Duality in Some Intuitionistic Paraconsistent Logics

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Keywords: Connexive Logic, Nelson's Paraconsistent Logic, Duality, Sequent Calculus, Cut-elimination Theorem, Embedding Theorem.

Abstract: Duality in constructive (or intuitionistic) logics is an important basic property since the dual counterpart of a given constructive logic can obtain a refutation or falsification of the information or knowledge which is described by the given logic. In this paper, duality in some intuitinistic paraconsistent logics is investigated. A constructive connexive logic (connexive logic for short) and Nelson's paraconsistent four-valued logic are addressed as an example of such intuitionistic paraconsistent logics. A new logic called dual connexive logic (dCN), which is the dual counterpart of the connexive logic (CN), is introduced as a Gentzen-type sequent calculus. Some theorems for embedding dCN into CN and vice versa, which represent the duality between them, are shown. Similar embedding results cannot be shown for Nelson's paraconsistent four-valued logic. But, similar embedding results can be shown for an extended Nelson logic with co-implication.

1 INTRODUCTION

Inconsistency-tolerant reasoning is of growing importance in AI since inconsistencies often appear and are inevitable in large and open intelligent systems. *Paraconsistent logics* are known to be useful for appropriately formalizing such inconsistency-tolerant reasoning (Priest, 2002; Kamide, 2014; Kamide, 2015a; Kamide, 2015b; Kamide and Koizumi, 2015). On the other hand, duality in constructive (or intuitionistic) logics is an important basic property since the dual counterpart of a given constructive logic can obtain a refutation or falsification of the information or knowledge which is described by the given logic. Finding or constructing the dual counterparts of some useful constructive logics is thus regarded as an important issue. But, the dual counterparts of some useful *intuitionistic paraconsistent logics* have not yet been found although the dual counterparts of some versions of intuitionistic logic have been found as *dual-intuitionistic logics* (Czermak, 1977; Goodman, 1981; Urbas, 1996). Thus, in this paper, duality in some intuitinistic paraconsistent logics is investigated. A *constructive connexive logic* (Wansing, 2005) and *Nelson's paraconsistent four-valued logic* (Almukdad and Nelson, 1984; Nelson, 1949) are addressed as an example of such intuitionistic paraconsistent logics.

Connexive logics are known to be a philosophically plausible paraconsistent logic (Angell, 1962; McCall, 1966; Wansing, 2005; Kamide and Wansing, 2011a; Wansing, 2014). A distinctive feature of connexive logics is that they validate the so-called *Boethius' theses*:

1. $(\alpha \to \beta) \to \sim(\alpha \to \sim\beta)$,
2. $(\alpha \to \sim\beta) \to \sim(\alpha \to \beta)$.

A *constructive connexive modal logic*, which is a constructive connexive analogue of the smallest normal modal logic K, was introduced in (Wansing, 2005) by extending a certain basic constructive connexive logic, which is a variant of Nelson's paraconsistent four-valued logic (Almukdad and Nelson, 1984; Nelson, 1949). A *classical connexive modal logic* called CS4, which is based on the positive normal modal logic S4, was introduced and studied in (Kamide and Wansing, 2011a) as a Gentzen-type sequent calculus. A survey on connexive logics can be found in (Wansing, 2014).

Nelson's paraconsistent four-valued logic, N4, is known to be an extension of the so called *useful four-valued logic* by Belnap (Belnap, 1977) and Dunn (Dunn, 1976), which has a wide range of applications to Computer Science. The logic N4 was originally motivated by the idea of defining a logic "in which falsity is conceived in a fashion analogous to that for intuitionistic truth" (Almukdad and Nelson, 1984). Moreover, a number of applications in Computer Sci-

Kamide, N.
Duality in Some Intuitionistic Paraconsistent Logics.
DOI: 10.5220/0005684202880297
In *Proceedings of the 8th International Conference on Agents and Artificial Intelligence (ICAART 2016) - Volume 2*, pages 288-297
ISBN: 978-989-758-172-4

ence, such as logic programming based on N4, have been proposed by several researchers. More information on Nelson's N4 can be found, for example, in (Wagner, 1991; Wansing, 1993; Kamide and Wansing, 2011b; Kamide and Wansing, 2012; Kamide and Wansing, 2015).

In this paper, the constructive connexive logic (connexive logic for short) and Nelson's N4 are studied from the point of view of duality. The notion of duality in constructive logics comes from the idea of dual-intuitionistic logics (Czermak, 1977; Goodman, 1981; Urbas, 1996). This notion can be considered as follows. For example, the duality between *positive intuitionistic logic* and the *positive dual-intuitionistic logic* holds, i.e., the former logic is embeddable into the latter logic and vice versa, by translating the logical connectives in the underlying logic into their dual connectives. The duality cannot be shown for N4, i.e., there is no dual counterpart of it. In this paper, the dual counterpart of the connexive logic, called *dual connexive logic*, and the dual counterpart of an extended Nelson logic with co-implication, called *dual Nelson logic with co-implication*, are constructed as a Gentzen-type sequent calculus.

Dual-intuitionistic logics are logics which have a Gentzen-type sequent calculus in which sequents have the restriction that the antecedent contains at most one formula (Czermak, 1977; Goodman, 1981; Urbas, 1996; Goré, 2000; Shramko, 2005). This restriction of being singular in the antecedent is dual to that in a Gentzen-type sequent calculus LJ for intuitionistic logic, which is singular in the consequent. Historically speaking, the logics containing Czermak's *dual-intuitionistic calculus* (Czermak, 1977), Goodman's *logic of contradiction* or *anti-intuitionistic logic* (Goodman, 1981), and Urbas's extensions of Czermak's and Goodman's logics (Urbas, 1996) were collectively referred to by Urbas as dual-intuitionistic logics.

The contents of this paper are then summarized as follows.

In Section 2, the duality in the connexive logic (CN) is investigated. Firstly, the logic CN is introduced as a Gentzen-type sequent calculus. Secondly, the dual connexive logic (dCN) is introduced as a Gentzen-type sequent calculus, and some theorems for embedding dCN into the positive dual-intuitionistic logic (DJ) are shown. Thirdly, some theorems for embedding dCN into CN and vice versa, which represent the duality between them, are shown. These embedding theorems for dCN and CN mean that dCN is indeed the dual counterpart of CN. By using these theorems, the cut-elimination and decidability theorems for dCN can be shown.

In Section 3, the duality in an extended Nelson logic with co-implication is investigated in a similar way as in Section 2. Firstly, the extended Nelson logic with co-implication (N4C) is introduced as a Gentzen-type sequent calculus. The co-implication connective in N4C is needed for showing the duality. Secondly, the dual Nelson logic with co-implication (dN4C) is introduced as a Gentzen-type sequent calculus. Some similar theorems for embedding dN4C into the positive dual-intuitionistic logic with co-implication can be shown. Thirdly, some theorems for embedding dN4C into N4C and vice versa, which represent the duality between them, are shown. By using these theorems, the cut-elimination and decidability theorems for dN4C can be shown.

In Section 4, this paper is concluded, and some remarks are addressed.

2 DUALITY IN CONNEXIVE LOGIC

2.1 Connexive Logic

The language of *(constructive) connexive logic* consists of logical connectives $\wedge_t$ (conjunction), $\vee_t$ (disjunction), $\rightarrow_t$ (implication) and $\sim_t$ (paraconsistent negation). An expression $\alpha \leftrightarrow \beta$ is used to denote $(\alpha{\rightarrow_t}\beta)\wedge(\beta{\rightarrow_t}\alpha)$. Lower case letters $p, q, ...$ are used for propositional variables, lower case Greek letters $\alpha, \beta, ...$ are used for formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used for finite (possibly empty) sequences of formulas. These letters are also used for other logics discussed in this paper. A *positive sequent* is an expression of the form $\Gamma \Rightarrow \gamma$ where γ denotes a single formula or the empty sequence. A *negative sequent* will also be defined.

An expression $L \vdash S$ is used to denote the fact that a positive or negative sequent S is provable in a sequent calculus L. An expression of the form $\alpha \Leftrightarrow \beta$ is used to represent both $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. A rule R of inference is said to be *admissible* in a sequent calculus L if the following condition is satisfied: for any instance

$$\frac{S_1 \ \cdots \ S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$. Since all logics discussed in this paper are formulated as sequent calculi, we will frequently identify a sequent calculus with the logic determined by it.

A Gentzen-type sequent calculus CN for connexive logic is defined as follows based on positive sequents.

Definition 2.1 (CN)**.** *The initial sequents of* CN *are of the following form, for any propositional variable p:*

$$p \Rightarrow p \qquad\qquad {\sim_t} p \Rightarrow {\sim_t} p.$$

The structural rules of CN *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma}\ (\text{t-cut})$$

$$\frac{\Delta, \beta, \alpha, \Gamma \Rightarrow \gamma}{\Delta, \alpha, \beta, \Gamma \Rightarrow \gamma}\ (\text{t-ex-l}) \qquad \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{t-co-l})$$

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma}\ (\text{t-we-l}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{t-we-r}).$$

The positive logical inference rules of CN *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge_t \beta, \Gamma \Rightarrow \gamma}\ (\wedge_t\text{l1}) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge_t \beta, \Gamma \Rightarrow \gamma}\ (\wedge_t\text{l2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge_t \beta}\ (\wedge_t\text{r}) \qquad \frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee_t \beta, \Gamma \Rightarrow \gamma}\ (\vee_t\text{l})$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee_t \beta}\ (\vee_t\text{r1}) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee_t \beta}\ (\vee_t\text{r2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \rightarrow_t \beta, \Gamma, \Delta \Rightarrow \gamma}\ (\rightarrow_t\text{l}) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow_t \beta}\ (\rightarrow_t\text{r}).$$

The negative logical inference rules of CN *are of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{{\sim_t}{\sim_t}\alpha, \Gamma \Rightarrow \gamma}\ ({\sim_t}\text{l}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow {\sim_t}{\sim_t}\alpha}\ ({\sim_t}\text{r})$$

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \gamma \quad {\sim_t}\beta, \Gamma \Rightarrow \gamma}{{\sim_t}(\alpha \wedge_t \beta), \Gamma \Rightarrow \gamma}\ ({\sim_t}\wedge_t\text{l})$$

$$\frac{\Gamma \Rightarrow {\sim_t}\alpha}{\Gamma \Rightarrow {\sim_t}(\alpha \wedge_t \beta)}\ ({\sim_t}\wedge_t\text{r1})$$

$$\frac{\Gamma \Rightarrow {\sim_t}\beta}{\Gamma \Rightarrow {\sim_t}(\alpha \wedge_t \beta)}\ ({\sim_t}\wedge_t\text{r2})$$

$$\frac{{\sim_t}\alpha, \Gamma \Rightarrow \gamma}{{\sim_t}(\alpha \vee_t \beta), \Gamma \Rightarrow \gamma}\ ({\sim_t}\vee_t\text{l1})$$

$$\frac{{\sim_t}\beta, \Gamma \Rightarrow \gamma}{{\sim_t}(\alpha \vee_t \beta), \Gamma \Rightarrow \gamma}\ ({\sim_t}\vee_t\text{l2})$$

$$\frac{\Gamma \Rightarrow {\sim_t}\alpha \quad \Gamma \Rightarrow {\sim_t}\beta}{\Gamma \Rightarrow {\sim_t}(\alpha \vee_t \beta)}\ ({\sim_t}\vee_t\text{r})$$

$$\frac{\Gamma \Rightarrow \alpha \quad {\sim_t}\beta, \Delta \Rightarrow \gamma}{{\sim_t}(\alpha \rightarrow_t \beta), \Gamma, \Delta \Rightarrow \gamma}\ ({\sim_t}{\rightarrow_t}\text{l})$$

$$\frac{\alpha, \Gamma \Rightarrow {\sim_t}\beta}{\Gamma \Rightarrow {\sim_t}(\alpha \rightarrow_t \beta)}\ ({\sim_t}{\rightarrow_t}\text{r}).$$

Some remarks are given as follows.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in CN. This fact can be shown by induction on α.
2. The negative logical inference rules for $\rightarrow_t$ in CN just correspond to the axiom scheme: ${\sim_t}(\alpha \rightarrow_t \beta) \leftrightarrow (\alpha \rightarrow_t {\sim_t}\beta)$.
3. A sequent calculus N4 for Nelson's paraconsistent four-valued logic (Almukdad and Nelson, 1984; Nelson, 1949) is obtained from CN by replacing $\{({\sim_t}{\rightarrow_t}\text{l}), ({\sim_t}{\rightarrow_t}\text{r})\}$ with the negative inference rules of the form:

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{{\sim_t}(\alpha \rightarrow_t \beta), \Gamma \Rightarrow \gamma} \qquad \frac{{\sim_t}\beta, \Gamma \Rightarrow \gamma}{{\sim_t}(\alpha \rightarrow_t \beta), \Gamma \Rightarrow \gamma}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow {\sim_t}\beta}{\Gamma \Rightarrow {\sim_t}(\alpha \rightarrow_t \beta)}$$

which correspond to the axiom scheme: ${\sim_t}(\alpha \rightarrow_t \beta) \leftrightarrow (\alpha \wedge_t {\sim_t}\beta)$.

Definition 2.2 (LJ)**.** *A sequent calculus* LJ *for positive intuitionistic logic is defined as the ${\sim_t}$-free fragment of* CN*, i.e., it is obtained from* CN *by deleting the negative initial sequents ${\sim_t} p \Rightarrow {\sim_t} p$ and the negative logical inference rules concerning ${\sim_t}$.*

2.2 Dual Connexive Logic

The language of *dual connexive logic* consists of logical connectives $\wedge_f$ (dual-conjunction), $\vee_f$ (dual-disjunction), $\leftarrow_f$ (dual-co-implication) and ${\sim_f}$ (dual-paraconsistent negation). A *negative sequent* is an expression of the form $\gamma \Rightarrow \Gamma$ where γ denotes a single formula or the empty sequence.

A Gentzen-type sequent calculus dCN for the dual connexive logic is defined as follows based on negative sequents.

Definition 2.3 (dCN)**.** *The initial sequents of* dCN *are of the following form, for any propositional variable p:*

$$p \Rightarrow p \qquad\qquad {\sim_f} p \Rightarrow {\sim_f} p.$$

The structural rules of dCN *are of the form:*

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \alpha \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta}\ (\text{f-cut})$$

$$\frac{\gamma \Rightarrow \Delta, \beta, \alpha, \Gamma}{\gamma \Rightarrow \Delta, \alpha, \beta, \Gamma}\ (\text{f-ex-r}) \qquad \frac{\gamma \Rightarrow \Gamma, \alpha, \alpha}{\gamma \Rightarrow \Gamma, \alpha}\ (\text{f-co-r})$$

$$\frac{\gamma \Rightarrow \Gamma}{\gamma \Rightarrow \Gamma, \alpha}\ (\text{f-we-r}) \qquad \frac{\Rightarrow \Gamma}{\alpha \Rightarrow \Gamma}\ (\text{f-we-l}).$$

The positive logical inference rules of dCN *are of the form:*

$$\frac{\alpha \Rightarrow \Gamma}{\alpha \wedge_f \beta \Rightarrow \Gamma}\ (\wedge_f\text{l1}) \qquad \frac{\beta \Rightarrow \Gamma}{\alpha \wedge_f \beta \Rightarrow \Gamma}\ (\wedge_f\text{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \alpha \wedge_f \beta}\ (\wedge_f\text{r}) \qquad \frac{\alpha \Rightarrow \Gamma \quad \beta \Rightarrow \Gamma}{\alpha \vee_f \beta \Rightarrow \Gamma}\ (\vee_f\text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \alpha \vee_f \beta}\ (\vee_f\text{r1}) \qquad \frac{\gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \alpha \vee_f \beta}\ (\vee_f\text{r2})$$

$$\frac{\alpha \Rightarrow \Gamma, \beta}{\alpha \leftarrow_f \beta \Rightarrow \Gamma}\ (\leftarrow_f\text{l}) \qquad \frac{\gamma \Rightarrow \Gamma, \alpha \quad \beta \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta, \alpha \leftarrow_f \beta}\ (\leftarrow_f\text{r}).$$

The negative logical inference rules of dCN *are of the form:*

$$\frac{\alpha \Rightarrow \Gamma}{\sim_f \sim_f \alpha \Rightarrow \Gamma}\ (\sim_f\text{l}) \qquad \frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \sim_f \sim_f \alpha}\ (\sim_f\text{r})$$

$$\frac{\sim_f \alpha \Rightarrow \Gamma \quad \sim_f \beta \Rightarrow \Gamma}{\sim_f(\alpha \wedge_f \beta) \Rightarrow \Gamma}\ (\sim_f \wedge_f\text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \wedge_f \beta)}\ (\sim_f \wedge_f\text{r1})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \wedge_f \beta)}\ (\sim_f \wedge_f\text{r2})$$

$$\frac{\sim_f \alpha \Rightarrow \Gamma}{\sim_f(\alpha \vee_f \beta) \Rightarrow \Gamma}\ (\sim_f \vee_f\text{l1})$$

$$\frac{\sim_f \beta \Rightarrow \Gamma}{\sim_f(\alpha \vee_f \beta) \Rightarrow \Gamma}\ (\sim_f \vee_f\text{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \gamma \Rightarrow \Gamma, \sim_f \beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \vee_f \beta)}\ (\sim_f \vee_f\text{r})$$

$$\frac{\sim_f \alpha \Rightarrow \Gamma, \beta}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma}\ (\sim_f \leftarrow_f\text{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \beta \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f \leftarrow_f\text{r}).$$

Some remarks are given as follows.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in dCN. This fact can be shown by induction on α.
2. The negative logical inference rules for $\leftarrow_f$ in dCN just correspond to the axiom scheme: $\sim_f(\alpha \leftarrow_f \beta) \leftrightarrow (\sim_f \alpha \leftarrow_f \beta)$.
3. A sequent calculus dN4 for a "dual-like" version of N4 can be obtained from dCN by replacing $\{(\sim_f \leftarrow_f\text{l}), (\sim_f \leftarrow_f\text{r})\}$ with the negative inference rules of the form:
$$\frac{\sim_f \alpha \Rightarrow \Gamma}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma} \qquad \frac{\beta \Rightarrow \Gamma}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma}$$
$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \leftarrow_f \beta)}$$
which correspond to the axiom scheme: $\sim_f(\alpha \leftarrow_f \beta) \leftrightarrow (\sim_f \alpha \wedge_f \beta)$.
4. A theorem for embedding dN4 into N4 dose not hold. Thus, dN4 is not regarded as a dual of N4.
5. To obtain the duality, an extended N4 with co-implication is needed. See Section 3.

Definition 2.4 (DJ)**.** *A sequent calculus* DJ *for positive dual intuitionistic logic is defined as the* $\sim_f$*-free fragment of* dCN*, i.e., it is obtained from* dCN *by deleting the negative initial sequents* $\sim_f p \Rightarrow \sim_f p$ *and the negative logical inference rules concerning* $\sim_f$.

The following result is known (Urbas, 1996).

Proposition 2.5 (Cut-elimination for DJ)**.** *The rule* (f-cut) *is admissible in cut-free* DJ.

In the following, we introduce a translation of dCN into DJ, and by using this translation, we show some theorems for embedding dCN into DJ. A similar translation has been used by Gurevich (Gurevich, 1977), Rautenberg (Rautenberg, 1979) and Vorob'ev (Vorob'ev, 1952) to embed Nelson's constructive logic (Almukdad and Nelson, 1984; Nelson, 1949) into intuitionistic logic.

Definition 2.6. *We fix a set* Φ *of propositional variables and define the set* $\Phi' := \{p' \mid p \in \Phi\}$ *of propositional variables. The language* $\mathcal{L}_{\text{dCN}}$ *of* dCN *is defined using* Φ, $\wedge_f, \vee_f, \leftarrow_f$ *and* $\sim_f$*. The language* $\mathcal{L}_{\text{DJ}}$ *of* DJ *is obtained from* $\mathcal{L}_{\text{dCN}}$ *by adding* Φ' *and deleting* $\sim_f$.

A mapping f *from* $\mathcal{L}_{\text{dCN}}$ *to* $\mathcal{L}_{\text{DJ}}$ *is defined inductively by:*

1. *for any* $p \in \Phi$, $f(p) := p$ *and* $f(\sim_f p) := p' \in \Phi'$,
2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ *where* $\circ \in \{\wedge_f, \vee_f, \leftarrow_f\}$,
3. $f(\sim_f \sim_f \alpha) := f(\alpha)$,
4. $f(\sim_f(\alpha \wedge_f \beta)) := f(\sim_f \alpha) \vee_f f(\sim_f \beta)$,
5. $f(\sim_f(\alpha \vee_f \beta)) := f(\sim_f \alpha) \wedge_f f(\sim_f \beta)$,
6. $f(\sim_f(\alpha \leftarrow_f \beta)) := f(\sim_f \alpha) \leftarrow_f f(\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$. The same notation is used for other mappings discussed later.

We then obtain a weak theorem for embedding dCN into DJ.

Theorem 2.7 (Weak embedding from dCN into DJ)**.** *Let* Γ *be a set of formulas in* $\mathcal{L}_{\text{dCN}}$*,* γ *be a formula in* $\mathcal{L}_{\text{dCN}}$ *or the empty sequence, and* f *be the mapping defined in Definition 2.6.*

1. *If* dCN $\vdash \gamma \Rightarrow \Gamma$*, then* DJ $\vdash f(\gamma) \Rightarrow f(\Gamma)$.
2. *If* DJ $-$ (f-cut) $\vdash f(\gamma) \Rightarrow f(\Gamma)$*, then* dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$.

Proof. (1): By induction on the proofs P of $\gamma \Rightarrow \Gamma$ in dCN. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\sim_f p \Rightarrow \sim_f p$): The last inference of P is of the form: $\sim_f p \Rightarrow \sim_f p$ for any $p \in \Phi$. In this case, we obtain DJ $\vdash f(\sim_f p) \Rightarrow f(\sim_f p)$, i.e., DJ $\vdash p' \Rightarrow p'$ ($p' \in \Phi'$), by the definition of f.

2. Case ($\sim_f \leftarrow_f$r): The last inference of P is of the form:

$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \beta \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta, \sim_f(\alpha \leftarrow_f \beta)}\ (\sim_f \leftarrow_f \mathrm{r}).$$

By induction hypothesis, we have DJ $\vdash f(\gamma) \Rightarrow f(\Gamma), f(|NF\alpha)$ and DJ $\vdash f(\beta) \Rightarrow f(\Delta)$. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ f(\gamma) \Rightarrow f(\Gamma), f(\sim_f \alpha)\end{matrix} \quad \begin{matrix}\vdots \\ f(\beta) \Rightarrow f(\Delta)\end{matrix}}{f(\gamma) \Rightarrow f(\Gamma), f(\Delta), f(\sim_f \alpha) \leftarrow_f f(\beta)}\ (\leftarrow_f \mathrm{r})$$

where $f(\sim_f \alpha) \leftarrow_f f(\beta)$ coincides with $f(\sim_f(\alpha \leftarrow_f \beta))$ by the definition of f.

3. Case ($\sim_f \leftarrow_f$l): The last inference of P is of the form:

$$\frac{\sim_f \alpha \Rightarrow \Gamma, \beta}{\sim_f(\alpha \leftarrow_f \beta) \Rightarrow \Gamma}\ (\sim_f \leftarrow_f \mathrm{l}).$$

By induction hypothesis, we have DJ $\vdash f(\sim_f \alpha) \Rightarrow f(\Gamma), f(\beta)$. Then, we obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ f(\sim_f \alpha) \Rightarrow f(\Gamma), f(\beta)\end{matrix}}{f(\sim_f \alpha) \leftarrow_f f(\beta) \Rightarrow f(\Gamma)}\ (\leftarrow_f \mathrm{l})$$

where $f(\sim_f \alpha) \leftarrow_f f(\beta)$ coincides with $f(\sim_f(\alpha \leftarrow_f \beta))$ by the definition of f.

(2): By induction on the proofs Q of $f(\gamma) \Rightarrow f(\Gamma)$ in DJ $-$ (f-cut). We distinguish the cases according to the last inference of Q, and show only the following cases.

1. Case ($\leftarrow_f$r): The last inference of Q is of the form:

$$\frac{f(\gamma) \Rightarrow f(\Gamma), f(\alpha) \quad f(\beta) \Rightarrow f(\Delta)}{f(\gamma) \Rightarrow f(\Gamma), f(\Delta), f(\alpha \leftarrow_f \beta)}\ (\leftarrow_f \mathrm{r})$$

where $f(\alpha \leftarrow_f \beta)$ coincides with $f(\alpha) \leftarrow_f f(\beta)$ by the definition of f. By induction hypothesis, we have dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma, \alpha$ and dCN $-$ (f-cut) $\vdash \beta \Rightarrow \Delta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \gamma \Rightarrow \Gamma, \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \beta \Rightarrow \Delta\end{matrix}}{\gamma \Rightarrow \Gamma, \Delta, \alpha \leftarrow_f \beta}\ (\leftarrow_f \mathrm{r}).$$

2. Case ($\wedge_f$r): The last inference of Q is ($\wedge_f$r).

(a) Subcase (1): The last inference of Q is of the form:

$$\frac{f(\gamma) \Rightarrow f(\Gamma), f(\alpha) \quad f(\gamma) \Rightarrow f(\Gamma), f(\beta)}{f(\gamma) \Rightarrow f(\Gamma), f(\alpha \wedge_f \beta)}\ (\wedge_f \mathrm{r})$$

where $f(\alpha \wedge_f \beta)$ coincides with $f(\alpha) \wedge_f f(\beta)$ by the definition of f. By induction hypothesis, we have dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma, \alpha$ and dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma, \beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \gamma \Rightarrow \Gamma, \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \gamma \Rightarrow \Gamma, \beta\end{matrix}}{\gamma \Rightarrow \Gamma, \alpha \wedge_f \beta}\ (\wedge_f \mathrm{r}).$$

(b) Subcase (2): The last inference of Q is of the form:

$$\frac{f(\gamma) \Rightarrow f(\Gamma), f(\sim_f \alpha) \quad f(\gamma) \Rightarrow f(\Gamma), f(\sim_f \beta)}{f(\gamma) \Rightarrow f(\Gamma), f(\sim_f(\alpha \vee_f \beta))}\ (\wedge_f \mathrm{r})$$

where $f(\sim_f(\alpha \vee_f \beta))$ coincides with $f(\alpha) \wedge_f f(\sim_f \beta)$ by the definition of f. By induction hypothesis, we have dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma, \sim_f \alpha$ and dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma, \sim_f \beta$. We thus obtain the required fact:

$$\frac{\begin{matrix}\vdots \\ \gamma \Rightarrow \Gamma, \sim_f \alpha\end{matrix} \quad \begin{matrix}\vdots \\ \gamma \Rightarrow \Gamma, \sim_f \beta\end{matrix}}{\gamma \Rightarrow \Gamma, \sim_f(\alpha \vee_f \beta)}\ (\sim_f \vee_f \mathrm{r}).$$

∎

Using Theorem 2.7 and the cut-elimination theorem for DJ, we obtain the following cut-elimination theorem for dCN.

Theorem 2.8 (Cut-elimination for dCN)**.** *The rule* (f-cut) *is admissible in cut-free* dCN.

Proof. Suppose dCN $\vdash \gamma \Rightarrow \Gamma$. Then, we have DJ $\vdash f(\gamma) \Rightarrow f(\Gamma)$ by Theorem 2.7 (1), and hence DJ $-$ (f-cut) $\vdash f(\gamma) \Rightarrow f(\Gamma)$ by the cut-elimination theorem for DJ. By Theorem 2.7 (2), we obtain dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$. ∎

Using Theorem 2.7 and the cut-elimination theorem for DJ, we obtain the following strong theorem for embedding dCN into DJ.

Theorem 2.9 (Strong embedding from dCN into DJ)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{dCN}}$, γ be a formula in $\mathcal{L}_{\mathrm{dCN}}$ or the empty sequence, and f be the mapping defined in Definition 2.6.*

1. dCN $\vdash \gamma \Rightarrow \Gamma$ *iff* DJ $\vdash f(\gamma) \Rightarrow f(\Gamma)$.

2. dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$ *iff* DJ $-$ (f-cut) $\vdash f(\gamma) \Rightarrow f(\Gamma)$.

Proof. (1): ($\Longrightarrow$): By Theorem 2.7 (1). ($\Longleftarrow$): Suppose DJ $\vdash f(\gamma) \Rightarrow f(\Gamma)$. Then we have DJ $-$ (f-cut) $\vdash f(\gamma) \Rightarrow f(\Gamma)$ by the cut-elimination theorem for DJ. We thus obtain dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$ by Theorem 2.7 (2). Therefore we have dCN $\vdash \gamma \Rightarrow \Gamma$.

(2): ($\Longrightarrow$): Suppose dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$. Then we have dCN $\vdash \gamma \Rightarrow \Gamma$. We then obtain DJ $\vdash f(\gamma) \Rightarrow f(\Gamma)$ by Theorem 2.7 (1). Therefore we obtain DJ $-$ (f-cut) $\vdash f(\gamma) \Rightarrow f(\Gamma)$ by the cut-elimination theorem for DJ. ($\Longleftarrow$): By Theorem 2.7 (2). ∎

Using Theorem 2.9, we can also obtain the decidability of dCN.

Theorem 2.10 (Decidability of dCN)**.** dCN *is decidable.*

Proof. By decidability of DJ, for each α, it is possible to decide if $f(\alpha)$ is provable in dCN. Then, by Theorem 2.9, dCN is also decidable. ∎

Some remarks are given as follow.

1. Similar theorems for embedding CN into LJ hold. Namely, we can define a translation from CN into LJ, and by using this translation, we can show similar theorems as Theorems 2.7 and 2.9.
2. By using these embedding theorems for CN, we can also show the cut-elimination and decidability theorems for CN.

2.3 Duality

Next, we introduce a translation from dCN into CN. The idea of the translation comes from (Czermak, 1977; Urbas, 1996).

Definition 2.11. *We fix a common set Φ of propositional variables. The language $\mathcal{L}_{\mathrm{dCN}}$ of* dCN *is defined using Φ, $\wedge_f, \vee_f, \leftarrow_f$ and $\sim_f$. The language $\mathcal{L}_{\mathrm{CN}}$ of* CN *is defined using Φ, $\wedge_t, \vee_t, \rightarrow_t$ and $\sim_t$.*

A mapping h from $\mathcal{L}_{\mathrm{dCN}}$ to $\mathcal{L}_{\mathrm{CN}}$ is defined inductively by:

1. $h(p) := p$ *for any* $p \in \Phi$,
2. $h(\alpha \wedge_f \beta) := h(\alpha) \vee_t h(\beta)$,
3. $h(\alpha \vee_f \beta) := h(\alpha) \wedge_t h(\beta)$,
4. $h(\alpha \leftarrow_f \beta) := h(\beta) \rightarrow_t h(\alpha)$,
5. $h(\sim_f \alpha) := \sim_t h(\alpha)$.

We then obtain a strong theorem for embedding dCN into CN.

Theorem 2.12 (Strong embedding from dCN into CN)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{dCN}}$, γ be a formula in $\mathcal{L}_{\mathrm{dCN}}$ or the empty sequence, and h be the mapping defined in Definition 2.11.*

1. dCN $\vdash \gamma \Rightarrow \Gamma$ *iff* CN $\vdash h(\Gamma) \Rightarrow h(\gamma)$.
2. dCN $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$ *iff* CN $-$ (t-cut) $\vdash h(\Gamma) \Rightarrow h(\gamma)$.

Proof. We show only (1) since (2) can be obtained as a subproof of (1). We show only the direction ($\Longrightarrow$) of (1) by induction on the proof P of $\gamma \Rightarrow \Gamma$ in dCN. We distinguish the cases according to the last inference of P, and show some cases.

1. Case ($\sim_f \leftarrow_f$l): The last inference of P is of the form:
$$\frac{\sim_f \alpha \Rightarrow \Gamma, \beta}{\sim_f (\alpha \leftarrow_f \beta) \Rightarrow \Gamma}\ (\sim_f \leftarrow_f \mathrm{l}).$$
By induction hypothesis, we have SI $\vdash h(\Gamma), h(\beta) \Rightarrow h(\sim_f \alpha)$ where $h(\sim_f \alpha)$ coincides with $\sim_t h(\alpha)$ by the definition of h. Then, we obtain the required fact:
$$\dfrac{\begin{array}{c}\vdots \\ h(\Gamma), h(\beta) \Rightarrow \sim_t h(\alpha) \\ \vdots\ (\text{t-ex-l}) \\ h(\beta), h(\Gamma) \Rightarrow \sim_t h(\alpha)\end{array}}{h(\Gamma) \Rightarrow \sim_t (h(\beta) \rightarrow_t h(\alpha))}\ (\sim_t \rightarrow_t \mathrm{r})$$
where $\sim_t (h(\beta) \rightarrow_t h(\alpha))$ coincides with $h(\sim_f (\alpha \leftarrow_f \beta))$ by the definition of h.
2. Case ($\sim_f \leftarrow_f$r): The last inference of P is of the form:
$$\frac{\gamma \Rightarrow \Gamma, \sim_f \alpha \quad \beta \Rightarrow \Delta}{\gamma \Rightarrow \Gamma, \Delta, \sim_f (\alpha \leftarrow_f \beta)}\ (\sim_f \leftarrow_f \mathrm{r}).$$
By induction hypothesis, we have SI $\vdash h(\Gamma), h(\sim_f \alpha) \Rightarrow h(\gamma)$ and SI $\vdash h(\Delta) \Rightarrow h(\beta)$ where $h(\sim_f \alpha)$ coincides with $\sim_t h(\alpha)$ by the definition of h. Then, we obtain the required fact:
$$\dfrac{\begin{array}{c}\vdots \\ h(\Gamma) \Rightarrow h(\beta)\end{array} \quad \begin{array}{c}\vdots \\ h(\Delta), \sim_t h(\alpha) \Rightarrow h(\gamma) \\ \vdots\ (\text{t-ex-l}) \\ \sim_t h(\alpha), h(\Delta) \Rightarrow h(\gamma)\end{array}}{\sim_t (h(\beta) \rightarrow_t h(\alpha)), h(\Gamma), h(\Delta) \Rightarrow h(\gamma)}\ (\sim_t \rightarrow_t \mathrm{l})$$
where $\sim_t (h(\beta) \rightarrow_t h(\alpha))$ coincides with $h(\sim_f (\alpha \leftarrow_f \beta))$ by the definition of h.

∎

We can introduce a translation from CN into dCN in a similar way.

Definition 2.13. Φ, *$\mathcal{L}_{\mathrm{dCN}}$ and $\mathcal{L}_{\mathrm{CN}}$ are the same as in Definition 2.11.*

A mapping k from $\mathcal{L}_{\mathrm{CN}}$ to $\mathcal{L}_{\mathrm{dCN}}$ is defined inductively as follows.

1. $k(p) := p$ *for any* $p \in \Phi$,
2. $k(\alpha \wedge_t \beta) := k(\alpha) \vee_f k(\beta)$,
3. $k(\alpha \vee_t \beta) := k(\alpha) \wedge_f k(\beta)$,

4. $k(\alpha \rightarrow_t \beta) := k(\beta) \leftarrow_f k(\alpha)$,

5. $k(\sim_t \alpha) := \sim_f k(\alpha)$.

We can obtain a strong theorem for embedding CN into dCN.

Theorem 2.14 (Strong embedding from CN into dCN)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\text{CN}}$, γ be a formula in $\mathcal{L}_{\text{CN}}$ or the empty sequence, and k be the mapping defined in Definition 2.13.*

1. CN $\vdash \Gamma \Rightarrow \gamma$ *iff* dCN $\vdash k(\gamma) \Rightarrow k(\Gamma)$.

2. CN − (t-cut) $\vdash \Gamma \Rightarrow \gamma$ *iff* dCN − (f-cut) $\vdash k(\gamma) \Rightarrow k(\Gamma)$.

Proof. Similar to Theorem 2.12. ∎

Some remarks are given as follows.

1. Theorems 2.12 and 2.14 represent the "duality" between CN and dCN.
2. It is well-known that similar theorems as Theorems 2.14 and 2.12 hold for LJ and DJ, i.e., the duality between LJ and DJ holds.
3. Some similar theorems for embedding dN4 into N4 and vice versa cannot be shown in a similar way. Thus, there is no duality between N4 and dN4, i.e., dN4 is not regarded as a dual of N4.
4. The cut-elimination theorems for CN and dCN can be obtained using Theorems 2.12 and 2.14.
5. The decidability of dCN can be obtained using Theorem 2.12.
6. The following hold for CN and dCN:
   (a) CN $\vdash hk(\Gamma) \Rightarrow hk(\gamma)$ iff CN $\vdash \Gamma \Rightarrow \gamma$,
   (b) dCN $\vdash kh(\gamma) \Rightarrow kh(\Gamma)$ iff dCN $\vdash \gamma \Rightarrow \Gamma$.

3 DUALITY IN NELSON LOGIC

3.1 Nelson Logic with Co-implication

The language of *Nelson's paraconsistent four-valued logic with co-implication* is obtained from that of connexive logic by adding $\leftarrow_t$ (co-implication).

A Gentzen-type sequent calculus N4C for Nelson's paraconsistent logic with co-implication is defined as follows based on positive sequents.

Definition 3.1 (N4C)**.** N4C *is obtained from* CN *by deleting the negative logical inference rules* $\{(\sim_t \rightarrow_t \text{l}), (\sim_t \rightarrow_t \text{r})\}$ *and adding the positive and negative logical inference rules of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \leftarrow_t \beta, \Gamma \Rightarrow \gamma}\ (\text{n}{\leftarrow_t}\text{l1}) \qquad \frac{\Gamma \Rightarrow \beta}{\alpha \leftarrow_t \beta, \Gamma \Rightarrow}\ (\text{n}{\leftarrow_t}\text{l2})$$

$$\frac{\Delta \Rightarrow \alpha \quad \beta, \Gamma \Rightarrow}{\Gamma, \Delta \Rightarrow \alpha \leftarrow_t \beta}\ (\text{n}{\leftarrow_t}\text{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\sim_t(\alpha \rightarrow_t \beta), \Gamma \Rightarrow \gamma}\ (\text{n}{\sim_t}{\rightarrow_t}\text{l1})$$

$$\frac{\sim_t \beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha \rightarrow_t \beta), \Gamma \Rightarrow \gamma}\ (\text{n}{\sim_t}{\rightarrow_t}\text{l2})$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim_t \beta}{\Gamma \Rightarrow \sim_t(\alpha \rightarrow_t \beta)}\ (\text{n}{\sim_t}{\rightarrow_t}\text{r})$$

$$\frac{\sim_t \alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\sim_t(\alpha \leftarrow_t \beta), \Gamma \Rightarrow \gamma}\ (\text{n}{\sim_t}{\leftarrow_t}\text{l})$$

$$\frac{\Gamma \Rightarrow \sim_t \alpha}{\Gamma \Rightarrow \sim_t(\alpha \leftarrow_t \beta)}\ (\text{n}{\sim_t}{\leftarrow_t}\text{r1})$$

$$\frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \sim_t(\alpha \leftarrow_t \beta)}\ (\text{n}{\sim_t}{\leftarrow_t}\text{r2}).$$

Some remarks are given as follows.

1. The $\leftarrow_t$-free fragment of N4C is just N4.
2. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in N4C. This fact can be shown by induction on α.
3. The negative logical inference rules for $\rightarrow_t$ and $\leftarrow_t$ in N4C just correspond to the following axiom schemes:
   (a) $\sim_t(\alpha \rightarrow_t \beta) \leftrightarrow (\alpha \wedge_t \sim_t \beta)$,
   (b) $\sim_t(\alpha \leftarrow_t \beta) \leftrightarrow (\sim_t \alpha \vee_t \beta)$.
4. We can show, in a similar way as for CN and LJ, the strong and weak theorems for embedding N4C into a sequent calculus LJC for the positive intuitionistic logic with co-implication. The translation f from N4C into LJC adopts the following conditions for negated implication and negated co-implication:
   (a) $f(\sim_t(\alpha \rightarrow_t \beta)) := f(\alpha) \wedge_t f(\sim_t \beta)$,
   (b) $f(\sim_t(\alpha \leftarrow_t \beta)) := f(\sim_t \alpha) \vee_t f(\beta)$.
5. Using such embedding theorems, we can show the cut-elimination and decidability theorems for N4C.

3.2 Dual Nelson Logic with Co-implication

The language of *dual Nelson logic with co-implication* is obtained from that of the dual connexive logic by adding $\rightarrow_f$ (dual-implication).

A Gentzen-type sequent calculus dN4C for the dual Nelson logic with co-implication is defined as follows based on negative sequents.

Definition 3.2 (dN4C)**.** dN4C *is obtained from* dCN *by deleting the negative logical inference rules* $\{(\sim_f\leftarrow_f\mathrm{l}), (\sim_f\leftarrow_f\mathrm{r})\}$ *and adding the positive and negative logical inference rules of the form:*

$$\frac{\alpha \Rightarrow \Delta \quad \Rightarrow \Gamma, \beta}{\alpha{\to_f}\beta \Rightarrow \Gamma, \Delta}\ (\mathrm{d}{\to_f}\mathrm{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \alpha{\to_f}\beta}\ (\mathrm{d}{\to_f}\mathrm{r1})$$

$$\frac{\beta \Rightarrow \Gamma}{\Rightarrow \Gamma, \alpha{\to_f}\beta}\ (\mathrm{d}{\to_f}\mathrm{r2})$$

$$\frac{\alpha \Rightarrow \Gamma}{\sim_f(\alpha{\to_f}\beta) \Rightarrow \Gamma}\ (\mathrm{d}{\sim_f}{\to_f}\mathrm{l1})$$

$$\frac{\sim_f\beta \Rightarrow \Gamma}{\sim_f(\alpha{\to_f}\beta) \Rightarrow \Gamma}\ (\mathrm{d}{\sim_f}{\to_f}\mathrm{l2})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \gamma \Rightarrow \Gamma, \sim_f\beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha{\to_f}\beta)}\ (\mathrm{d}{\sim_f}{\to_f}\mathrm{r})$$

$$\frac{\sim_f\alpha \Rightarrow \Gamma \quad \beta \Rightarrow \Gamma}{\sim_f(\alpha{\leftarrow_f}\beta) \Rightarrow \Gamma}\ (\mathrm{d}{\sim_f}{\leftarrow_f}\mathrm{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \sim_f\alpha}{\gamma \Rightarrow \Gamma, \sim_f(\alpha{\leftarrow_f}\beta)}\ (\mathrm{d}{\sim_f}{\leftarrow_f}\mathrm{r1})$$

$$\frac{\gamma \Rightarrow \Gamma, \beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha{\leftarrow_f}\beta)}\ (\mathrm{d}{\sim_f}{\leftarrow_f}\mathrm{r2}).$$

We can obtain the following theorems in a similar way as for dCN.

Theorem 3.3 (Cut-elimination for dN4C)**.** *The rule* (f-cut) *is admissible in cut-free* dN4C.

Theorem 3.4 (Decidability of dN4C)**.** dN4C *is decidable.*

Some remarks are given as follows.

1. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in dN4C. This fact can be shown by induction on α.
2. The negative logical inference rules for $\to_f$ and $\leftarrow_f$ in dN4C just correspond to the following axiom schemes:
   (a) $\sim_f(\alpha{\to_f}\beta) \leftrightarrow (\alpha{\wedge_f}{\sim_f}\beta)$,
   (b) $\sim_f(\alpha{\leftarrow_f}\beta) \leftrightarrow ({\sim_f}\alpha{\vee_f}\beta)$.
3. We can show, in a similar way as for dCN and DJ, the strong and weak theorems for embedding dN4C into a sequent calculus DJC for the positive dual intuitionistic logic with co-implication. The translation f from N4C into DJC adopts the following conditions for negated implication and negated co-implication:
   (a) $f(\sim_f(\alpha{\to_f}\beta)) := f(\alpha){\wedge_f}f(\sim_f\beta)$,
   (b) $f(\sim_f(\alpha{\leftarrow_f}\beta)) := f(\sim_f\alpha){\vee_f}f(\beta)$.

3.3 Duality

Next, we introduce a translation from dN4C into N4C.

Definition 3.5. *We fix a common set* Φ *of propositional variables. The language* $\mathcal{L}_{\mathrm{dN4C}}$ *of* dN4C *is defined using* Φ, $\wedge_f, \vee_f, \to_f, \leftarrow_f$ *and* $\sim_f$. *The language* $\mathcal{L}_{\mathrm{N4C}}$ *of* N4C *is defined using* Φ, $\wedge_t, \vee_t, \to_t, \leftarrow_t$ *and* $\sim_t$.

A mapping h *from* $\mathcal{L}_{\mathrm{dN4C}}$ *to* $\mathcal{L}_{\mathrm{N4C}}$ *is defined inductively by:*

1. $h(p) := p$ *for any* $p \in \Phi$,
2. $h(\alpha{\wedge_f}\beta) := h(\alpha){\vee_t}h(\beta)$,
3. $h(\alpha{\vee_f}\beta) := h(\alpha){\wedge_t}h(\beta)$,
4. $h(\alpha{\to_f}\beta) := h(\beta){\leftarrow_t}h(\alpha)$,
5. $h(\alpha{\leftarrow_f}\beta) := h(\beta){\to_t}h(\alpha)$,
6. $h(\sim_f\alpha) := \sim_t h(\alpha)$.

We then obtain a strong theorem for embedding dN4C into N4C.

Theorem 3.6 (Strong embedding from dN4C into N4C)**.** *Let* Γ *be a set of formulas in* $\mathcal{L}_{\mathrm{dN4C}}$, γ *be a formula in* $\mathcal{L}_{\mathrm{dN4C}}$ *or the empty sequence, and* h *be the mapping defined in Definition 3.5.*

1. dN4C $\vdash \gamma \Rightarrow \Gamma$ *iff* N4C $\vdash h(\Gamma) \Rightarrow h(\gamma)$.
2. dN4C $-$ (f-cut) $\vdash \gamma \Rightarrow \Gamma$ *iff* N4C $-$ (t-cut) $\vdash h(\Gamma) \Rightarrow h(\gamma)$.

Proof. We show only (1) since (2) can be obtained as a subproof of (1). We show only the direction ($\Longrightarrow$) of (1) by induction on the proofs P of $\gamma \Rightarrow \Gamma$ in dN4C. We distinguish the cases according to the last inference of P, and show some cases.

1. Case $(\mathrm{d}{\sim_f}{\to_f}\mathrm{r})$: The last inference of P is of the form:

$$\frac{\gamma \Rightarrow \Gamma, \alpha \quad \gamma \Rightarrow \Gamma, \sim_f\beta}{\gamma \Rightarrow \Gamma, \sim_f(\alpha{\to_f}\beta)}\ (\mathrm{d}{\sim_f}{\to_f}\mathrm{r}).$$

By induction hypothesis, we have N4C $\vdash h(\Gamma), h(\alpha) \Rightarrow h(\gamma)$ and N4C $\vdash h(\Gamma), h(\sim_f\beta) \Rightarrow h(\gamma)$ where $h(\sim_f\beta)$ coincides with $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ h(\Gamma), \sim_t h(\beta) \Rightarrow h(\gamma)\end{array}}{\sim_t h(\beta), h(\Gamma) \Rightarrow h(\gamma)}\ \vdots\,(\mathrm{t\text{-}ex\text{-}l}) \quad \dfrac{\begin{array}{c}\vdots\\ h(\Gamma), h(\alpha) \Rightarrow h(\gamma)\end{array}}{h(\alpha), h(\Gamma) \Rightarrow h(\gamma)}\ \vdots\,(\mathrm{t\text{-}ex\text{-}l})}{\dfrac{\sim_t(h(\beta){\leftarrow_t}h(\alpha)), h(\Gamma) \Rightarrow h(\gamma)}{h(\Gamma), \sim_t(h(\beta){\leftarrow_t}h(\alpha)) \Rightarrow h(\gamma)}\ \vdots\,(\mathrm{t\text{-}ex\text{-}l})}\ (\mathrm{n}{\sim_t}{\leftarrow_t}\mathrm{l})$$

where $\sim_t(h(\beta){\leftarrow_t}h(\alpha))$ coincides with $h(\sim_f(\alpha{\to_f}\beta))$ by the definition of h.

2. Case (d$\sim_f\rightarrow_f$l2): The last inference of P is of the form:

$$\frac{\sim_f\beta \Rightarrow \Gamma}{\sim_f(\alpha\rightarrow_f\beta) \Rightarrow \Gamma}\ (\mathrm{d}\sim_f\rightarrow_f\mathrm{l2}).$$

By induction hypothesis, we have N4C $\vdash h(\Gamma) \Rightarrow h(\sim_f\beta)$ where $h(\sim_f\beta)$ coincides with $\sim_t h(\beta)$ by the definition of h. Then, we obtain the required fact:

$$\frac{\begin{array}{c}\vdots\\ h(\Gamma) \Rightarrow \sim_t h(\beta)\end{array}}{h(\Gamma) \Rightarrow \sim_t(h(\beta)\leftarrow_t h(\alpha))}\ (\mathrm{n}\sim_t\leftarrow_t\mathrm{r1})$$

where $\sim_t(h(\beta)\leftarrow_t h(\alpha))$ coincides with $h(\sim_f(\alpha\rightarrow_f\beta))$ by the definition of h.

∎

We can introduce a translation from N4C into dN4C in a similar way.

Definition 3.7. *Φ, $\mathcal{L}_{\mathrm{dN4C}}$ and $\mathcal{L}_{\mathrm{N4C}}$ are the same as in Definition 3.5.*

A mapping k from $\mathcal{L}_{\mathrm{N4C}}$ to $\mathcal{L}_{\mathrm{dN4C}}$ is defined inductively by:

1. $k(p) := p$ *for any* $p \in \Phi$,
2. $k(\alpha\wedge_t\beta) := k(\alpha)\vee_f k(\beta)$,
3. $k(\alpha\vee_t\beta) := k(\alpha)\wedge_f k(\beta)$,
4. $k(\alpha\rightarrow_t\beta) := k(\beta)\leftarrow_f k(\alpha)$,
5. $k(\alpha\leftarrow_t\beta) := k(\beta)\rightarrow_f k(\alpha)$,
6. $k(\sim_t\alpha) := \sim_f k(\alpha)$.

We can obtain a strong theorem for embedding N4C into dN4C.

Theorem 3.8 (Strong embedding from N4C into dN4C)**.** *Let Γ be a set of formulas in $\mathcal{L}_{\mathrm{N4C}}$, γ be a formula in $\mathcal{L}_{\mathrm{N4C}}$ or the empty sequence, and k be the mapping defined in Definition 3.7.*

1. N4C $\vdash \Gamma \Rightarrow \gamma$ *iff* dN4C $\vdash k(\gamma) \Rightarrow k(\Gamma)$.
2. N4C $-$ (t-cut) $\vdash \Gamma \Rightarrow \gamma$ *iff* dN4C $-$ (f-cut) $\vdash k(\gamma) \Rightarrow k(\Gamma)$.

Proof. Similar to Theorem 3.6. ∎

Some remarks are given as follows.

1. The cut-elimination theorems for N4C and dN4C can be obtained using Theorems 3.6 and 3.8.
2. The decidability of dN4C can be obtained using Theorem 3.6.
3. The following hold for N4C and dN4C:
   (a) N4C $\vdash hk(\Gamma) \Rightarrow hk(\gamma)$ iff N4C $\vdash \Gamma \Rightarrow \gamma$,
   (b) dN4C $\vdash kh(\gamma) \Rightarrow kh(\Gamma)$ iff dN4C $\vdash \gamma \Rightarrow \Gamma$.

4 CONCLUSIONS AND REMARKS

4.1 Conclusions

In this paper, the dual connexive logic dCN, which is the dual counterpart of the connexive logic CN, and the dual Nelson logic dN4C with co-implication, which is the dual counterpart of the extended Nelson logic N4C with co-implication, were constructed as a Gentzen-type sequent calculus. The cut-elimination and decidability theorems for dCN and dN4C were proved using some theorems for embedding dCN and dN4C into their negation-free fragments. Some theorems for embedding dCN (dN4C) into CN (N4C, respectively) and vice versa, which represent the duality between them, were shown. Similar duality result cannot be shown for Nelson's paraconsistent four-valued logic N4, and hence the logics CN, dCN, N4C and dN4C are regarded as a novel extension of the positive intuitionistic logic or the positive dual-intuitionistic logic. Thus, in this paper, we have found the novel dual counterparts dCN and dN4C of the plausible and useful intuitionistic paraconsistent logics CN and N4C, respectively.

4.2 Remarks

Some remarks on the extensions CNN, dCNN, N4CN and dN4CN of CN, dCNN, N4C and dN4C, respectively, by adding $\neg_t$ (negation) or $\neg_f{}'$ (dual-co-negation) are addressed below. CNN and N4CN are respectively obtained from CN and N4C by adding the logical inference rules of the form:

$$\frac{\Gamma \Rightarrow \alpha}{\neg_t\alpha, \Gamma \Rightarrow}\ (\neg_t\mathrm{l}) \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg_f\alpha}\ (\neg_t\mathrm{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\sim_t\neg_t\alpha, \Gamma \Rightarrow \gamma}\ (\sim_t\neg_t\mathrm{l}) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim_t\neg_t\alpha}\ (\sim_t\neg_t\mathrm{r}).$$

dCNN and dN4CN are respectively obtained from dCN and dN4C by adding the logical inference rules of the form:

$$\frac{\Rightarrow \Gamma, \alpha}{\neg_f{}'\alpha \Rightarrow \Gamma}\ (\neg_f{}'\mathrm{l}) \qquad \frac{\alpha \Rightarrow \Gamma}{\Rightarrow \Gamma, \neg_f{}'\alpha}\ (\neg_f{}'\mathrm{r})$$

$$\frac{\alpha \Rightarrow \Gamma}{\sim_f\neg_f{}'\alpha \Rightarrow \Gamma}\ (\sim_f\neg_f{}'\mathrm{l})$$

$$\frac{\gamma \Rightarrow \Gamma, \alpha}{\gamma \Rightarrow \Gamma, \sim_f\neg_f{}'\alpha}\ (\sim_f\neg_f{}'\mathrm{r}).$$

The same results as CN, dCN, N4C and dN4C such as the cut-elimination, decidability, embedding and duality properties, can be obtained for CNN, dCNN, N4CN and dN4CN in a similar way as for CN, dCN, N4C and dN4C.

ACKNOWLEDGEMENTS

We would like to thank anonymous referees for their valuable comments. This work was supported by JSPS KAKENHI Grant (C) 26330263 and by Grant-in-Aid for Okawa Foundation for Information and Telecommunications.

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