Ranking of Interval Type-2 Fuzzy Numbers based on Centroid Point and Spread

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Abstract: A concept of interval type-2 fuzzy numbers is introduced in decision making analysis as this concept is capable to effectively deal with the uncertainty in the information about a decision. It considers two types of uncertainty namely inter and intra personal uncertainties, in enhancing the representation of type-1 fuzzy numbers in the literature of fuzzy sets. As interval type-2 fuzzy numbers are crucial in decision making, this paper proposes a methodology for ranking interval type-2 fuzzy numbers. This methodology consists of two parts namely the interval type-2 fuzzy numbers reduction methodology as the first part and ranking of type-1 fuzzy numbers as the second part. In this study, established reduction methodology of interval type-2 fuzzy numbers into type-1 fuzzy numbers is extended to reduction into standardised generalised type-1 fuzzy numbers as the extension complements the capability of the methodology on dealing with both positive and negative data values. It is worth adding here that this methodology is analysed using thorough empirical comparison with some established ranking methods for consistency evaluation. This methodology is considered as a generic decision making procedure, especially when interval type-2 fuzzy numbers are applied to real decision making problems.

1 INTRODUCTION

Fuzzy set theory serves as the basis of formal decision making analysis when uncertainty factors are involved in human decision making. This is expressed through ability of human in making logical decisions using imprecise and incomplete information which leads to uncertainty in terms of decision informativeness. Fuzzy number or type-1 fuzzy number is the first numerical representation of fuzzy sets (or type-1 fuzzy sets) introduced in the literature of fuzzy sets (Zadeh, 1965). Among decision making situations that considered type-1 fuzzy numbers in the evaluations are fuzzy risk analysis by Chen et al. (2012), supply chain management (Wu et al., 2013), fuzzy portfolio (Bermudez et al., 2012), selection of construction project (Ebrahimnejad et al., 2012) and decision making on water resources (Morais and Almeida, 2012). Main reason of those authors utilised type-1 fuzzy numbers in their chosen decision making situations is due to the capabilities of type-1 fuzzy numbers to appropriately deal with imprecise numerical quantities and subjective preferences of decision makers (Deng, 2014). Although, type-1 fuzzy numbers are appropriate for decision making purposes, it is not easy to clearly determine which type-1 fuzzy number is larger or smaller than another (Kumar et al., 2010). This is because type-1 fuzzy numbers are represented by possibility distributions which indicate that they potentially overlap with each other (Zimmermann, 2000; Kumar et al., 2010). Thus, decision makers need to compare or ranking them correctly so that an effective outcome of a decision making is obtained.

In the literature of fuzzy sets, a concept of ranking type-1 fuzzy numbers is introduced by Jain (1976) as a way to differentiate type-1 fuzzy numbers effectively. In order to do so, several ranking methods are recently suggested in literature of ranking type-1 fuzzy numbers namely ranking method based on different heights and spreads (Chen and Chen, 2009), method using similarity measure with centroid (Bakar et al., 2010), ranking based on area of fuzzy numbers (Chen and Sanguatsan, 2011), ranking of fuzzy numbers based on distance (Bakar et al., 2012), centroid based ranking method (Dat et al., 2012) and ranking method using epsilon degree (Yu et al., 2013). Based on these ranking methods, two common processes involve when ranking type-1

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fuzzy numbers are identified. They are evaluation of each type-1 fuzzy number considered and comparison among type-1 fuzzy numbers under consideration.

In a recent research work by Bakar and Gegov (2014), a new ranking of type-1 fuzzy numbers method is proposed where the method utilises two intuition based approaches namely the centroid point and spread (*CPS*). The capabilities of this method in ranking type-1 fuzzy numbers are shown when the method effectively solved many main problems of type-1 fuzzy numbers faced by recently established ranking methods of Kumar et al. (2010), Chen and Chen (2009), Chen and Sanguatsan (2011) and Dat et al. (2012) in embedded fuzzy numbers of different spread, ranking embedded fuzzy numbers of different shapes but having same centroid and embedded normal and non-normal fuzzy numbers respectively. Even though, the *CPS* ranking method capable to solve all shortcomings faced by the established ranking methods, limitation of type-1 fuzzy number to adequately representing the uncertainty affects the role played by methods developed for ranking type-1 fuzzy numbers, including the *CPS* ranking method when dealing with complex decision making.

Due to this, issue regarding the representation adequacy of type-1 fuzzy numbers on the uncertainty becomes one of the crucial problems in decision making environment (Zadeh, 1975; Wallsten and Budescu, 1995). According to Zadeh (1975) and Wallsten and Budescu (1995), there are two kinds of uncertainties that are supposedly related to linguistic characteristics which are often used in human decision making namely the intra-personal uncertainty and inter-personal uncertainty. Nonetheless, only one kind of uncertainty which is the intra-personal uncertainty is considered in the representation of type-1 fuzzy numbers. Thus, a concept of type-2 fuzzy sets is introduced by Zadeh (1975) in the literature of fuzzy sets as the extension of type-1fuzzy sets with capability of representing both kinds of uncertainty appropriately.

If the numerical representation for type-1 fuzzy set is called type-1 fuzzy number (Tsoukalas and Urigh, 1997), then the numerical representation for type-2 fuzzy set is known as type-2 fuzzy number (Coupland and John, 2003). As far as the investigations on utilising type-2 fuzzy numbers are concerned, many decision making problems are solved such as radiographic tibia image clustering (John, 2000), signal processing problem (Nagy and Takács, 2008), pattern recognition (Wu and Mendel, 2009) and oversea minerals investment problem (Hu et al., 2013).

Even though, type-2 fuzzy number is better than type-1 fuzzy number in terms of uncertainty representation (Agüero and Vargas, 2007), less coverage on type-2 fuzzy numbers are given in the literature of fuzzy sets. Therefore, this paper suggests a methodology for ranking interval type-2 fuzzy numbers where the first part covers on the reduction of interval type-2 fuzzy numbers into type-1 fuzzy numbers and the second part is on the application of the *CPS* ranking method (Bakar and Gegov, 2014). It is worth mentioning here that interval type-2 fuzzy numbers is used in this methodology, instead of type-2 fuzzy number as it is viewed as a special case and requires less computational works (Hu et al., 2013). Along with this study, an extension of interval type-2 fuzzy numbers into standardised generalised interval type-2 fuzzy numbers is introduced for the first time in the literature of fuzzy sets due to the fact that the extension creates generic representation for established interval type-2 fuzzy numbers which are suitable for generic decision making purposes.

The remainder of the paper is organised as follows: Section 2 discusses the theoretical preliminaries, Section 3 views on the proposed work. Validation of the proposed work is given in Section 4 and at last, a conclusion is made in section 5.

2 THEORETICAL PRELIMINARIES

2.1 Type-1 Fuzzy Sets

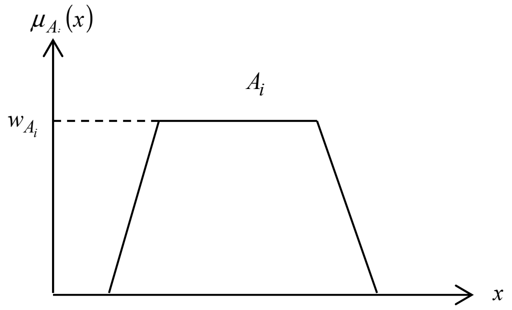
A type-1 fuzzy set A_i in a universe of discourse X is characterised by a membership function $\mu_{A_i}(x)$ which maps each element x in X such that x is real number in the interval [0, 1] (Cheng, 1998).

Membership function for A_i, $\mu_{A_i}(x)$ is given as

$$A_i = \{(x, \mu_{A_i}(x)) | \mu_{A_i}(x) \in [0,1] \forall x \in X\} \quad (1)$$

When type-1 fuzzy set is in the numerical representation, hence it is called as type-1 fuzzy numbers with membership function shown as follows.

$$\mu_{A_i}(x) = (a_{i1}, a_{i2}, a_{i3}, a_{i4}) = \begin{cases} \frac{x - a_{i1}}{a_{i2} - a_{i1}} & if \quad a_{i1} \le x \le a_{i2} \\ 1 & if \quad a_{i2} \le x \le a_{i3} \\ \frac{a_{i4} - x}{a_{i4} - a_{i3}} & if \quad a_{i3} \le x \le a_{i4} \\ 0 & otherwise \end{cases}$$



$A_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}; w_{A_i})$

Figure 1: Type-1 trapezoidal fuzzy number.

For a type-1 trapezoidal fuzzy numbers as in Figure 1, if $a_{i2} = a_{i3}$, then the type-1fuzzy number is in the form of type-1triangular fuzzy number. Whereas, if $a_{i1} = a_{i2} = a_{i3} = a_{i4}$ for both type-1 triangular and type-1 trapezoidal fuzzy numbers, then both type-1 fuzzy numbers are said to be in the form of type-1 singleton fuzzy number. Length between a_{i1} and a_{i4} is known as the support of the type-1fuzzy numbers (Chen and Chen, 2009).

2.2 Standardised Generalised Type-1 Fuzzy Numbers

If type-1 fuzzy number A_i has the property such that $-1 \le a_{i1} \le a_{i2} \le a_{i3} \le a_{i4} \le 1$, then $\widetilde{A}_i$ is called a standardised generalised type-1 trapezoidal fuzzy number (Chen and Chen, 2009) and is denoted as

$$\widetilde{A}_i = \left(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \widetilde{a}_{i3}, \widetilde{a}_{i4}; w_{\widetilde{A}_i}\right)$$

Furthermore, if $\widetilde{a}_{i2} = \widetilde{a}_{i3}$ then $\widetilde{A}_i$ is known as a standardised generalised type-1 triangular fuzzy number. Any type-1 fuzzy numbers are transformed into a standardised generalised type-1 fuzzy numbers by normalisation step which is described in (2).

$$\begin{aligned} \widetilde{A}_i &= \left(\frac{a_{i1}}{|k|}, \frac{a_{i2}}{|k|}, \frac{a_{i3}}{|k|}, \frac{a_{i4}}{|k|}; w_{\widetilde{A}_i}\right) \\ &= \left(\widetilde{a}_{i1}, \widetilde{a}_{i2}, \widetilde{a}_{i3}, \widetilde{a}_{i4}; w_{\widetilde{A}_i}\right) \end{aligned} \quad (2)$$

where $|k| = \max(a_{i1}, a_{i2}, a_{i3}, a_{i4})$.

It should be noted here that the normalisation process in equation (2), only the components of type-1 fuzzy numbers are changed where a_{i1}, a_{i2}, a_{i3}, a_{i4} are changed to $\widetilde{a}_{i1}, \widetilde{a}_{i2}, \widetilde{a}_{i3}, \widetilde{a}_{i4}$ while the height of the type-1 fuzzy number remains unchanged (Chen and Chen, 2009).

2.3 Type-2 Fuzzy Sets

If $P(U)$ is the set for fuzzy set U, then a type-2 fuzzy set A_i' in universe of discourse X is characterised by membership grades which are fuzzy (Zadeh, 1975). This implies that $\mu_{A_i'}(x)$ is a fuzzy set in U for all x given as

$$A_i' = \left\{ \left(x, \mu_{A_i'}(x)\right) \middle| \mu_{A_i'}(x) \in P(U) \forall x \in X \right\} \quad (3)$$

This follows that $\forall x \in X \exists J_x \subseteq U$ such that $\mu_{A_i'}(x) : J_x \to U$. Using equation (1), the following is obtained.

$$\mu_{A_i'}(x) = \left\{ \left(u, \mu_{A_i'}(x)(u)\right) \middle| \mu_{A_i'}(x)(u) \in U \forall u \in J_x \subseteq U \right\} \quad (4)$$

where X and J_x are the primary domain and primary membership of x respectively while U and $\mu_{A_i'}(x)$ are the secondary domain and secondary membership of x (Greenfield and Chiclana, 2013).
Using (3) and (4), the following is obtained.

$$A_i' = \left\{ \left(x, \left(u, \mu_{A_i'}(x)(u)\right)\right) \middle| _{A_i'}(x)(u) \in U, \forall x \in X \wedge \forall u \in J_x \subseteq U \right\} \quad (5)$$

2.4 Interval Type-2 Fuzzy Sets

According to Greenfield and Chiclana (2013), an interval type-2 fuzzy set is a type- 2 fuzzy set whose secondary membership grades are all 1. Thus, in the case of interval, equation (5) can be reduced to the following equation (6).

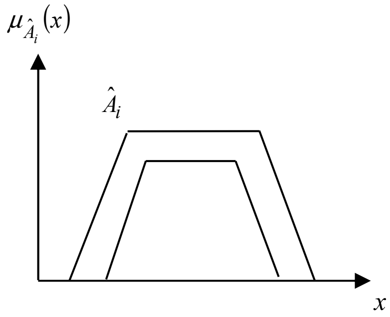
$$A_i' = \left\{ \left(x, (u, 1)\right), \forall x \in X \wedge \forall u \in J_x \subseteq U \right\} \quad (6)$$

Therefore, based on equations (1) and (3), the interval type-2 fuzzy set is called a trapezoidal interval type-2 fuzzy set when upper membership function (secondary) and lower membership function (primary) are depicted as

$$\hat{A}_i = \left[\left(\hat{a}_{i1}^U, \hat{a}_{i2}^U, \hat{a}_{i3}^U, \hat{a}_{i4}^U; 1\right), \left(\left(\hat{a}_{i1}^L, \hat{a}_{i2}^L, \hat{a}_{i3}^L, \hat{a}_{i4}^L; w_{\hat{a}_i}^L\right)\right)\right] \quad (7)$$

where $\hat{a}_{ij}^U, j = 1,2,3,4$ and $\hat{a}_{ij}^L, j = 1,2,3,4$ are secondary and primary membership functions values for $\hat{A}_i$.

Therefore, the numerical domain representation of trapezoidal interval type-2 fuzzy number is illustrated as follows,



$$\hat{A}_i = \left[\left(\hat{a}_{i1}^U,\hat{a}_{i2}^U,\hat{a}_{i3}^U,\hat{a}_{i4}^U;1\right),\left(\left(\hat{a}_{i1}^L,\hat{a}_{i2}^L,\hat{a}_{i3}^L,\hat{a}_{i4}^L;w_{\hat{a}_i}^L\right)\right)\right]$$

Figure 2: An interval type-2 fuzzy number.

3 PROPOSED METHODOLOGY

In this section, details with regard to procedure involved in the proposed methodology are described. Full descriptions on the methodology are as follows.

This study first proposes a concept of standardised generalised interval type-2 fuzzy numbers in replacing the interval type-2 fuzzy numbers for easy computation. This is because the proposed concept provides generic representation of interval type-2 fuzzy numbers that are suitable for decision making purposes such as consideration of both positive and negative values. Thus, the definition of standardised generalised interval type-2 fuzzy number introduced in this study is given as the following.

If an interval type-2 fuzzy number $\hat{A}_i$ has the property such that $-1<\hat{a}_{i1}^U<\hat{a}_{i2}^U<\hat{a}_{i3}^U<\hat{a}_{i4}^U<1$ and $-1<\hat{a}_{i1}^L<\hat{a}_{i2}^L<\hat{a}_{i3}^L<\hat{a}_{i4}^L<1$, then A_i' is called as a standardised generalised interval type-2 fuzzy number denoted as

$$A_i' = \left[\left(a_{i1}'^U,a_{i2}'^U,a_{i3}'^U,a_{i4}'^U;1\right)\left(a_{i1}'^L,a_{i2}'^L,a_{i3}'^L,a_{i4}'^L;w_{a_i}^L\right)\right] \quad (8)$$

Any interval type-2 fuzzy numbers are transformed into a standardised generalised interval type-2fuzzy numbers by normalisation process shown as follows.

$$A_i' = \left[\left(\frac{\hat{a}_{i1}^U}{|k|},\frac{\hat{a}_{i2}^U}{|k|},\frac{\hat{a}_{i3}^U}{|k|},\frac{\hat{a}_{i4}^U}{|k|};1\right),\left(\left(\frac{\hat{a}_{i1}^L}{|k|},\frac{\hat{a}_{i2}^L}{|k|},\frac{\hat{a}_{i3}^L}{|k|},\frac{\hat{a}_{i4}^L}{|k|};w_{\hat{a}_i}^L\right)\right)\right]$$

$$= \left[\left(a_{i1}'^U,a_{i2}'^U,a_{i3}'^U,a_{i4}'^U;1\right),\left(\left(a_{i1}'^L,a_{i2}'^L,a_{i3}'^L,a_{i4}'^L;w_{a_i}^L\right)\right)\right] \quad (9)$$

where $|k| = \max\left(\hat{a}_{i1}^U,\hat{a}_{i2}^U,\hat{a}_{i3}^U,\hat{a}_{i4}^U\right)$

It should be noted here that the normalisation process only changes the components of interval type-2 fuzzy numbers where $\hat{a}_{i1}^U,\hat{a}_{i2}^U,\hat{a}_{i3}^U,\hat{a}_{i4}^U$ and $\hat{a}_{i1}^L,\hat{a}_{i2}^L,\hat{a}_{i3}^L,\hat{a}_{i4}^L$ are changed to $a_{i1}'^U,a_{i2}'^U,a_{i3}'^U,a_{i4}'^U$ and $a_{i1}'^L,a_{i2}'^L,a_{i3}'^L,a_{i4}'^L$ respectively while the heights of interval type-2 fuzzy number remain unchanged.

After the interval type-2 fuzzy numbers are transformed into standardized generalised interval type-2 fuzzy numbers, the standardized generalised interval type-2 fuzzy numbers obtained are then ranked using the following procedure.

- Reduce the standardised generalised interval type-2 fuzzy number using Nie and Tan (2008) reduction method.
- Extend the *CPS* ranking method (Bakar and Gegov, 2014) to ranking the reduced standardised generalised interval type-2 fuzzy number.

3.1 Part One

According to Greenfield and Chiclana, 2013), the reduction algorithm is a process of reducing type-2 fuzzy sets into type-1 fuzzy sets. The process generalises the operations defined for crisp numbers to type-1 fuzzy sets mathematically which is in line with the Extension Principle developed by Zadeh (1965). In an analysis on accuracy by Greenfield and Chiclana (2013), Nie and Tan (2008) reduction method outperforms Wu and Mendel (2002) and Enhanced Iterative Algorithm with Stop Condition, EIASC(Wu and Nie, 2011) methods.

Since, it is shown that equation (6) is obtained from equation (5), hence the reduction algorithm developed for type-2 fuzzy sets by Nie and Tan (2008) is also applicable to interval type-2 fuzzy sets. In this case, the numerical representation of interval type-2 fuzzy sets which is the interval type-2 fuzzy numbers is reduced into type-1 fuzzy numbers. Therefore, without loss of generality, the reduction of standardised generalised interval type-2 fuzzy numbers into type-1 fuzzy numbers using Nie and Tan (2008) is as follows.

$$\mu_{A_i'} = \frac{1}{2}\left(\mu_{A_i'}^L + \mu_{A_i'}^U\right)$$

$$= \left[\frac{1}{2}\left(a_{i1}'^U + a_{i1}'^L, a_{i2}'^U + a_{i2}'^L, a_{i3}'^L + a_{i3}'^U, a_{i4}'^U + a_{i4}'^L\right)\right] \quad (10)$$

where

$$a_{i1}'^U, a_{i1}'^L, a_{i2}'^U, a_{i2}'^L, a_{i3}'^L, a_{i3}'^U, a_{i4}'^U, a_{i4}'^L \in [-1,1].$$

Note that, Nie andTan (2008) reduction method in equation (10) neglects the non-normal interval type-2 fuzzy sets in their analysis as the work assumes that the heights for both primary and secondary components are always 1, $\mu^L_{A'_i}, \mu^U_{A'_i} = 1$. However, Wu and Mendel (2009) indicate that the height of secondary component is $1, \mu^L_{A'_i} = 1$ but the height of the primary component can be between 0 and $1, \mu^U_{A'_i} \in [0,1]$. Therefore, based on this reason, this study extends the work by Nie andTan (2008) where the extension is as follows.

$$\mu_{A'_i} = \frac{1}{2}\left(\mu^L_{A'_i} + \mu^U_{A'_i}\right)$$

$$= \left[\frac{1}{2}\left(a'^U_{i1} + a'^L_{i1}, a'^U_{i2} + a'^L_{i2}, a'^L_{i3} + a'^U_{i3}, a'^U_{i4} + a'^L_{i4}; 1 + w^L_{A'_i}\right)\right] \quad (11)$$

where $a'^U_{i1}, a'^L_{i1}, a'^U_{i2}, a'^L_{i2}, a'^L_{i3}, a'^U_{i3}, a'^U_{i4}, a'^L_{i4} \in [-1,1]$

$w^L_{A'_i} \in [0,1]$

It is worth mentioning here that, the main different between equations (10) and (11) is the latter considers the height in the reduction process while this is neglected by the former. This extension is introduced in this study as it provides a more generic valuation for the height of the primary component and standardised generalised interval type-2 fuzzy numbers. This process is crucial for evaluating standardised generalised interval type-2 fuzzy numbers, especially when they are applied for decision making. It is worth mentioning here that the extension is introduced in accordance to equation (6) given by Wu and Nie (2011). Note that, the reduction process in equation (11) reduces standardised generalised interval type-2 fuzzy numbers and not the interval type-2 fuzzy numbers.

3.2 Part Two

As mentioned in the introduction section, the *CPS* ranking method introduced by Bakar and Gegov (2014) caters limitations faced by existing established methods in ranking type-1 fuzzy numbers. Effectiveness of this method in ranking various type-1 fuzzy numbers cases proves this method is applicable for practical usage. Note that, as standardised generalised interval type-2 fuzzy numbers are used in previous part, the reduction step in (11) reduces standardised generalised interval type-2 fuzzy numbers into standardised generalised type-1 fuzzy numbers. Therefore, the *CPS* ranking method proposed by Bakar and Gegov (2014) is extended to the CPS_2 ranking method to indicate that the CPS_2 ranking method is utilised for ranking standardised generalised type-1 fuzzy numbers after reduction from standardised generalised interval type-2 fuzzy numbers. Details on the procedure in the CPS_2 ranking method are as follows.

Let A be standardised generalised type-1 fuzzy numbers after reduction using equation (11) described as $A = (a_1, a_2, a_3, a_4; w_A)$,

Step 1: Compute the centroid point value for standardised generalised type-1 fuzzy number A using Shieh (2007) formula such that the horizontal – x centroid value of A, x_A is calculated as

$$x_A = \frac{\int_{-\infty}^{\infty} x f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} \quad (12)$$

and vertical – y cenrtroid of A, yA is

$$y_A = \frac{\int_0^{w_A} \alpha\left|A^\alpha\right| d\alpha}{\int_0^{w_A} \left|A^\alpha\right| d\alpha} \quad (13)$$

where

$\left|A^\alpha\right|$ is length of α – cuts of standardised generalised type-1 fuzzy number A, $x_A \in [-1 , 1]$ and $y_A \in [0 ,w_A]$.

Step 2: Calculate the spread value for standardised generalised type-1 fuzzy number A by considering the distance along x – axis from x_A defines as

$$i_A = dist(a_4 - a_1) = \left|a_4 - x_A\right| + \left|x_A - a_1\right|$$
$$= \left|a_4 - a_1\right| \quad (14)$$

and the distance along vertical y – axis defines as

$$ii_A = y_A \quad (15)$$

Therefore, the spread of A, s_A is defined as

$$s_A = i_A \text{x } ii_A \quad (16)$$

where i_A and ii_A are $dist(a_4 - a_1)$ and y_A respectively. $s_A, i_A, ii_A, dist(a_4 - a_1) \in [0 ,1]$.

Step 3: Determine the ranking value for standardised generalised type-1 fuzzy number A using the following equation as

$$CPS_2(A) = x_A \times y_A \times (1 - s_A) \quad (17)$$

where

x_A is the horizontal – x centroid for A

y_A is the vertical – y centroid for A

s_A is the spread for A

$CPS_2(A) \in [-1, 1]$

If $CPS_2(A) > CPS_2(B)$, then $A \succ B$ (i.e. A is greater than B).

If $CPS_2(A) < CPS_2(B)$, then $A \prec B$ (i.e. A is lesser than B).

If $CPS_2(A) = CPS_2(B)$, then $A \approx B$ (i.e. A and B are equal).

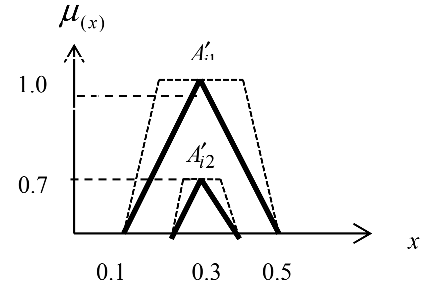
4 VALIDATION OF RESULTS

In this validation, relevant benchmarking examples of standardised generalised interval type-2 fuzzy numbers are introduced for the first time in the literature of fuzzy sets. It has to be noted here that validation in this section is a comparative – based analysis which compares the CPS_2 ranking method with some established ranking methods of ranking type-1 fuzzy numbers to ranking standardised generalised interval type-2 fuzzy numbers. This is because there are inadequate methods for ranking standardised generalised interval type-2 fuzzy numbers in the literature of fuzzy sets. Moreover, the standardised generalised interval type-2 fuzzy numbers are reduced into standardised generalised type-1 fuzzy numbers, thus established ranking methods considered in this study are suitable for ranking standardised generalised type-1 fuzzy numbers

Benchmarking examples developed in this study involve cases that are related with decision making problems. If a ranking method produces correct ranking result such that the result is consistent with human intuition, then the ranking result is signified as consistent (Y), otherwise, it is inconsistent (N). As mentioned, all established existing ranking methods used in this section are methods established for ranking standardised generalised type-1 fuzzy numbers. Thus, these established methods are added '2' (e.g. 2-Cheng (1998)) to signify that they are methods for ranking standardised generalised type-1 fuzzy numbers but are extended to ranking standardised generalised interval type-2 fuzzy number for the first time. Note that, these methods are applicable to ranking standardised generalised interval type-2 fuzzy numbers only if all standardised generalised interval type-2 fuzzy numbers considered in this validation are reduced standardised generalised type-1 fuzzy numbers using equation (11). Therefore, cases of standardised generalised interval type-2 fuzzy numbers considered in this study that are potentially appeared in decision making environment are as follows.

Case 1: Embedded standardised generalised interval type-2 fuzzy numbers of different shapes.

Consider two standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} shown in Figure 3. The correct ranking order such that the ranking result is consistent with human intuition for this case is $A'_{i1} \succ A'_{i2}$ because the centroid point of A'_{i1} is greater than A'_{i2}. Using Chen and Chen (2009) ranking method, an unreasonable ranking order is produced such that the result is inconsistent with human intuition $A'_{i2} \succ A'_{i1}$ as they treat type-2 fuzzy numbers with smaller centroid point as greater than the other. Kumar et al. (2010) and Chen and Sanguatsan (2011) ranking methods on the other hand treat both type-2 fuzzy numbers as equal ($A'_{i1} \approx A'_{i2}$) which is also incorrect such that the result is inconsistent with human intuition. It is also shown in Table 1 where ranking methods by Cheng (1998) and Chu and Tsao (2002) unable to give any ranking result for this case as they are only applicable to normal case of standardised generalised interval type-2 fuzzy numbers. Using the CPS_2 ranking method, the ranking order produced is the same as Dat et al. (2012) ranking method where both ranking methods produce correct ranking order for this case such that the result is consistent with human intuition by ranking the standardised generalised interval type-2 fuzzy numbers with higher centroid point as higher ranking order.



$$A'_{i1} = [(0.1,0.2,0.4,0.5;1;1),((0.2,0.25,0.35,0.4;0.7;0.7))]$$
$$A'_{i2} = [(0.1,0.3,0.3,0.5;1;1),((0.2,0.3,0.3,0.4;0.7;0.7))]$$

Figure 3: Standardised generalised interval type -2 fuzzy numbers A'_{i1} and A'_{i2} of Case 1.

Case 2: Embedded standardised generalised interval type-2 fuzzy numbers of different spreads

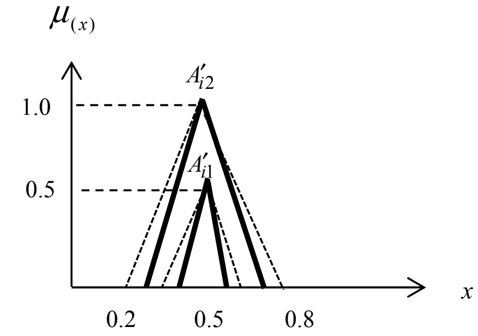
Consider two standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} shown in Figure 4. The correct ranking order such that the ranking

Table 1: Comparative results for Case 1.

Methods	Type-2 fuzzy numbers		Ranking Results	Evaluation
	A'_{i1}	A'_{i2}		
2-Cheng (1998)	-	-	x	N
2-Chu and Tsao (2002)	-	-	x	N
2-Chen and Chen (2009)	0.2243	0.2272	$A'_{i1} \prec A'_{i2}$	N
2-Kumar et al. (2010)	0.2400	0.2400	$A'_{i1} \approx A'_{i2}$	N
2-(Chen and Sanguatsan, 2011)	0.3000	0.3000	$A'_{i1} \approx A'_{i2}$	N
2-Dat et al. (2012)	0.3333	0.2220	$A'_{i1} \succ A'_{i2}$	Y
2-Yu et al. (2013), $\alpha = 0$	1.0000	1.0000	$A'_{i1} \prec A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 0.5$	1.0000	1.0000	$A'_{i1} \approx A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 1$	1.0000	1.0000	$A'_{i1} \succ A'_{i2}$	Y
CPS_2	0.0136	0.0077	$A'_{i1} \succ A'_{i2}$	Y

'x' denotes ranking method as unable to rank the standardised generalised interval type-2 fuzzy numbers
'-' denotes no ranking result are obtained.
'Y' denotes the ranking result is consistent
'N' denotes the ranking result is inconsistent.

result is consistent with human intuition for this case is $A'_{i2} \succ A'_{i1}$. This is due to ranking order for any standardised generalised interval type-2 fuzzy numbers with lower spread value is greater than others provided that the centroid point value of each standardised generalised interval type-2 fuzzy number under considerationis the same. In this case, Kumar et al. (2010), Chen and Sanguatsan (2011) and Dat et al. (2012) ranking methods are unable to differentiate the standardised generalised interval type-2 fuzzy numbers where they produce equal ranking $(A'_{i1} \approx A'_{i2})$ such that the result is inconsistent with human intuition. Cheng (1998) and Chu andTsao (2002) ranking methods in this case produce no ranking result as they both are not applicable when dealing with non – normal standardised generalised interval type-2 fuzzy numbers. Ranking method by Yu et al. (2013) on the other hand, captures the actual decision makers' preference by utilising the degree of optimisms in obtaining the ranking order for the standardised generalised interval type-2 fuzzy numbers. Thus, this method produces many ranking result for this case. Another incorrect ranking order such that the result is inconsistent with human intuition is produced by Chen and Chen (2009) ranking method where it gives $A'_{i1} \prec A'_{i2}$. Based on Table 2, only the CPS_2 ranking method produces correct ranking order such that the result is consistent with human intuition by giving priority towards standardised generalised interval type-2 fuzzy numbers with lower spread as higher ranking. It is also shown in this case where most of the latest presented ranking methods are unable to solve this case appropriately.



$$A'_{i1} = [(0.2,0.5,0.5,0.8;1;1),((0.4,0.5,0.5,0.6;0.5;0.5))]$$
$$A'_{i2} = [(0.25,0.5,0.5,0.75;1;1),((0.45,0.5,0.5,0.55;0.5;0.5))]$$

Figure 4: Standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} of Case 2.

Table 2: Comparative results for Case 2.

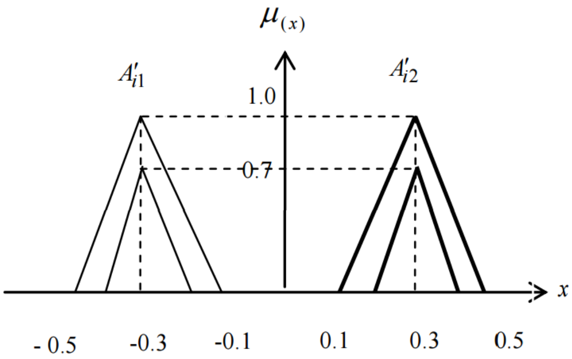
Methods	Type-2 fuzzy numbers		Ranking Results	Evaluation
	A'_{i1}	A'_{i2}		
2-Cheng (1998)	-	-	x	N
2-Chu and Tsao (2002)	-	-	x	N
2-Chen and Chen (2009)	0.3819	0.4770	$A'_{i1} \prec A'_{i2}$	N
2-Kumar et al. (2010)	0.5000	0.5000	$A'_{i1} \approx A'_{i2}$	N
2-(Chen and Sanguatsan, 2011)	0.3000	0.3000	$A'_{i1} \approx A'_{i2}$	N
2-Dat et al. (2012)	0.1111	0.1111	$A'_{i1} \approx A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 0$	1.0000	1.0000	$A'_{i1} \prec A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 0.5$	1.0000	1.0000	$A'_{i1} \approx A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 1$	1.0000	1.0000	$A'_{i1} \succ A'_{i2}$	Y
CPS_2	0.0135	0.0115	$A'_{i1} \succ A'_{i2}$	Y

'x' denotes ranking method as unable to rank the standardised generalised interval type-2 fuzzy numbers
'-' denotes no ranking result are obtained.
'Y' denotes the ranking result is consistent
'N' denotes the ranking result is inconsistent.

Case 3: Reflection of standardised generalised interval type-2 fuzzy numbers.

Consider two standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} shown in Figure

5. It is obvious that A'_{i2} is situated on the farthest right of A'_{i1}, where the correct ranking order such that for this case such that the result is consistent with human intuitions is $A'_{i2} \succ A'_{i1}$. Cheng (1998) and Chu and Tsao (2002) ranking methods again produce no ranking result for this case while Kumar et al. (2010) ranking method is incapable to differentiate both standardised generalised interval type-2 fuzzy numbers, hence produces incorrect ranking result such that the result is inconsistent



$$A'_{i1} = [(0.1,0.2,0.4,0.5;1;1),((0.2,0.25,0.35,0.4;0.7;0.7))]$$
$$A'_{i2} = [(-0.5,-0.4,-0.2,-0.1;1;1),((-0.4,-0.35,-0.25,-0.2;0.7;0.7))]$$

Figure 5: Standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} of Case 3.

Table 3: Comparative results for Case 3.

Methods	Type-2 fuzzy numbers		Ranking Results	Evaluation
	A'_{i1}	A'_{i2}		
2-Cheng (1998)	-	-	x	N
2-Chu and Tsao (2002)	-	-	x	N
2-Chen and Chen (2009)	– 0.2272	0.2272	$A'_{i1} \prec A'_{i2}$	Y
2-Kumar et al. (2010)	0	0	$A'_{i1} \approx A'_{i2}$	N
2-(Chen and Sanguatsan, 2011)	– 0.3000	0.3000	$A'_{i1} \prec A'_{i2}$	Y
2-Dat et al. (2012)	0.1333	0.1500	$A'_{i1} \prec A'_{i2}$	Y
2-Yu et al. (2013), $\alpha = 0$	1.0000	1.0000	$A'_{i1} \prec A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 0.5$	1.0000	1.0000	$A'_{i1} \approx A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 1$	1.0000	1.0000	$A'_{i1} \succ A'_{i2}$	Y
CPS$_2$	– 0.0077	0.0077	$A'_{i1} \prec A'_{i2}$	Y

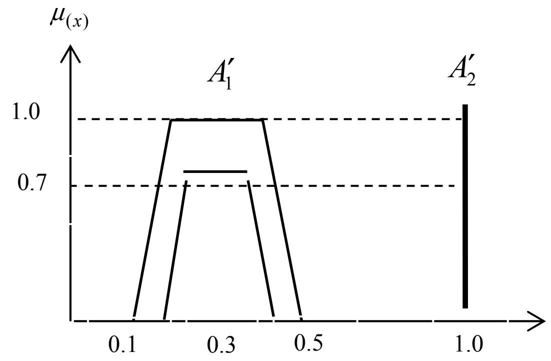
'x' denotes ranking method as unable to rank the standardised generalised interval type- 2 fuzzy numbers
'-' denotes no ranking result are obtained.
'Y' denotes the ranking result is consistent
'N' denotes the ranking result is inconsistent.

with human intuition. While for the *CPS*$_2$ ranking method, the ranking order obtained is the same as Chen and Chen (2009), Chen and Sanguatsan (2011) and Dat et al. (2012) where the ranking order is correct such that consistent with human intuitions.

Case 4: Non – overlapping standardised generalised interval type-2 fuzzy numbers of different shapes.

Consider different shape case of two non – overlapping standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} shown in Figure 6. Using the same explanation in Case 3, the correct ranking order such that the ranking result is consistent with human intuition is $A'_{i2} \succ A'_{i1}$. This is because a crisp value is always treated greater than any standardised generalised interval type-2 fuzzy numbers as it represent the actual value. Based on Table 4, only some ranking methods are capable to rank this case correctly such that the result is consistent with human intuitions. They are Chen and Chen (2009), Chen and Sanguatsan (2011), Dat et al. (2012) and the *CPS*$_2$ ranking method. While, for other remaining ranking methods under consideration, they are incapable to give any ranking order for this case. Therefore, this case indicates that the *CPS*$_2$ ranking method not only capable to give consistent ranking order towards standardised generalised interval type-2 fuzzy numbers but also to crisp value.

It is notable that each presented method of ranking standardised generalised interval type-2 fuzzy numbers has its own strengths and weaknesses. Although, all methods use for comparing standardised generalised interval type-2 fuzzy numbers in this section are actually methods for ranking type-1 fuzzy numbers, the above analysis is provided to illustrate the capability of the established ranking methods in ranking standardised generalised interval type-2 fuzzy numbers rather than ranking type-1 fuzzy numbers only. Based on the analysis provided, there are some methods deals with cases of fuzzy numbers effectively while some produce irrelevant results for certain cases. Nevertheless, in each case examined above, the *CPS*$_2$ ranking method is the most effective ranking method compared to other ranking methods under consideration where it provides correct ranking order such that the result is consistent with human intuition in all cases of standardised generalised interval type-2 fuzzy numbers considered in this study.



$A'_{i1} = [(0.1,0.2,0.4,0.5;1;1),((0.2,0.25,0.35,0.4;0.7;0.7))]$
$A'_{i2} = [(1.0,1.0,1.0,1.0;1;1),((1.0,1.0,1.0,1.0;0.7;0.7))]$

Figure 6: Standardised generalised interval type-2 fuzzy numbers A'_{i1} and A'_{i2} of Case 4.

Table 4: Comparative results for Case 4.

Methods	Type-2 fuzzy numbers		Ranking Results	Evaluation
	A'_{i1}	A'_{i2}		
2-Cheng (1998)	x	x	-	N
2-Chu and Tsao (2002)	x	x	-	N
2-Chen and Chen (2009)	0.2243	0.8500	$A'_{i1} \prec A'_{i2}$	Y
2-Kumar et al. (2010)	x	x	-	N
2-(Chen and Sanguatsan, 2011)	0.3000	1.000	$A'_{i1} \prec A'_{i2}$	N
2-Dat et al. (2012)	x	x	-	N
2-Yu et al. (2013), $\alpha = 0$	1.0000	1.0000	$A'_{i1} \prec A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 0.5$	1.0000	1.0000	$A'_{i1} \approx A'_{i2}$	N
2-Yu et al. (2013), $\alpha = 1$	1.0000	1.0000	$A'_{i1} \succ A'_{i2}$	Y
CPS_2	0.0136	0.283	$A'_{i1} \prec A'_{i2}$	Y

'x' denotes ranking method as unable to rank the standardised generalised interval type-2 fuzzy numbers
'-' denotes no ranking result are obtained.
'Y' denotes the ranking result is consistent
'N' denotes the ranking result is inconsistent.

Since, the proposed methodology have been analysed through empirical validations proposed in this study, hence this implies that the proposed methodology is relevant and reliable for solving any real decision making problems involving standardised generalised interval type-2 fuzzy numbers.

5 CONCLUSION

This study proposes a novel method for ranking standardised generalised interval type-2 fuzzy numbers which consists of centroid point and spread approaches, CPS_2. In this paper, it is shown that the CPS_2 ranking methodology is analysed and produced results that are correct such that the results are consistent with human intuition. Furthermore, the introduction of the standardised generalised interval type-2 fuzzy numbers in replacing conventional interval type-2 fuzzy numbers improves the capability of interval type-2 fuzzy numbers when being applied to decision making problems. In conclusion, the proposed method possesses intuitional concepts for ranking standardised generalised interval type-2 fuzzy numbers as well as for decision making analysis. Therefore, it is expected that this method can be further improved for decision making purposes.

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