truth constants $\bar{c}$, $c \in (0,1)$. We have modified translation of a formula to an equivalent satisfiable finite order clausal theory, consisting of order clauses. An order clause is a finite set of order literals of the form $\varepsilon_1 \diamond \varepsilon_2$ where ε_i is an atom or a quantified atom, and $\diamond$ is the connective $\eqcirc$ or $\prec$. $\eqcirc$ and $\prec$ are interpreted by the equality and standard strict linear order on $[0,1]$, respectively. We have investigated the so-called canonical standard completeness, where the semantics of Gödel logic is given by the standard *G*-algebra and truth constants are interpreted by 'themselves'. The modified hyperresolution calculus is refutation sound and complete for a countable order clausal theory if the set of truth constants occurring in the theory, is admissible with respect to suprema and infima. This condition covers the case of finite order clausal theories. We have solved the deduction problem of a formula from a countable theory. As an interesting consequence, we get an affirmative solution to the open problem of recursive enumerability of unsatisfiable formulae in Gödel logic with truth constants and the equality, $\eqcirc$, strict order, $\prec$, projection, $\mathbf{\Delta}$, operators.

Corollary 5.1. *The set of unsatisfiable formulae of $\mathcal{L}$ is recursively enumerable.*

Proof. Let $\phi \in OrdForm_{\mathcal{L}}$. Then ϕ contains a finite number of truth constants and $tcons(\{\phi\})$ is admissible with respect to suprema and infima. ϕ is unsatisfiable if and only if $\{\phi\} \models 0$. Hence, the problem that ϕ is unsatisfiable can be reduced to the deduction problem $\{\phi\} \models 0$ after a constant number of steps. Let $n_0 \in \mathbb{N}$. By Corollary 4.5 for n_0, 0, $\{\phi\}$, there exist $J^0_{\{\phi\}} \subseteq \{(i,j) \,|\, i \geq n_0\}$, $S^0_{\{\phi\}} \subseteq SimOrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J^0_{\{\phi\}}\}}$ and $tcons(S^0_{\{\phi\}})$ is admissible with respect to suprema and infima, $\{\phi\} \models 0$ if and only if $\Box \in clo^{\mathcal{H}}(S^0_{\{\phi\}})$; if $\{\phi\} \models 0$, then $\Box \in clo^{\mathcal{H}}(S^0_{\{\phi\}})$ and we can decide it after a finite number of steps. This straightforwardly implies that the set of unsatisfiable formulae of $\mathcal{L}$ is recursively enumerable. The corollary is proved. $\Box$

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