Local Point Control of a New Rational Quartic Interpolating Spline

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Abstract: A new rational quartic interpolating spline based on function values is constructed. The rational quartic interpolating spline curves have simple and explicit representation with parameters. The monotonicity-preserving, C^2 continuity and boundedness of rational quartic interpolating spline curves are confirmed. Function value control and derivative value control of rational quartic interpolation spline are given respectively. The advantage of these control methods is that they can be applied to modifying the local shape of interpolating curve only by selecting suitable parameters according to the practical requirements.

1 INTRODUCTION

In engineering and science, one often has a number of data points, obtained by sampling or experimentation. It is often required to interpolate the value and derivatives of that original function. In the mathematical field of numerical analysis, interpolation is a method of constructing new function. The polynomial interpolation methods include Lagrange interpolation, Newton interpolation, Hermite interpolation, etc. However, once the interpolation condition is determined, the interpolation curve will be fixed uniquely. The classical Vandermode interpolation does not allows to control the curve, but it is worthy to say that there are another methods of controlling the shape. We know the augmented, generalized interpolation based on the so-called confluent Vandermonde matrices (Respondek, 2011; 2013; 2016). They enable to control the slope and convexity of the curve in other way.

In order to meet the need of the ever-increasing modeling complexity and to incorporate manufacturing requirements, shape control becomes more and more important as curves and surfaces are constructed. Given the interpolation condition, how to control the shape of the curve to meet the practical application is a very meaningful and urgent problem.

Spline interpolation is a useful and powerful tool in CAGD and CAD. Spline methods have been widely used in geometric modeling. The rational interpolating splines with shape parameters can modify curves locally or globally, and it is very convenient for interaction design in geometric modeling. Their application in shape control has attracted a great deal of interest. In recent years, univariate rational spline interpolations with the parameters have been receiving more attentions. A rational cubic spline based on function values is constructed (Duan et al., 1998), which can be used to control the position and shape of curve or surface. Duan and Wang constructed rational cubic interpolation spline (Duan and Wang, 2005a) and weighted rational cubic interpolation spline (Duan et al., 2005b) based on function values. Meanwhile, convexity-preserving, monotonicity-preserving, error approximation property and region control property have been given. The interpolation spline often is required to satisfy some geometric characteristics (positivity, monotonity, convexity) of data points in industrial design. A shape-preserving rational cubic spline with three parameters has been developed (Abbas et al., 2012; Zhang et al., 2007), and the convexity control of interpolating surfaces had been treated. The region control and convexity control of rational interpolation curves with quadratic denominators have been achieved (Gregory, 1986; Sarfraz, 2000). However, rational quartic interpolating curves have been ignored due to the complexity of calculation. With the in-depth research, Wang and Tan constructed a class of rational quartic interpolation with linear denominators (Wang and Tan, 2004), and discussed

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monotonicity-preserving, C^2 continuity and error approximation property. Duan and Bao proposed the method of local point control of rational cubic interpolating spline with linear and quadratic denominators based on function value respectively (Bao et al., 2009; Duan et al., 2009; Bao et al., 2010). The methods of local point control of rational cubic interpolation spline with linear, quadratic and cubic denominators respectively were discussed (Duan et al., 2010; Pan et al., 2013). The above methods can modify shape of curves at a place flexible by selecting suitable parameters. Duan and Bao constructs rational cubic interpolating spline with difference quotient, but their methods can't show the expression of the spline curve or the point control on the last subinterval (Bao et al., 2009; Duan et al., 2009).

The rational quartic interpolating spline based on function values is constructed and studied in this paper. In section 2, the rational interpolation spline with parameters based on function values will be constructed. In this section, monotonicity-preserving, C^2 continuity and boundedness of rational quartic interpolating curves are proved. The method of function value control and derivative value control of rational quartic interpolation spline $P(t)$ will be discussed in section 3. In section 4, for the end subintervals, point control of rational quartic interpolation curves $P(t)$ are given. Finally, some examples of local point control methods are shown.

2 RATIONAL QUARTIC INTERPOLATING SPLINE AND PROPERTIES

Let $\{(t_i, f_i), i = 0,1,\cdots,n\}$ be a set of given data points, where

$$a = t_0 < t_1 < \cdots < t_n = b,$$

and f_i is the value of the function being interpolated at the knot t_i. Denote

$$h_i = t_{i+1} - t_i, \; \theta = \frac{t - t_i}{h_i}, \; \Delta_i = \frac{f_{i+1} - f_i}{h_i}.$$

And let δ_i be positive parameters, where $i = 0,1,\cdots,n-1$. Then C^1-continuous, piecewise rational quartic splines with the quadratic denominator are defined on the interpolating subinterval $[t_i, t_{i+1}]$ as follow

$$P(t)\big|_{[t_i, t_{i+1}]} = p_i(t)/q_i(t), \; i = 0,1,\cdots,n-1, \tag{1}$$

where

$$p_i(t) = (1-\theta)^4 \delta_i f_i + \theta(1-\theta)^3 U_i + \theta^2 (1-\theta)^2 V_i + \theta^3 (1-\theta) W_i + \theta^4 f_{i+1},$$

$$q_i(t) = (1-\theta)^2 \delta_i + \theta^2,$$

and

$$\begin{aligned} U_i &= \delta_i (f_i + f_{i+1}), & i &= 1,2,\cdots,n-1, \\ V_i &= \delta_i f_i + f_{i+1}, & i &= 0,1,\cdots,n-1, \\ W_i &= 2 f_{i+1} - \Delta_{i+1} h_i, & i &= 0,1,\cdots,n-2, \end{aligned}$$

U_0 and W_{n-1} are free variables. It is easy to prove that the rational quartic spline $P(t)$ satisfies interpolation condition:

$$P(t_i) = f_i, \; i = 0,\cdots,n,$$

and

$$P'(t_i) = \Delta_i, \; i = 1,2,\cdots,n-1.$$

Let

$$P''(t_i +) = P''(t_i -), \; i = 1,2,\cdots,n-1.$$

According to the interpolation condition, the following conclusion can be obtained.

Theorem 1. (C^2-continuous) When parameters δ_i satisfy the equations as follow:

$$h_i \delta_i \delta_{i-1} \Delta_{i-1} = (h_i \delta_i + h_{i-1} \delta_i - h_{i-1}) \Delta_i,$$

$i = 1,2,\cdots,n-1$, the rational quartic spline (1) keep C^2-continuous at interval $[t_0, t_n]$.

Specially, consider equidistant knots case, that is $h_i = h_j$ for all $i, j \in \{0,1,\cdots,n-1\}$. The C^2-continuous condition of rational quartic spline can be simplified as follow:

$$\delta_i \delta_{i-1} \Delta_{i-1} = (2\delta_i - 1) \Delta_i. \tag{2}$$

Example 1. Set

$$f(t) = e^t, \; t \in [-1.5, 1.5]$$

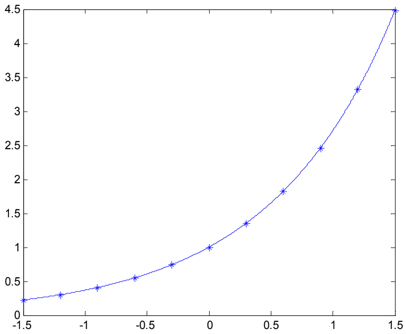
as Fig. 1(a). The C^2-continuous rational quartic spline curve $P(t)$ with $\delta_0 = 1$ where

$$t_i = -1.5 + ih, \; i = 0,1,\cdots,10,$$

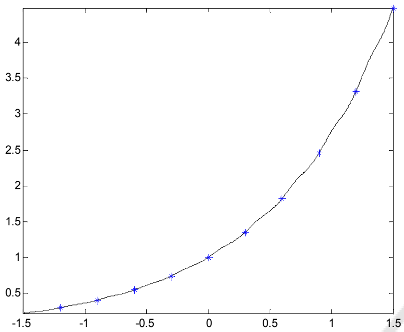
$$h = 0.3,$$

$$U_0 = 0.5243, \; W_9 = 5.4787$$

is given as Fig. 1(b).



(a) The original function.



(b) The rational quartic interpolation curve.

Figure 1: The C^2-continuous rational quartic interpolation.

Theorem 2. (Monotonicity-preserving) If the data points $\{(t_i, f_i), i = 0,1,\cdots,n\}$ satisfy the condition $\Delta_i > 0$, the rational quartic splines (1) satisfy $P'(t) > 0$ when $0 < \delta_i < 5/6$, $\Delta_{i+1}/\Delta_i \le 2\delta_i$ or $5/6 \le \delta_i \le 2$, $\Delta_{i+1}/\Delta_i \le 5/3$.

Proof. The $P'(t)$ can be presented in the simpler form as

$$P'(t) = \frac{1}{q_i^2(t)} \sum_{k=0}^{5} Q_{i,k} (1-\theta)^{5-k} \theta^k \tag{3}$$

$t \in [t_i, t_{i+1}]$, $i = 0,1,\cdots,n-2$, where

$$\begin{cases} Q_{i,0} = \delta_i^2 \Delta_i \\ Q_{i,1} = 2\delta_i \Delta_i - \delta_i^2 \Delta_i \\ Q_{i,2} = 5\delta_i \Delta_i - 3\delta_i \Delta_{i+1} \\ Q_{i,3} = 3\delta_i \Delta_i - \delta_i \Delta_{i+1} \\ Q_{i,4} = 2\delta_i \Delta_i - \Delta_{i+1} \\ Q_{i,5} = \Delta_{i+1} \end{cases}$$

Obviously, inequality $Q_{i,k} > 0$ $(k = 0,\cdots,5)$ are true when $\Delta_i > 0$, $\Delta_{i+1}/\Delta_i \le 5/3$ and $\Delta_{i+1}/\Delta_i \le 2\delta_i \le 4$. So, $P'(t) > 0$ when $0 < \delta_i < 5/6$, $\Delta_{i+1}/\Delta_i \le 2\delta_i$ or $5/6 \le \delta_i \le 2$, $\Delta_{i+1}/\Delta_i \le 5/3$.

According to the above conclusion, the rational quartic spline (1) is monotonicity-preserving if and only if the shape parameters δ_i satisfies $\Delta_i > 0$, $0 < \delta_i < 5/6$, $\Delta_{i+1}/\Delta_i \le 2\delta_i$ or $5/6 \le \delta_i \le 2$, $\Delta_{i+1}/\Delta_i \le 5/3$.

To make it easier to analyze the properties of the rational quartic splines, Eq. (1) can be rewritten as

$$P(t) = \omega_0(\theta,\delta_i) f_i + \omega_1(\theta,\delta_i) f_{i+1} + \omega_2(\theta,\delta_i) f_{i+2} \tag{4}$$

where

$$\begin{cases} \omega_0(\theta,\delta_i) = \left[(1-\theta)^4 + \theta(1-\theta)^3 + \theta^2(1-\theta)^2\right] \\ \qquad \delta_i / \left[(1-\theta)^2 \delta_i + \theta^2\right], \\ \omega_1(\theta,\delta_i) = \left[\theta(1-\theta)^3 \delta_i + \theta^2(1-\theta)^2\right. \\ \qquad \left. +3\theta^3(1-\theta) + \theta^4\right] / \left[(1-\theta)^2 \delta_i + \theta^2\right], \\ \omega_2(\theta,\delta_i) = -\theta^3(1-\theta) / \left[(1-\theta)^2 \delta_i + \theta^2\right]. \end{cases} \tag{5}$$

For all $\theta \in [0,1]$, the basis function $\omega_j(\theta,\delta_i)$ satisfy

$$\sum_{j=0}^{2} \omega_j(\theta,\delta_i) = 1.$$

For the given data, no matter what the parameters δ_i might be, the interpolating function defined by (1) are bounded in the interpolation interval as described by the following Theorem 3.

Theorem 3. (Boundedness) Given interpolation data $\{(t_i, f_i), i = 0,1,\cdots,n\}$ and all $\delta_i > 0$, where the knots are equidistant. Let $P(t)$ be the interpolating functions defined by (1) and define $N = \max_{j=i}^{i+2} |f_j|$. The values of $P(t)$ in $[t_i, t_{i+1}]$ satisfy $|P(t)| < 3N/2$.

Proof. For all $t \in [t_i, t_{i+1}]$, $i = 0,1,\cdots,n-2$, $\theta \in [0,1]$, it is easy to show that

$$\begin{aligned} |P(t)| &\le |\omega_0(\theta,\delta_i) f_i| + |\omega_1(\theta,\delta_i) f_{i+1}| \\ &\quad + |\omega_2(\theta,\delta_i) f_{i+2}| \end{aligned}.$$

According to Eq. (5),

$$\begin{aligned}\sum_{j=0}^{2}\left|\omega_j(\theta,\delta_i)\right| &= \frac{(1-\theta)^2\delta_i+\theta^2+2\theta^3(1-\theta)}{(1-\theta)^2\delta_i+\theta^2}\\ &=1+\frac{2\theta^3(1-\theta)}{(1-\theta)^2\delta_i+\theta^2}\\ &\le 1+2\theta(1-\theta)\end{aligned}$$

thus

$$|P(t)|<3N/2 \tag{6}$$

So, the proof is completed.

3 LOCAL POINT CONTROL OF RATIONAL INTERPOLATION QUARTIC SPLINES

In general, the common spline interpolation is the fixed interpolation which means the shape of the interpolating curve or surface is fixed for the given interpolating data. However, for the quartic rational interpolation splines defined by Eq (1), although the interpolation conditions remain unchanged, we can still adjust the value of shape parameters δ_i to obtain the ideal shape. Thus, function value control and derivative value control can be carried out at any point on the quartic rational interpolation curve.

The curve through a fixed point is often demanded in geometric design. Let θ^* be the local coordinate of a point $t^*\in[t_i,t_{i+1}]$, $i=1,\cdots,n-2$. The point control of rational quartic interpolation curve on the end subinterval will be discussed in section 4.

In the practical design, it is often been required that the function value of the curve at the point t^* to be equal to a real number M^* $\left(f_i<M^*<f_{i+1}\right)$. Let

$$\begin{aligned}M^*=\omega_0(\theta^*,\delta_i)f_i+\omega_1(\theta^*,\delta_i)f_{i+1}\\ +\omega_2(\theta^*,\delta_i)f_{i+2}\end{aligned} \tag{7}$$

The above equation is called a control equation; it is equivalent to

$$A\delta_i+B=0 \tag{8}$$

where

$$\begin{cases}A=\left(1-\theta^*\right)^2\begin{bmatrix}\left(1-\theta^*+\theta^{*2}\right)f_i\\ +\theta^*\left(1-\theta^*\right)f_{i+1}-M^*\end{bmatrix},\\ B=\theta^{*2}\left[\left(1+\theta^*-\theta^{*2}\right)f_{i+1}-\theta^*\left(1-\theta^*\right)f_{i+2}-M^*\right].\end{cases}$$

If there exist parameters δ_i satisfying Eq. (8) when $A,B\neq 0$, there must exist positive δ_i satisfying Eq. (7). Therefore, we have the following function value control theorem.

Theorem 4. Let $P(t)$ be interpolation functions over $[t_i,t_{i+1}]$ defined in (1), and let $t^*\in[t_i,t_{i+1}]$, $i=1,\cdots,n-2$. The sufficient condition for existence of positive parameters δ_i satisfying $P(t^*)=M^*$ is $AB<0$.

If $P(t^*)<M^*$ is required, it is equivalent to $A\delta_i+B<0$. Thus, $P(t^*)<M^*$ can set up if and only if $A\ge 0,\ B\ge 0$ can't set up at the same time.

On the other hand, in the practical design, it is often been required that the first derivative of the interpolation at the point $\bar{t}\in[t_i,t_{i+1}]$ to be equal to a real number $\bar{M}$.

Let

$$\bar{M}=\begin{bmatrix}\omega_0(\theta,\delta_i)f_i+\omega_1(\theta,\delta_i)f_{i+1}\\ +\omega_2(\theta,\delta_i)f_{i+2}\end{bmatrix}'_{t=\bar{t}} \tag{9}$$

Then the control equation (9) is equivalent to

$$A_0\delta_i^2+A_1\delta_i+A_2=0 \tag{10}$$

where

$$\begin{cases}A_0=\left[\left(1-2\bar{\theta}\right)\left(f_{i+1}-f_i\right)-h_i\bar{M}\right]\left(1-\bar{\theta}\right)^4,\\ A_1=\left[\left(1+2\bar{\theta}\right)f_i+\left(2-4\bar{\theta}\right)f_{i+1}+\left(2\bar{\theta}-3\right)f_{i+2}\right.\\ \quad\left.-2h_i\bar{M}\right]\bar{\theta}^2\left(1-\bar{\theta}\right)^2+2\left(f_{i+1}-f_i\right)\bar{\theta}\left(1-\bar{\theta}\right),\\ A_2=\left[\left(2\bar{\theta}-1\right)\left(f_{i+2}-f_{i+1}\right)-h_i\bar{M}\right]\bar{\theta}^4.\end{cases}$$

If there exist positive parameters δ_i satisfying Eq. (10), then (9) holds. This can be stated as the following derivative value control theorem.

Theorem 5. Let $P(t)$ be the interpolation function over $[t_i,t_{i+1}]$ defined in (1), and let $\bar{t}\in[t_i,t_{i+1}]$. The sufficient condition for existence of the positive parameters δ_i satisfying $P'(\bar{t})=\bar{M}$ is that Eq. (10) has positive roots.

Example 2. Without loss of generality, consider the interpolation on $[0,1]$. Let $f(t)$ be the interpolated function satisfying $f(0)=1$, $f(1)=2.5$ $f(2)=4$ and $P(t)$ be interpolation functions defined by (1) in the interpolating interval $[0,1]$. It is obvious that $\theta=(t-0)/(1-0)=t$. The function value control is shown in Figure 2(a).

Let $\delta_i = 1.5$. It can be computed that $P(0.4) = 1.5383$. If $P(0.4) = 1.60$ is required, then Eq. (8) should be satisfied, and $\delta_i = 1$ which satisfy Eq. (8). Thus, the interpolation function becomes

$$P_1(t) = \left[(1-t)^4 + 3.5t(1-t)^3 + 3.5t^2(1-t)^2 \right.$$
$$\left. +3.5t^3(1-t) + 2.5t^4\right] / \left[(1-t)^2 + t^2\right],$$

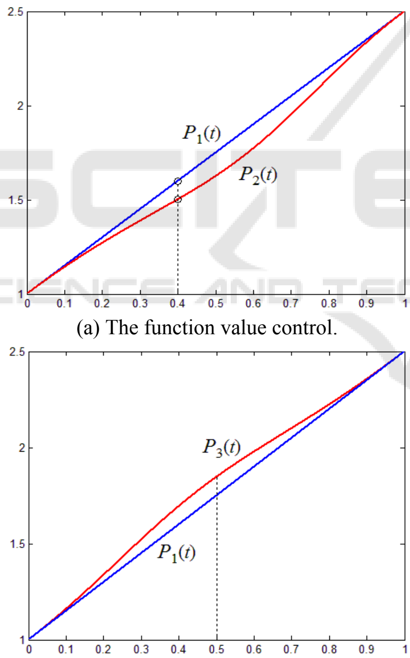
$t \in [0,1]$. Furthermore, if

$$P(0.4) = 413/275 < 1.5383$$

is required, $\delta_i = 2$ can be selected from Eq. (8) and the interpolation function becomes

$$P_2(t) = \left[2(1-t)^4 + 7t(1-t)^3 + 4.5t^2(1-t)^2 \right.$$
$$\left. +3.5t^3(1-t) + 2.5t^4\right] / \left[2(1-t)^2 + t^2\right],$$

$t \in [0,1]$.



(a) The function value control.

(b) The derivative value control.

Figure 2: Local point control of rational interpolation quartic splines.

Example 3. For the same interpolation conditions in Example 2, let $\delta_i = 0.75$. It can be calculated that $P'(0.5) = 1.4694$. Let $t = 0.5$, $\bar{M} = 1.40$ in Eq. (10). It is can be obtained that $7\delta_i^2 - 16\delta_i + 7 = 0$. Solving the above equation $\delta_i = \dfrac{8 \pm \sqrt{15}}{7}$ can be obtained. For $\delta_i = \dfrac{8 - \sqrt{15}}{7}$, the interpolation function becomes

$$P_3(t) = \left[(8-\sqrt{15})(1-t)^4 + (28 - 3.5\sqrt{15})t(1-t)^3 \right.$$
$$\left. +(25.5 - \sqrt{15})t^2(1-t)^2 + 24.5t^3(1-t) + 17.5t^4\right]$$
$$/\left[\left(8-\sqrt{15}\right)(1-t)^2 + 7t^2\right],$$

$t \in [0,1]$. If $P'(0.5) = 1.50$ is required, according to Eq. (10), we can obtain the equation $\delta_i^2 - 2\delta_i + 1 = 0$, calculating that $\delta_i = 1$, then the interpolation function becomes $P_1(t)$. The derivative value control is shown in Figure 2(b).

4 THE END SUBINTERVAL POINT CONTROL OF RATIONAL QUARTIC INTERPOLATION CURVES

As discussed earlier, the given interval $[a,b]$ is divided into n subintervals $[t_i, t_{i+1}]$, $(i = 1, 2, \cdots, n)$. Unlike in (Bao et al., 2009), We could construct $P(t)$ in every subintervals, including end subintervals. We use examples to illustrate the point control of rational interpolation curves $P(t)$. Without loss of generality, we take $n = 3$. Namely, there are three subintervals and the curve has three sections. The general principles are given in Theorem 4 and 5.

Example 4. Let $P(t)$ be defined by (1), $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$ and $f_0 = 1$, $f_1 = 3$, $f_2 = 2$, $f_3 = 4$, $W_2 = 13$, we discuss the function value control of $P(t)$ on the last subinterval, $t \in [2,3]$. The function value control is shown in Figure 3(a).

Let $\delta_2 = 2.0$. The interpolation function is

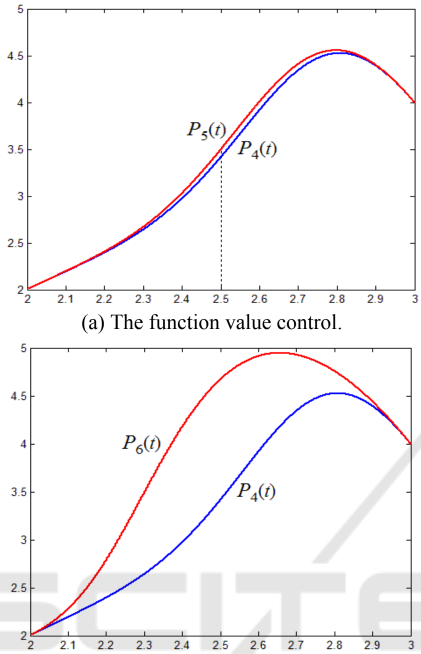
$$P_4(t) = \left[4(3-t)^4 + 12(t-2)(3-t)^3 + 8(t-2)^2(3-t)^2 \right.$$
$$\left. +13(t-2)^3(3-t) + 4(t-2)^4\right]$$
$$/\left[2(3-t)^2 + (t-2)^2\right],$$

$t \in [2,3]$. It can be computed that $P_4(2.5) = 3.4167$.

Furthermore, if $P_4(2.5) = 3.5$ is required, then $\delta_2 = 1.75$ can be selected from Theorem 4 and the interpolation function becomes

$$P_5(t)=\left[14(3-t)^4+42(t-2)(3-t)^3+30(t-2)^2(3-t)^2 +52(t-2)^3(3-t)+16(t-2)^4\right] /\left[7(3-t)^2+4(t-2)^2\right],$$

$t\in[2,3]$.



(a) The function value control.

(b) The derivative value control.

Figure 3: Local point control of rational interpolation quartic splines on the last subinterval.

Example 5. For Example 4, let $\delta_2=2.0$. It can be computed that $P'(0.5)=1.4694$. Let $t=2.5$, it can be computed that $P'(2.5)=7.7778$. If $P'(2.5)=8.0$ is required, it can be obtained that $\delta_2{}^2+11\delta_2-3=0$ from Theorem 5. After calculating, we get $\delta_2=\frac{-11\pm\sqrt{133}}{7}$, $\delta_i>0$, we take $\delta_2=\frac{-11+\sqrt{133}}{7}$ and the interpolation function becomes

$$P_6(t)=\left[\left(-22+2\sqrt{133}\right)(3-t)^4 +\left(-66+6\sqrt{133}\right)(t-2)(3-t)^3 +\left(-14+2\sqrt{133}\right)(t-2)^2(3-t)^2 +26(t-2)^3(3-t)+8(t-2)^4\right] /\left[\left(-11+\sqrt{133}\right)(3-t)^2+2(t-2)^2\right],$$

$t\in[2,3]$. The function value control is shown in Figure 3(b).

5 CONCLUSIONS

In this paper, a C^2 rational quartic function has been developed for the smooth and pleasing visualization of provided data. It can be testified that rational quartic interpolation spline is C^2-continuous, monotonicity-preserving and bounded. Function value control and derivative value control of rational quartic interpolation splines with difference quotient are given. Rational quartic interpolation splines can be changed locally by selecting the corresponding parameters. Thus, they can meet the needs of the practical design.

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