Nonlinear Second Cumulant/H-infinity Control with Multiple Decision Makers

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Abstract: This paper studies a second cumulant/h-infinity control problem with multiple players for a nonlinear stochastic system on a finite-horizon. The second cumulant/h-infinity control problem, which is a generalization of the higher-order multi-objective control problem, involves a control method with multiple performance indices. The necessary condition for the existence of Nash equilibrium strategies for the second cumulant/h-infinity control problem is given by the coupled Hamilton-Jacobi-Bellman (HJB) equations. In addition, a three-player Nash strategy is derived for the second cumulant/h-infinity control problem. A simulation example is given to illustrate the application of the proposed theoretical formulations.

1 INTRODUCTION

Higher-order control problems (Won et al., 2010) for stochastic systems have been investigated in recent years and related to multi-objective control theoretical game formulations (Lee et al., 2010). In multi-objective control problems, the control method must concern itself with multiple performance indices. A typical multi-objective control problem for both stochastic and deterministic systems can be formulated as mixed H_2/H_∞ control, where the control wishes to minimize an H_2 norm while keeping the H_∞ norm constrained. In fact, H_2/H_∞ control problem is a robust control method which requires a controller to minimize the H_2 performance while attenuating the worst case external disturbance. This approach was investigated in (Bernstein and Hassas, 1989), while the Nash game approach to the problem was given in (Limebeer et al., 1994). In (Basar and Olsder, 1999), a two-player game involving control and disturbance was analyzed, where both players wished to optimize their respective performance indices when the other player plays their equilibrium strategy.

In this paper, mixed second cumulant/h-infinity (second cumulant/H_∞) control problem with multiple players is investigated for a nonlinear stochastic system. Why second cumulant/H_∞ as compared to first cumulant/H_∞ or (H_2/H_∞). Earlier studies in (Won et al., 2010) have shown that higher-order cumulants offer the control engineer additional degrees of freedom to improve system performance through the shaping of the cost function distribution. As a result of this opportunity, there is need to investigate higher-order cumulant to worst case disturbance effects on dynamic systems. The second cumulant/h-infinity control problem involves simultaneous optimization of the higher-order statistical properties of each individual player's cost function distribution through cumulants while keeping the H_∞ norm constrained. The optimization of cost function distribution through cost cumulant was initiated by Sain (Sain, 1966), (Sain and Liberty, 1971). Linear quadratic statistical game with related application such as satellite systems was investigated in (Lee et al., 2010) while an output feedback approach to higher-order statistical game was studied in (Aduba and Won, 2015).

As an extension of the foregoing studies in (Lee et al., 2010), (Aduba and Won, 2015) and the references there in, a nonlinear system of three players with quadratic cost function which is a non trivial extension is considered. Typical multi-objective control problem applications are in large-scale systems such as computer communications networks, electric power grid networks and manufacturing plant networks (Bauso et al., 2008), (Charilas and Panagopoulos, 2010) while the higher-order multi-objective control application has been reported for satellite network (Lee et al., 2010). The rest of this paper is organized as follows. In Section 2, the mathematical preliminaries and second cumulant/h-infinity control problem for a completely observed nonlinear system with multiple players; which is formulated as a nonzero-

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sum differential game problem are given. Section 3 states and proves the necessary condition for the existence of Nash equilibrium strategies while Section 4 derives the optimal players strategy based on solving coupled Hamilton-Jacobi-Bellman equations which is the main result of this paper. Section 5 gives the numerical approximate method for solving the coupled Nash game Hamilton-Jacobi-Bellman equations while a numerical example is demonstrated in Section 6. Finally, the conclusions are drawn in Section 7.

2 PROBLEM FORMULATION

Consider a 3-player nonlinear stochastic state dynamics given by the following itô-type differential equation:

$$dx(t) = f(t,x(t),u_1(t),u_2(t),v(t))dt + \sigma(x(t))dw(t),$$
$$z(t) = Cx(t) + D_1u_1(t) + D_2u_2(t), \tag{1}$$

where $t \in [t_0,t_F] = T$, $x(t) \in \mathbb{R}^n$ is the state and $x(t_0) = x_0$, $u_k(t) \in \mathbb{U}_k \subset \mathbb{R}^m$ is the k-th player strategy, $k = 1,2$, $v_k(t) \in \mathbb{V}_k \subset \mathbb{R}^m$ is the external disturbance player and $dw(t)$ is a Gaussian random process of dimension d with zero mean, covariance of $W(t)dt$. Let $Q_0 = [t_0,t_F) \times \mathbb{R}^n$ and $\bar{Q}_0$ is the closure of Q_0.

f and σ are Borel measurable functions given as $f : C^1(\bar{Q}_0 \times \mathbb{U}_k \times \mathbb{U}_k \times \mathbb{V}_k)$ and $\sigma : C^1(\bar{Q}_0)$. In addition, f and σ satisfy Lipschitz and linear growth conditions (Arnold, 1974) while $z(t)$ is the regulated output of the stochastic system. Let $u_k(t) = \mu_k(t,x), v(t) = \nu(t,x), t \in T$ be memoryless state feedback strategies with $\mu_k(t,x), \nu(t,x)$ satisfying Lipschitz and linear growth condition and thus are admissible strategies. It is shown in (Fleming and Rishel, 1975) that a process $x(t)$ from (1) having admissible strategies together with polynomial growth condition ensures that $E\|x(t)\|^2$ is finite.

The backward evolution operator, $O(\mu_1,\mu_2,\nu)$ (Sain et al., 2000): $O = O_1 + O_2$ is introduced

$$O_1(\mu_1,\mu_2,\nu) = \frac{\partial}{\partial t} + f'(t,x,\mu_1,\mu_2,\nu)\frac{\partial}{\partial x},$$
$$O_2(\mu_1,\mu_2,\nu) = \frac{1}{2}\mathrm{tr}\left(\sigma W \sigma' \frac{\partial^2}{\partial x^2}\right), \tag{2}$$

where tr is the trace operator The cost function (J_k) for the k-th player is given as:

$$J_k(t,x,\mu_1,\mu_2,\nu) = \int_t^{t_F} L_k(s,x(s),\mu_1,\mu_2,\nu)ds + \psi_k(x(t_F)) \text{ or} \tag{3}$$
$$J_k(t,x,\mu_1,\mu_2,\nu) = \int_t^{t_F} z_k'(t)z_k(t)ds + \psi_k(x(t_F)),$$

where $k = 1,2$, L_k is the running cost and ψ_k is the terminal cost with both (L_k,ψ_k) satisfying polynomial growth condition. The z_k in (3) is defined as $z_k(t) = x'(t)Q(t)x(t) + u_k'R_ku_k(t)$, $Q(t) = Q'(t) \geq 0$, $R_k = R_k' > 0$.

The cost function (J) for ν is given as:

$$J(t,x,\mu_1,\mu_2,\nu) = \int_t^{t_F} L(s,x(s),\mu_1,\mu_2,\nu)ds + \psi(x(t_F)) \text{ or}$$
$$J(t,x,\mu_1,\mu_2,\nu) = \int_t^{t_F} \Big(\rho^2\nu'(t)\nu(t) - z'(t)z(t)\Big)ds + \psi(x(t_F)), \tag{4}$$

where L is the running cost and ψ is the terminal cost with both (L,ψ) satisfying polynomial growth condition. Also, $\rho > 0$ is the constraint on the H_∞ of the system.

To study the cumulant game of cost function, the m-th moments of cost functions M_m^k of the k-th player is defined as:

$$M_m^k(t,x,\mu_1,\mu_2) = E\Big\{(J^k)^m(t,x,\mu_1,\mu_2)|x(t) = x\Big\}, \tag{5}$$

where $m = 1,2$. The m-th cost cumulant function $V_m^k(t,x)$ of the k-th player is defined by (Smith, 1995),

$$V_m^k(t,x) = M_m^k - \sum_{i=0}^{m-2}\frac{(m-1)!}{i!(m-1-i)!}M_{m-1-i}^kV_{i+1}^k, \tag{6}$$

where $t \in T = [t_0,t_F]$, $x(t_0) = x_0$, $x(t) \in \mathbb{R}^n$. Next, the following definitions are given:

Definition 2.1: A function $M_i^k, V_i^k : Q_0 \to \mathbb{R}^+$ is an admissible i-th moment cost function if there exists a strategy μ_k such that

$$M_i^k(t,x) = M_i^k(t,x;\mu_1,\mu_2,\nu),$$
$$V_i^k(t,x) = V_i^k(t,x;\mu_1,\mu_2,\nu), \tag{7}$$

for $t \in T, x \in \mathbb{R}^n$, $i = 1,2$.

Definition 2.2: The players equilibrium strategy μ_1^*, μ_2^* is such that

$$M_i^{1*}(t,x) = M_i^1(t,x,\mu_1^*,\mu_2^*,\nu^*) \leq M_i^1(t,x,\mu_1^*,\mu_2,\nu),$$
$$V_i^{1*}(t,x) = V_i^1(t,x,\mu_1^*,\mu_2^*,\nu^*) \leq V_i^1(t,x,\mu_1^*,\mu_2,\nu),$$
$$M_i^{2*}(t,x) = M_i^1(t,x,\mu_1^*,\mu_2^*,\nu^*) \leq M_i^1(t,x,\mu_1,\mu_2^*,\nu),$$
$$V_i^{2*}(t,x) = V_i^1(t,x,\mu_1^*,\mu_2^*,\nu^*) \leq V_i^1(t,x,\mu_1,\mu_2^*,\nu). \tag{8}$$

The moment (5), moment-cumulant relationship (6), definition 2.1 (7) and definition 2.2 (8) all hold for the external disturbance player (ν) as well.

Problem Definition: Consider an open set $Q \subset Q_0$ and let the k-th player and disturbance cost cumulant functions $V_1^k(t,x), \bar{V}_1(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be an admissible cumulant function. Assume the existence of optimal players strategies μ_1^*, μ_2^*, ν^* and optimal players value functions $V_2^{k*}(t,x), \bar{V}_2^*(t,x)$, thus, the multi-player second cumulant/H_∞ control problem is to find the Nash strategies μ_1^*, μ_2^*, ν^* which result in the minimal second value functions $V_2^{k*}(t,x), \bar{V}_2^*(t,x)$ while satisfying the system H_∞ constraint. Thus, μ_k^* is the second cumulant/H_∞ optimal strategy and ν^* is the external disturbance strategy.

Remark: To find the Nash strategies μ_1^*, μ_2^*, ν^*, we constrain the candidates of the optimal players strategy to $U_{M^1}, U_{M^2}, U_{\bar{M}}$ and the optimal value functions $V_2^{k*}(t,x), \bar{V}_2^*(t,x)$ are found with the assumption that lower order cumulants, $V_1^k, \bar{V}_1$ are admissible.

3 SECOND CUMULANT HJB EQUATION

Theorem 3.1: Let $M_j^k(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be the admissible moment cost function, if there exists an optimal k-th player strategy μ_k^* such that $M_j^{k*}(t,x) = M_j^k(t,x,\mu_1^*,\mu_2^*,\nu^*)$, $t \in T = [t_0, t_F]$ then,

$$O\left[M_j^{k*}(t,x)\right] + jM_{j-1}^k(t,x)L_k(t,x,\mu_1,\mu_k,\nu) = 0, \tag{9}$$

where $M_j^k(t_F,x) = \psi_k^j(x(t_F))$, $j = 1,2$ and $k = 1,2$.

Remark: This theorem is an extension of Theorem 3.1 in (Won et al., 2010) which considered only a single player in a statistical optimal control problem. This theorem is applied in this multi-player game.

Theorem 3.2: The necessary condition for Nash equilibrium using the k-th player; $k = 1,2$ as reference is stated and proven. However, similar prove holds with disturbance ν as reference. Consider a 3-player nonlinear system (1) with cost functional (3),(4) of fixed duration $[t_0, t_F]$. Let $V_1^k(t,x), V_2^k(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be admissible value functions for the k-th player.

Similarly, let $\bar{V}_1(t,x), \bar{V}_2(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be admissible value functions for the disturbance player. Assume the existence of optimal player strategy μ_k^* and an optimal value function $V_2^{k*}(t,x)$. Then, the minimal 2^{nd} value function $V_2^{k*}(t,x)$ satisfies in compact form the following HJB equation for the k-th player.

$$0 = \min_{\mu_k \in U_{M^k}} \Bigg\{ O(\mu_1^*, \mu_2^*, \nu^*)\left[V_2^{k*}(t,x)\right] + \left(\frac{\partial V_1^k(t,x)}{\partial x}\right)' \sigma(t,x)W(t)\sigma(t,x)' \left(\frac{\partial V_1^k(t,x)}{\partial x}\right) \Bigg\}, \tag{10}$$

with $V_j^k(t_F, x_F) = 0$, $j = 1,2$, $x(t) \in \mathbb{R}^n$.

Proof: Let V_2^k be a class of $C_p^{1,2}(Q) \cap C(\bar{Q})$ where the arguments for the cumulant and moment functions are suppressed. From (2), (6), the second cost cumulant V_2^{k*} satisfies

$$O\left[V_2^{k*}\right] = O\left[M_2^k\right] - O\left[(V_1^k)^2\right]. \tag{11}$$

From (9), the function M_2^k and running cost L_k satisfy

$$O\left[M_2^k\right] + 2M_1^k L_k(t,x,\mu_1,\mu_2,\nu) = 0. \tag{12}$$

Using (12) in (11) gives

$$O\left[V_2^{k*}\right] + O\left[(V_1^k)^2\right] + 2M_1^k L_k(t,x,\mu_1,\mu_2,\nu) = 0. \tag{13}$$

Replacing $(M_1^k)^2$ with $(V_1^k)^2$ in (13) gives

$$O\left[V_2^{k*}\right] + O\left[(V_1^k)^2\right] + 2V_1^k L_k(t,x,\mu_1,\mu_2,\nu) = 0. \tag{14}$$

Further expansion of (14) gives

$$O\left[V_2^{k*}\right] + V_1^k O\left[V_1^k\right] + V_1^k O\left[V_1^k\right] + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) + 2V_1^k L_k(t,x,\mu_1,\mu_2,\nu) = 0. \tag{15}$$

Then, applying (9) to (15) gives

$$O\left[V_2^{k*}\right] - 2V_1^k L_k(t,x,\mu_1,\mu_2,\nu) + \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) + 2V_1^k L_k(t,x,\mu_1,\mu_2,\nu) = 0. \tag{16}$$

Rearranging and eliminating terms in (16) gives

$$0 = \min_{\mu_k \in U_{M^k}} \Bigg\{ O(\mu_1^*, \mu_k^*, \nu^*)\left[V_2^{k*}(t,x)\right] + \left(\frac{\partial V_1^k(t,x)}{\partial x}\right)' \sigma(t,x)W(t)\sigma(t,x)' \left(\frac{\partial V_1^k(t,x)}{\partial x}\right) \Bigg\}, \tag{17}$$

The theorem is proved. □

Remark: The HJB equation (17) provides a necessary condition for the existence of equilibrium solution of the 3-player 2^{nd} cost cumulant game. The equilibrium solution is achieved under the constraint that $V_1^1, V_1^2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ are admissible value functions.

4 3-PLAYER NASH STRATEGY

Theorem 4: Let $V_1^k(t,x), \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be admissible value functions for the k-th player; $k =$

$1,2$. Also, $\bar{V}_1(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ is the admissible value function for the external disturbance (ν). The players full state-feedback Nash strategies are given as

$$\begin{aligned} \mu_k^*(t,x) &= -\frac{1}{2} R_k^{-1} B_k' \left(\frac{\partial V_1^k}{\partial x} + \gamma_2^k(t) \frac{\partial V_2^k}{\partial x} \right), \\ \nu^*(t,x) &= -\frac{1}{2\rho^2} B_3' \left(\frac{\partial \bar{V}_1}{\partial x} + \gamma(t) \frac{\partial \bar{V}_2}{\partial x} \right), \end{aligned} \tag{18}$$

with $V_j^k(t_F,x_F) = \bar{V}_j(t_F,x_F) = 0$ where $j = 1,2$, $\rho > 0$ and $\gamma_2^k(t), \gamma(t)$ are the Lagrange multipliers. From (1), $f(.) = g(x(t)) + B_1(x)u_1(t) + B_2(x)u_2(t) + B_3(x)v(t)$ and from (3), $L_k = x(t)'Q(t)x(t) + \mu_k'(x)R_k(t)\mu_k(x)$, with $g : \bar{Q}_0 \to \mathbb{R}^n$ is $C^1(\bar{Q}_0)$, $B_i(x(t))$, $i = 1,2,3$ are continuous real matrices and $R_k(t) > 0$ are symmetric matrices.

In addition, $L = \rho^2 \nu'(t)\nu(t) - z'(t)z(t)$ in (4) with $z(t)$ given in (1) and the matrices C, D_1, D_2 are continuous real matrices of appropriate dimensions with $C'C = D_1'D_1, D_2'D_2 = I$ and $D_1'C = D_2'C = D_2'D_1 = 0$.

Proof: The minimal 3-player 2^{nd} value functions $V_2^{1*}(t,x), V_2^{2*}(t,x), \bar{V}_2(t,x)$ satisfy (10) with the constraint condition that $V_1^1(t,x), V_2^1(t,x), \bar{V}_1(t,x)$ are admissible value functions. Then, the value functions V_1^1, V_2^1 satisfy the following coupled partial differential equations for first player μ_1:

$$\begin{aligned} &O(\mu_1^*,\mu_2,\nu)\left[V_1^1(t,x)\right] + L_1(t,x,\mu_1,\mu_2,\nu) = 0, \\ &O(\mu_1^*,\mu_2,\nu)\left[V_2^1(t,x)\right] + \left(\frac{\partial V_1^1}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^1}{\partial x}\right) = 0, \end{aligned} \tag{19}$$

with $V_1^1(t_F,x_F) = V_2^1(t_F,x_F) = 0$. Similarly, the value functions V_1^2, V_2^2 satisfy the following coupled partial differential equations for second player μ_2:

$$\begin{aligned} &O(\mu_1,\mu_2^*,\nu)\left[V_1^2(t,x)\right] + L_2(t,x,\mu_1,\mu_2,\nu) = 0, \\ &O(\mu_1,\mu_2^*,\nu)\left[V_2^2(t,x)\right] + \left(\frac{\partial V_1^2}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^2}{\partial x}\right) = 0, \end{aligned} \tag{20}$$

with $V_1^2(t_F,x_F) = V_2^2(t_F,x_F) = 0$. Similarly, the value functions $\bar{V}_1, \bar{V}_2$ satisfy the following coupled partial differential equations for disturbance player ν:

$$\begin{aligned} &O(\mu_1,\mu_2,\nu^*)[\bar{V}_1(t,x)] + L(t,x,\mu_1,\mu_2,\nu) = 0, \\ &O(\mu_1,\mu_2,\nu^*)[\bar{V}_2(t,x)] + \left(\frac{\partial \bar{V}_1}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial \bar{V}_1}{\partial x}\right) = 0, \end{aligned} \tag{21}$$

with $\bar{V}_1(t_F,x_F) = \bar{V}_2(t_F,x_F) = 0$.

Applying Lagrange multiplier method, let $\mathcal{G}_1(\mu_1,\mu_2,\nu)$ be formulated by converting the constrained coupled HJB equations (19) to unconstrained coupled HJB equations as follows:

$$\begin{aligned} \mathcal{G}_1(\mu_1,\mu_2,\nu) &= O\left[V_2^1\right] + \left(\frac{\partial V_1^1}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^1}{\partial x}\right) \\ &+ \lambda_1^1(t)\Big(O\left[V_1^1\right] + L_1(t,x,\mu_1,\mu_2,\nu)\Big), \end{aligned} \tag{22}$$

where $\lambda_1^1(t)$ is time-varying Lagrange multiplier. Similarly, let $\mathcal{G}_2(\mu_1,\mu_2,\nu)$ be formulated by converting the constrained coupled HJB equation (20) to unconstrained coupled HJB equations as follows:

$$\begin{aligned} \mathcal{G}_2(\mu_1,\mu_2,\nu) &= O\left[V_2^2\right] + \left(\frac{\partial V_1^2}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^2}{\partial x}\right) \\ &+ \lambda_1^2(t)\Big(O\left[V_1^2\right] + L_2(t,x,\mu_1,\mu_2,\nu)\Big), \end{aligned} \tag{23}$$

where $\lambda_1^2(t)$ is time-varying Lagrange multiplier.

Similarly, let $\mathcal{G}(\mu_1,\mu_2,\nu)$ be formulated by converting the constrained coupled HJB equation (21) to unconstrained coupled HJB equations as follows:

$$\begin{aligned} \mathcal{G}(\mu_1,\mu_2,\nu) &= O[\bar{V}_2] + \left(\frac{\partial \bar{V}_1}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial \bar{V}_1}{\partial x}\right) \\ &+ \lambda(t)\Big(O[\bar{V}_1] + L(t,x,\mu_1,\mu_2,\nu)\Big), \end{aligned} \tag{24}$$

where $\lambda(t)$ is time-varying Lagrange multiplier.

At equilibrium state, the stationary conditions are given by the partial derivative of $\mathcal{G}_1(\mu_1,\mu_2,\nu), \mathcal{G}_2(\mu_1,\mu_2,\nu), \mathcal{G}(\mu_1,\mu_2,\nu)$ in (22), (23), (24), with respect to $\mu_1, \lambda_1^1(t), \mu_2, \lambda_1^2(t), \nu, \lambda(t)$, which is zero. Thus, the full-state feedback Nash strategies μ_1^*, μ_2^*, ν^* become

$$\begin{aligned} \mu_1^*(t,x) &= -\frac{1}{2} R_1^{-1} B_1' \left(\frac{\partial V_1^1}{\partial x} + \frac{1}{\lambda_1^1(t)} \frac{\partial V_2^1}{\partial x} \right), \\ \mu_2^*(t,x) &= -\frac{1}{2} R_2^{-1} B_2' \left(\frac{\partial V_1^2}{\partial x} + \frac{1}{\lambda_1^2(t)} \frac{\partial V_2^2}{\partial x} \right), \\ \nu^*(t,x) &= -\frac{1}{2\rho^2} B_3' \left(\frac{\partial \bar{V}_1}{\partial x} + \frac{1}{\lambda(t)} \frac{\partial \bar{V}_2}{\partial x} \right). \end{aligned} \tag{25}$$

Now, let the Lagrange multipliers in (25) be defined as

$$\gamma_2^1(t) = \frac{1}{\lambda_1^1(t)}, \gamma_2^2(t) = \frac{1}{\lambda_1^2(t)}, \gamma(t) = \frac{1}{\lambda(t)}. \tag{26}$$

Then, substituting (26) in (25) gives

$$\begin{aligned} \mu_1^*(t,x) &= -\frac{1}{2} R_1^{-1} B_1' \left(\frac{\partial V_1^1}{\partial x} + \gamma_2^1(t) \frac{\partial V_2^1}{\partial x} \right), \\ \mu_2^*(t,x) &= -\frac{1}{2} R_2^{-1} B_2' \left(\frac{\partial V_1^2}{\partial x} + \gamma_2^2(t) \frac{\partial V_2^2}{\partial x} \right), \\ \nu^*(t,x) &= -\frac{1}{2\rho^2} B_3' \left(\frac{\partial \bar{V}_1}{\partial x} + \gamma(t) \frac{\partial \bar{V}_2}{\partial x} \right). \end{aligned} \tag{27}$$

Thus, substituting for μ_1^*, μ_2^*, ν^* to the 3-player 2^{nd} cost cumulant HJB equations (10) gives the closed loop system form of the second cumulant/H_∞ control. The theorem is proved. □

Remark: The coupled cost cumulant HJB equation (10) provides the necessary condition for the Nash equilibrium solution of the 3-player second cumulant/H_∞ control.

However, substituting for μ_k^* in (19) or (20) for the first cumulant HJB equation (first line of (19) or (20)) gives

$$\begin{aligned}
&\frac{\partial V_1^k}{\partial t} + g'(x)\left(\frac{\partial V_1^k}{\partial x}\right) + \frac{1}{4}\left(\frac{\partial V_1^k}{\partial x}\right)' B_k R_k^{-1} B_k' \left(\frac{\partial V_1^k}{\partial x}\right) \\
&- \frac{1}{2}\left(\left(\frac{\partial V_1^1}{\partial x}\right)' + \gamma_2^1 \left(\frac{\partial V_2^1}{\partial x}\right)'\right) B_1 R_1^{-1} B_1' \left(\frac{\partial V_1^k}{\partial x}\right) \\
&- \frac{1}{2}\left(\left(\frac{\partial V_1^2}{\partial x}\right)' + \gamma_2^2 \left(\frac{\partial V_2^2}{\partial x}\right)'\right) B_2 R_2^{-1} B_2' \left(\frac{\partial V_1^k}{\partial x}\right) \\
&- \frac{1}{2\rho^2}\left(\frac{\partial \bar{V}_1}{\partial x}\right)' B_3 B_3' \left(\frac{\partial V_1^k}{\partial x}\right) - \frac{\gamma}{2\rho^2}\left(\frac{\partial \bar{V}_2}{\partial x}\right)' \times \\
&B_3 B_3' \left(\frac{\partial V_1^k}{\partial x}\right) + \frac{(\gamma_2^k)^2}{4}\left(\frac{\partial V_2^k}{\partial x}\right)' B_k R_k^{-1} B_k' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&+ \frac{\gamma_2^k}{2}\left(\frac{\partial V_1^k}{\partial x}\right)' B_k R_k^{-1} B_k' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&+ x' Q x + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 V_1^k}{\partial x^2}\right) = 0.
\end{aligned} \tag{28}$$

Also, substituting for μ_k^* in (19) or (20) for the second cumulant HJB equation (second line of (19) or (20)) gives

$$\begin{aligned}
&\frac{\partial V_2^k}{\partial t} + g'(x)\left(\frac{\partial V_2^k}{\partial x}\right) - \frac{1}{2\rho^2}\left(\frac{\partial \bar{V}_1}{\partial x}\right)' B_3 B_3' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&- \frac{\gamma}{2\rho^2}\left(\frac{\partial \bar{V}_2}{\partial x}\right)' B_3 B_3' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&- \frac{1}{2}\left(\left(\frac{\partial V_1^1}{\partial x}\right)' + \gamma_2^1 \left(\frac{\partial V_2^1}{\partial x}\right)'\right) B_1 R_1^{-1} B_1' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&- \frac{1}{2}\left(\left(\frac{\partial V_1^2}{\partial x}\right)' + \gamma_2^2 \left(\frac{\partial V_2^2}{\partial x}\right)'\right) B_2 R_2^{-1} B_2' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&- \frac{\gamma_2^k}{2}\left(\frac{\partial V_2^k}{\partial x}\right)' B_k R_k^{-1} B_k' \left(\frac{\partial V_2^k}{\partial x}\right) \\
&+ \left(\frac{\partial V_1^k}{\partial x}\right)' \sigma W \sigma' \left(\frac{\partial V_1^k}{\partial x}\right) + \frac{1}{2} tr\left(\sigma W \sigma' \frac{\partial^2 V_2^k}{\partial x^2}\right) = 0.
\end{aligned} \tag{29}$$

Similarly, substituting ν^* in (21) for the first and second cumulant HJB equations will yield closed-loop equations as in (28) and (29). Thus, the resulting six (6) coupled HJB equations are solved for the value functions $V_1^k, V_2^k, \bar{V}_1, \bar{V}_2$.

Remark: The minimal second cumulant strategies are found under constrained first cumulant at constrained worst case disturbance related by the cost function (4).

5 APPROXIMATE SOLUTION

The analytical solutions of HJB equations (19), (20), (21) are difficult to find for nonlinear systems. Several approximate methods such as power series, spectral and pseudo-spectral, wavelength, path integral and neural network methods have been utilized to solve coupled HJB equations (Al'brekht, 1961), (Beard et al., 1998), (Song and Dyke, 2011), (Kappen, 2005), (Chen et al., 2007). In this paper, neural network approximate method is applied to solve the HJB equation. A polynomial series function is utilized to approximate the value function using the method of least squares on a pre-defined region. The value functions V_i^k, $\bar{V}_i$ in (19), (20), (21) can be approximated as $V_i^k(t,x) = V_{iL}^k(t,x) = \mathbf{w}_{iL}'(t)\Lambda_{iL}(x) = \sum_{i=1}^{L} w_i(t)\gamma_i(x)$ on t over a compact set $\Omega \to \mathbb{R}^n$. Using the approximated value functions $V_{iL}^k(t,x)$ in the HJB equations result in residual error equations. Then weighted residual method (Finlayson, 1972) is applied to minimize the residual error equations and then numerically solve for the least square $\mathbf{w}_{iL}'(t)$ weights. See (Chen et al., 2007) for details.

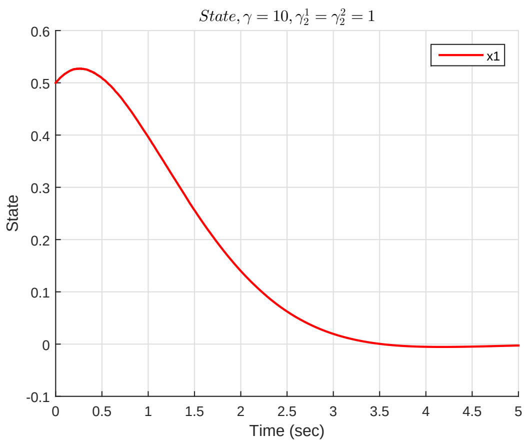
6 SIMULATION RESULTS

Consider a 3-player nonlinear stochastic system with full-state feedback information. The stochastic system is represented as

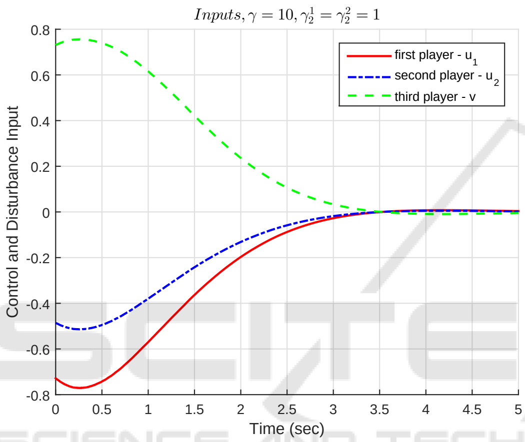
$$\begin{aligned}
dx(t) = &\Big(5x(t) + x^3(t) + 3u_1(t) + 2u_2(t) + 1.5v(t)\Big)dt \\
&+ x(t)dw(t),
\end{aligned} \tag{30}$$

with the state variable defined as $x(t)$. The three players are $u_1(t), u_2(t), v(t)$, where $u_1(t), u_2(t)$ are the controls while $v(t)$ is the external disturbance. The initial state condition is given as $x(0) = 0.5$ and $dw(t)$ in (30) is a Gaussian process with mean $E\{dw(t)\} = 0$, and covariance $E\{dw(t)dw(t)'\} = 0.01$. The first player cost function J_1 is

$$J_1(t_0, x(t), u_1(t)) = \int_{t_0}^{t_F} \left[x^2(t) + u_1^2(t)\right] dt + \psi_1(x(t_F)), \tag{31}$$



(a) State trajectory



(b) Input trajectory

Figure 1: 3-Player State Trajectory and Optimal Input Strategies.

where $\psi_1(x(t_F)) = 0$ is the terminal cost and the second player cost function J_2 is

$$J_2(t_0, x(t), u_2(t)) = \int_{t_0}^{t_F} \left[x^2(t) + u_2^2(t)\right] dt + \psi_2(x(t_F)), \quad (32)$$

where $\psi_2(x(t_F)) = 0$ is the terminal cost and the third player cost function J is

$$J(t_0, x(t), u_1(t), u_2(t), v(t)) = \int_{t_0}^{t_F} \left[\rho^2 v^2 - \left(x^2(t) + u_1^2(t) + u_2^2(t)\right)\right] dt + \psi(x(t_F)), \quad (33)$$

where $\psi(x(t_F)) = 0$ is the terminal cost. The attenuation level is set at $\rho = 1$. In the simulation, the asymptotic stability region for state was arbitrarily chosen as $-1 \leq x \leq 1$. The final time t_F was 5 seconds and external disturbance was $v(t)$ = 0.5cos(t)exp(-t).

Fig. 1(a) shows the state trajectory for noise influence with variance σ^2 = 0.01 for the 2^{nd} cumulant/H_∞ game control. The state is bounded and converged to value close to the origin. It should be noted from Fig. 1(b), that the Nash equilibrium controls for the two player is solved by selecting γ, γ_2^1 and γ_2^2 where the value functions are minimum which in our case were $\gamma = 10$, $\gamma_2^1 = 1$ and $\gamma_2^2 = 1$. In addition, we have the design freedom in γ_2^k and γ values selection to enhance system performance at the chosen attenuation level.

Remark: The second cumulant Nash strategy is found within all admissible first cumulant strategy. A closer look at the state trajectory 1(a) and players trajectory 1(b) show that convergence to the origin is gradual. Additional investigation is required to verify convergence rate at different attenuation levels.

7 CONCLUSION

In this paper, finite-time higher-order control with multiple players was investigated for a nonlinear stochastic system. The second cumulant/H_∞ control problem which is a generalization of higher-order multi-objective control problem was analyzed and the necessary condition for the existence of Nash equilibrium solution was given. A 3-player optimal strategy was derived where a Nash game approach was taken to minimize the different orders of the cost cumulants of the players. A nonlinear example problem was solved to evaluate the theoretical concepts. As a future work, a more practical system example and improved numerical approaches for fast convergence will be explored.

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