Existence Conditions of Asymptotically Stable 2-D Feedback Control Systems on the Basis of Block Matrix Diagonalization

Giido Izuta
Yonezawa Women's Junior College, Yonezawa, Yamagata, Japan

Keywords: 2-D Systems, Asymptotic Stability, Feedback Control, Block Matrix Diagonalization.

Abstract: This work is concerned with the existence of asymptotically stable 2-D (2-dimensional) systems by means of a feedback control model represented by the system of partial difference equations and their Lagrange solutions. Thus, the goal is to establish a controller that provides a feedback control system with state variables depending solely on its Lagrange solution in the sense that the solution to the variable state is not a linear combination of other Lagrange solutions. Roughly speaking, the results showed that, to achieve such a control system, the controller has to diagonalize the block matrices of the matrices composing the system description model. Finally, a numerical example is presented to show how the controller is designed in order to generate a stable feedback control with given Lagrange solutions.

1 INTRODUCTION

Discrete 2-D control systems have been a subject of research for over half a century now; and many approaches have been proposed to study the stability and design of feedback control systems. As far as the mathematical model is concerned with, either the Roesser system description (Givone and Roesser, 1972) or the FM model (Attasi, 1973; Fornasini and Marchesini, 1978; Fornasini and Marchesini, 1980), which are essentially equivalent to one another in the sense that it is possible to transform the partial difference equations describing one model into the other (Kaczorek, 1985), is taken as the system representation to handle this type of control systems. In addition, the formalism adopted has spread over a large range of mathematical fields. Among those, the $z-$transform showed to be a very effective strategy to investigate systems with single input and single output corresponding to what is called the class of 'delayless systems' in the ordinary 1-D system counterpart theory; and a great deal of stability criteria and controller design procedures were accomplished for 2-D control systems (Anderson and Jury, 1974; Bose, 1982; Tzafestas, 1986; Lim, 1990).

During the time period around the turning of the century, a remarkable shift in the paradigm from handling those problems on the grounds of analytical mathematics into the incorporation of computational methods, which in control system theory is closely associated to the energy method framework, took place. In fact, this approach relies basically on the Lyapunov stability theory as well as the optimization algorithms, and has undergone tremendous developments hand in hand with the advances in computational techniques; so that the formalization of the control system design based on the linear matrix inequalities gained widespread popularity and established its status as a standard procedure to handle not only the ordinary 1-D but also multi-input multi-output n-D systems (Piekarski, 1977; Du and Xie, 2002; Pazke et al., 2004; Izuta, 2007; Rogers et al., 2007). However, this procedure has been pointed out to yield conservative controllers in the control theory sense. Also it is worth noting that many other techniques have flourished and contributed to the broadening of the field (Jerri, 1996; Zerz, 2000; Cheng, 2003; Elaydi, 2005; Izuta, 2012; Izuta, 2014a).

Unlike the theoretical accounts aforementioned, this paper examines the existence conditions of asymptotically stable 2-D feedback control systems from the Lagrange method frame of reference, which has shed light on the stability analysis problem (Izuta, 2010; Izuta, 2014b). In these reports, the Lagrange method was used in conjunction with Jordan matrix transformation in order to transform the original system of partial difference equations into a system composed by only diagonal matrices, which allow the transformed system be solved analytically.

Motivated by these results, this paper aims to fig-

Izuta, G.
Existence Conditions of Asymptotically Stable 2-D Feedback Control Systems on the Basis of Block Matrix Diagonalization.
DOI: 10.5220/0005975604630470
In *Proceedings of the 13th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2016) - Volume 1*, pages 463-470
ISBN: 978-989-758-198-4

ure out the conditions that the controller has to fulfill so as to form a feedback control system having defined asymptotically stable Lagrange solutions. Moreover, the kind of systems concerned with here corresponds to the class of 'systems with delays' in the 1-D case.

Finally, the paper is organized as follows: in section 2, the 2-d control system and the concepts necessary throughout the paper are presented; the asymptotic stability conditions are establish in section 3; a numerical example is provided in 4; and some final remarks are enunciated in section 5 .

2 PRELIMINARIES

In this section, the concepts and definitions used throughout this paper are presented. Firstly, the 2-d control system to be investigated is presented here.

Definition 2.1. *Let the 2-d control system be given by the following system of partial difference equations.*

$$\begin{bmatrix} \mathbf{x_p}(\mathbf{i}+\mathbf{1},\mathbf{j}) \\ \mathbf{x_q}(\mathbf{i},\mathbf{j}+\mathbf{1}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x_p}(i,j) \\ \mathbf{x_q}(i,j) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x_p}(\mathbf{i}-\delta_\mathbf{p},\mathbf{j}-\sigma_\mathbf{p}) \\ \mathbf{x_q}(\mathbf{i}-\delta_\mathbf{q},\mathbf{j}-\sigma_\mathbf{q}) \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u_p}(\mathbf{i},\mathbf{j}) \\ \mathbf{u_q}(\mathbf{i},\mathbf{j}) \end{bmatrix} \tag{1}$$

where $\mathbf{x_p}(\mathbf{i},\mathbf{j})$ *and* $\mathbf{x_q}(\mathbf{i},\mathbf{j})$ *are vectors representing the states of the system and given by*

$$\mathbf{x_p}(\mathbf{i}+\mathbf{1},\mathbf{j}) \overset{def}{=} \begin{bmatrix} x_1(i+1,j) \\ \vdots \\ x_n(i+1,j) \end{bmatrix} \quad \mathbf{x_q}(\mathbf{i},\mathbf{j}+\mathbf{1}) \overset{def}{=} \begin{bmatrix} \hat{x}_1(i,j+1) \\ \vdots \\ \hat{x}_n(i,j+1) \end{bmatrix} \tag{2}$$

with the indices i and j (i, j $\in \mathbb{Z}$*) indicating that the system is discrete and doubly indexed. In addition, the inputs of the system* $\mathbf{u_p}(\mathbf{i},\mathbf{j})$ *and* $\mathbf{u_q}(\mathbf{i},\mathbf{j})$ *are and are defined similarly to (2).*

Assumption 2.2. *Hereafter the matrices* C_{11}, C_{12}, C_{21}, C_{22} *are all real valued non-singular matrices.*

In order to simplify the notations, the following definitions are adopted.

$$\star(\mathbf{i},\mathbf{j}) = \begin{bmatrix} \star_\mathbf{p}(\mathbf{i},\mathbf{j}) \\ \star_\mathbf{q}(\mathbf{i},\mathbf{j}) \end{bmatrix} \overset{def}{=} \begin{bmatrix} \star_1(i,j) \\ \vdots \\ \star_n(i,j) \\ \hat{\star}_1(i,j) \\ \vdots \\ \hat{\star}_n(i,j) \end{bmatrix}, \quad \begin{array}{l} \star = x \\ \text{or} \\ \star = u \end{array}$$

$$\mathbf{x}(\mathbf{i}+\mathbf{1},\mathbf{j}+\mathbf{1}) \overset{def}{=} \begin{bmatrix} x_1(i+1,j) \\ \vdots \\ x_n(i+1,j) \\ \hat{x}_1(i,j+1) \\ \vdots \\ \hat{x}_n(i,j+1) \end{bmatrix} \tag{3}$$

$$\mathbf{x}(\mathbf{i}-\delta,\mathbf{j}-\sigma) \overset{def}{=} \begin{bmatrix} x_1(i-\delta_1,j-\sigma_1) \\ \vdots \\ x_n(i-\delta_n,j-\sigma_n) \\ \hat{x}_1(i-\hat{\delta}_1,j-\hat{\sigma}_1) \\ \vdots \\ \hat{x}_n(i-\hat{\delta}_n,j-\hat{\sigma}_n) \end{bmatrix}$$

Thus, a compact notation for (1) is represented by

$$\begin{aligned} \mathbf{x}(\mathbf{i}+\mathbf{1},\mathbf{j}+\mathbf{1}) &= A\mathbf{x}(\mathbf{i},\mathbf{j}) + B\mathbf{x}(\mathbf{i}-\delta,\mathbf{j}-\sigma) \\ &+ C\mathbf{u}(\mathbf{i},\mathbf{j}) \end{aligned} \tag{4}$$

Remark 2.3. *Small bold faced letters stand for vectors whereas their non-bold faced counterparts mean the variables. Capital letters describe either matrices or block matrix.*

Definition 2.4. *A 2-d feedback control system is a system accomplished from (1) by setting the inputs variables be the states of the system itself. That is to say*

$$\mathbf{x}(\mathbf{i}+\mathbf{1},\mathbf{j}+\mathbf{1}) = \bar{A}\mathbf{x}(\mathbf{i},\mathbf{j}) + \bar{B}\mathbf{x}(\mathbf{i}-\delta,\mathbf{j}-\sigma) \tag{5}$$

in which

$$\begin{aligned} \underline{\mathbf{u}}(\mathbf{i},\mathbf{j}) &= -F\mathbf{x}(\mathbf{i},\mathbf{j}) - G\mathbf{x}(\mathbf{i}-\delta,\mathbf{j}-\sigma) \\ \bar{A} &= A - CF \\ \bar{B} &= B - CG \end{aligned} \tag{6}$$

Remark 2.5. *Under the notations of (1), (6) yields*

$$\begin{aligned} \bar{A}_{11} &= A_{11} - C_{11}F_{11} - C_{12}F_{21} \\ \bar{A}_{12} &= A_{12} - C_{11}F_{12} - C_{12}F_{22} \\ \bar{A}_{21} &= A_{21} - C_{21}F_{11} - C_{22}F_{21} \\ \bar{A}_{22} &= A_{22} - C_{21}F_{12} - C_{22}F_{22} \\ \bar{B}_{11} &= B_{11} - C_{11}G_{11} - C_{12}G_{21} \\ \bar{B}_{12} &= B_{12} - C_{11}G_{12} - C_{12}G_{22} \\ \bar{B}_{21} &= B_{21} - C_{21}G_{11} - C_{22}G_{21} \\ \bar{B}_{22} &= B_{22} - C_{21}G_{12} - C_{22}G_{22} \end{aligned} \tag{7}$$

In this paper, the system is asymptotically stable if all the state variables vanish as the indices increase.

Definition 2.6. *2-D feedback control system (5) is said to be asymptotically stable if all the solutions $x_1(i,j)$, ..., $\hat{x}_n(i,j)$ fulfill the conditions described by*

$$\begin{cases} \lim\limits_{(i+j)\to\infty} \mid x_1(i,j) \mid \to 0 \\ \vdots \\ \lim\limits_{(i+j)\to\infty} \mid \hat{x}_n(i,j) \mid \to 0 \end{cases} \tag{8}$$

Furthermore, an asymptotically stable Lagrange solution is an analytic solution given by

Definition 2.7. *A non-null asymptotically stable Lagrange solution to the state variable is the equation*

$$x_\star(i,j) = \sum_{k=1}^{n} \mathrm{I}_k \alpha_k^i \beta_k^j + \sum_{k=1}^{n} \hat{\mathrm{I}}_k \hat{\alpha}_k^i \hat{\beta}_k^j \tag{9}$$

where $I_\star$'s and $\hat{I}_\star$'s are the initial values and non-null real numbers α_*'s, $\hat{\alpha}_*$'s , $\beta_\star$'s and $\hat{\beta}_\star$'s are such that $|\alpha_*|, |\hat{\alpha}_*|, |\beta_*|, |\hat{\beta}_*| < 1$.

Finally, this paper handles the following problem.

Problem 2.8. *To establish conditions on the feedback controller matrices for the feedback control system (5) have asymptotically stable Lagrange solution given by*

$$\begin{bmatrix} x_1(i,j) \\ \vdots \\ x_t(i,j) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_1 \alpha_1^i \beta_1^j \\ \vdots \\ \mathrm{I}_n \alpha_n^i \beta_n^j \\ \hat{\mathrm{I}}_1 \hat{\alpha}_1^i \hat{\beta}_1^j \\ \vdots \\ \hat{\mathrm{I}}_m \hat{\alpha}_m^i \hat{\beta}_m^j \end{bmatrix} \tag{10}$$

such that the numbers $|\alpha_r| < 1$, $|\hat{\alpha}_r| < 1$, $|\beta_s| < 1$ *and* $|\hat{\beta}_s| < 1$, $\forall$ *r, s.*

Remark 2.9. *The initial condition for (10) yield*

$$\begin{aligned} \mathbf{x}(\mathbf{0},\mathbf{0}) &= \begin{bmatrix} \mathbf{x_p}(\mathbf{0},\mathbf{0}) \\ \mathbf{x_q}(\mathbf{0},\mathbf{0}) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{K_p} & 0 \\ 0 & \mathrm{K_q} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \end{aligned} \tag{11}$$

where

$$\mathrm{K_p} = \begin{bmatrix} \mathrm{I}_1 & 0 \\ 0 & \mathrm{I_n} \end{bmatrix}, \ \mathrm{K_q} = \begin{bmatrix} \hat{\mathrm{I}}_1 & 0 \\ 0 & \hat{\mathrm{I}}_\mathrm{n} \end{bmatrix} \tag{12}$$

3 RESULTS

Henceforth the solution to the problem is presented gathered in a theorem. However, before doing it, a very basic result is given for the sake of the computational procedures that are to be followed in the numerical example.

Proposition 3.1. *Let the controller be as in (6) and (7). Then, their defining matrices are computed as*

$$\begin{aligned} \mathrm{F}_{11} &= \mathrm{C}_{11}^{-1}(\mathrm{A}_{11} - \bar{\mathrm{A}}_{11}) - \\ &\quad \mathrm{C}_{11}^{-1}\mathrm{C}_{12}(\mathrm{C}_{11}^{-1}\mathrm{C}_{12} - \mathrm{C}_{21}^{-1}\mathrm{C}_{22})^{-1} \\ &\quad \times \left[\mathrm{C}_{11}^{-1}(\mathrm{A}_{11} - \bar{\mathrm{A}}_{11}) - \mathrm{C}_{21}^{-1}(\mathrm{A}_{21} - \bar{\mathrm{A}}_{21})\right] \\ \mathrm{F}_{21} &= (\mathrm{C}_{11}^{-1}\mathrm{C}_{12} - \mathrm{C}_{21}^{-1}\mathrm{C}_{22})^{-1} \\ &\quad \times \left[\mathrm{C}_{11}^{-1}(\mathrm{A}_{11} - \bar{\mathrm{A}}_{11}) - \mathrm{C}_{21}^{-1}(\mathrm{A}_{21} - \bar{\mathrm{A}}_{21})\right] \\ \mathrm{F}_{12} &= (\mathrm{C}_{22}^{-1}\mathrm{C}_{21} - \mathrm{C}_{12}^{-1}\mathrm{C}_{11})^{-1} \\ &\quad \times \left[\mathrm{C}_{22}^{-1}(\mathrm{A}_{22} - \bar{\mathrm{A}}_{22}) - \mathrm{C}_{12}^{-1}(\mathrm{A}_{12} - \bar{\mathrm{A}}_{12})\right] \\ \mathrm{F}_{22} &= \mathrm{C}_{22}^{-1}(\mathrm{A}_{22} - \bar{\mathrm{A}}_{22}) - \\ &\quad \mathrm{C}_{22}^{-1}\mathrm{C}_{21}(\mathrm{C}_{22}^{-1}\mathrm{C}_{21} - \mathrm{C}_{12}^{-1}\mathrm{C}_{11})^{-1} \\ &\quad \times \left[\mathrm{C}_{22}^{-1}(\mathrm{A}_{22} - \bar{\mathrm{A}}_{22}) - \mathrm{C}_{12}^{-1}(\mathrm{A}_{12} - \bar{\mathrm{A}}_{12})\right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} \mathrm{G}_{11} &= \mathrm{C}_{11}^{-1}(\mathrm{B}_{11} - \bar{\mathrm{B}}_{11}) - \\ &\quad \mathrm{C}_{11}^{-1}\mathrm{C}_{12}(\mathrm{C}_{11}^{-1}\mathrm{C}_{12} - \mathrm{C}_{21}^{-1}\mathrm{C}_{22})^{-1} \\ &\quad \times \left[\mathrm{C}_{11}^{-1}(\mathrm{B}_{11} - \bar{\mathrm{B}}_{11}) - \mathrm{C}_{21}^{-1}(\mathrm{B}_{21} - \bar{\mathrm{B}}_{21})\right] \\ \mathrm{G}_{21} &= (\mathrm{C}_{11}^{-1}\mathrm{C}_{12} - \mathrm{C}_{21}^{-1}\mathrm{C}_{22})^{-1} \\ &\quad \times \left[\mathrm{C}_{11}^{-1}(\mathrm{B}_{11} - \bar{\mathrm{B}}_{11}) - \mathrm{C}_{21}^{-1}(\mathrm{B}_{21} - \bar{\mathrm{B}}_{21})\right] \\ \mathrm{G}_{12} &= (\mathrm{C}_{22}^{-1}\mathrm{C}_{21} - \mathrm{C}_{12}^{-1}\mathrm{C}_{11})^{-1} \\ &\quad \times \left[\mathrm{C}_{22}^{-1}(\mathrm{B}_{22} - \bar{\mathrm{B}}_{22}) - \mathrm{C}_{12}^{-1}(\mathrm{B}_{12} - \bar{\mathrm{B}}_{12})\right] \\ \mathrm{G}_{22} &= \mathrm{C}_{22}^{-1}(\mathrm{B}_{22} - \bar{\mathrm{B}}_{22}) - \\ &\quad \mathrm{C}_{22}^{-1}\mathrm{C}_{21}(\mathrm{C}_{22}^{-1}\mathrm{C}_{21} - \mathrm{C}_{12}^{-1}\mathrm{C}_{11})^{-1} \\ &\quad \times \left[\mathrm{C}_{22}^{-1}(\mathrm{B}_{22} - \bar{\mathrm{B}}_{22}) - \mathrm{C}_{12}^{-1}(\mathrm{B}_{12} - \bar{\mathrm{B}}_{12})\right] \end{aligned} \tag{14}$$

Proof. It follows straightforwardly from assumption 2.2. □

Now, the main result reads.

Theorem 3.2. *Let the 2-D feedback control system be given by (5). If there exist diagonal matrices $\bar{A}_{11}$, $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$, $\bar{B}_{11}$, $\bar{B}_{12}$, $\bar{B}_{21}$ and $\bar{B}_{22}$ in the sense of the feedback controller (14) and such that the following conditions all are satisfied then the control system is asymptotically stable and its Lagrange solutions can be established.*

1. *the diagonal entries of the diagonal matrices $\bar{A}_{11} + \bar{B}_{11}\bar{B}_{21}^{-1}\bar{A}_{21}$, and $\bar{A}_{22} + \bar{B}_{22}\bar{B}_{12}^{-1}\bar{A}_{12}$ have all non-null absolute values less then unit.*
2. *the diagonal entries of diagonal matrices*

$$\bar{A}_{21}^{-1}\bar{B}_{21}\begin{bmatrix}\alpha_1^{-\delta_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix}\hat{\beta}_1^{-\hat{\sigma}_1} & & 0\\ & \ddots & \\ 0 & & \hat{\beta}_n^{-\hat{\sigma}_n}\end{bmatrix}\bar{A}_{12}^{-1}\bar{B}_{12} \tag{16}$$

have all positive values less then unit.

Proof. Firstly, note that (5) can be rewritten as

$$\begin{aligned}\mathbf{x_p}(\mathbf{i+1},\mathbf{j}) &= \bar{A}_{11}\mathbf{x_p}(\mathbf{i},\mathbf{j}) + \bar{A}_{12}\mathbf{x_q}(\mathbf{i},\mathbf{j})\\ &+\bar{B}_{11}\mathbf{x_p}(\mathbf{i}-\delta,\mathbf{j}-\sigma) + \bar{B}_{12}\mathbf{x_q}(\mathbf{i}-\delta,\mathbf{j}-\sigma)\\ \mathbf{x_q}(\mathbf{i},\mathbf{j+1}) &= \bar{A}_{21}\mathbf{x_p}(\mathbf{i},\mathbf{j}) + \bar{A}_{22}\mathbf{x_q}(\mathbf{i},\mathbf{j})\\ &+\bar{B}_{21}\mathbf{x_p}(\mathbf{i}-\delta,\mathbf{j}-\sigma) + \bar{B}_{22}\mathbf{x_q}(\mathbf{i}-\delta,\mathbf{j}-\sigma)\end{aligned} \tag{17}$$

Now substituting (10) into the first equation in (17) leads to the subsystem described by

$$\begin{aligned}&\begin{bmatrix}I_1\alpha_1^{i+1}\beta_1^j\\ \vdots\\ I_n\alpha_n^{i+1}\beta_n^j\end{bmatrix} - \bar{A}_{11}\begin{bmatrix}I_1\alpha_1^{i}\beta_1^j\\ \vdots\\ I_n\alpha_n^{i}\beta_n^j\end{bmatrix}\\ &-\bar{B}_{11}\begin{bmatrix}I_1\alpha_1^{i-\delta_1}\beta_1^{j-\sigma_1}\\ \vdots\\ I_n\alpha_n^{i-\delta_n}\beta_n^{j-\sigma_n}\end{bmatrix}\\ &= \bar{A}_{12}\begin{bmatrix}\hat{I}_1\hat{\alpha}_1^{i}\hat{\beta}_1^j\\ \vdots\\ \hat{I}_n\hat{\alpha}_n^{i}\hat{\beta}_n^j\end{bmatrix} - \bar{B}_{12}\begin{bmatrix}\hat{I}_1\hat{\alpha}_1^{i-\hat{\delta}_1}\hat{\beta}_1^{j-\hat{\sigma}_1}\\ \vdots\\ \hat{I}_n\hat{\alpha}_n^{i-\hat{\delta}_n}\hat{\beta}_n^{j-\hat{\sigma}_n}\end{bmatrix}\end{aligned} \tag{18}$$

whereas the second equation in (17) provides the set of equations gathered as

$$\begin{aligned}&\begin{bmatrix}\hat{I}_1\bar{\alpha}_1^{i}\bar{\beta}_1^{j+1}\\ \vdots\\ \hat{I}_n\bar{\alpha}_n^{i}\bar{\beta}_n^{j+1}\end{bmatrix} - \bar{A}_{22}\begin{bmatrix}\bar{I}_1\bar{\alpha}_1^{i}\bar{\beta}_1^j\\ \vdots\\ \bar{I}_n\bar{\alpha}_n^{i}\bar{\beta}_m^j\end{bmatrix}\\ &-\bar{B}_{22}\begin{bmatrix}\hat{I}_1\bar{\alpha}_1^{i-\bar{\delta}_1}\bar{\beta}_1^{j-\bar{\sigma}_1}\\ \vdots\\ \hat{I}_n\bar{\alpha}_n^{i-\bar{\delta}_n}\bar{\beta}_n^{j-\bar{\sigma}_n}\end{bmatrix}\\ &= \bar{A}_{21}\begin{bmatrix}I_1\alpha_1^{i}\beta_1^j\\ \vdots\\ I_n\alpha_n^{i}\beta_n^j\end{bmatrix} - \bar{B}_{21}\begin{bmatrix}I_1\alpha_1^{i-\delta_1}\beta_1^{j-\sigma_1}\\ \vdots\\ I_n\alpha_n^{i-\delta_n}\beta_n^{j-\sigma_n}\end{bmatrix}\end{aligned} \tag{19}$$

Splitting the matrices further in order to single out the terms related to the index, (18) yields

$$\begin{aligned}&K_p\begin{bmatrix}\alpha_1^{i+1}\beta_1^j\\ \vdots\\ \alpha_n^{i+1}\beta_n^j\end{bmatrix} - \bar{A}_{11}K_p\begin{bmatrix}\alpha_1^{i}\beta_1^j\\ \vdots\\ \alpha_n^{i}\beta_n^j\end{bmatrix}\\ &-\bar{B}_{11}K_p\begin{bmatrix}\alpha_1^{i-\delta_1}\beta_1^{j-\sigma_1}\\ \vdots\\ \alpha_n^{i-\delta_n}\beta_n^{j-\sigma_n}\end{bmatrix}\\ &= \bar{A}_{12}K_q\begin{bmatrix}\hat{\alpha}_1^{i}\hat{\beta}_1^j\\ \vdots\\ \hat{\alpha}_n^{i}\hat{\beta}_n^j\end{bmatrix} - \bar{B}_{12}K_q\begin{bmatrix}\hat{\alpha}_1^{i-\hat{\delta}_1}\hat{\beta}_1^{j-\hat{\sigma}_1}\\ \vdots\\ \hat{\alpha}_n^{i-\hat{\delta}_n}\hat{\beta}_n^{j-\hat{\sigma}_n}\end{bmatrix}\end{aligned} \tag{20}$$

on the other hand (19) produces

$$\begin{aligned}&K_q\begin{bmatrix}\hat{\alpha}_1^{i}\hat{\beta}_1^{j+1}\\ \vdots\\ \hat{\alpha}_n^{i}\hat{\beta}_n^{j+1}\end{bmatrix} - \bar{A}_{22}K_q\begin{bmatrix}\hat{\alpha}_1^{i}\hat{\beta}_1^j\\ \vdots\\ \hat{\alpha}_n^{i}\hat{\beta}_n^j\end{bmatrix}\\ &-\bar{B}_{22}K_q\begin{bmatrix}\hat{\alpha}_1^{i-\hat{\delta}_1}\hat{\beta}_1^{j-\hat{\sigma}_1}\\ \vdots\\ \hat{\alpha}_n^{i-\hat{\delta}_n}\hat{\beta}_n^{j-\hat{\sigma}_n}\end{bmatrix}\\ &= \bar{A}_{21}K_p\begin{bmatrix}\alpha_1^{i}\beta_1^j\\ \vdots\\ \alpha_t^{i}\beta_t^j\end{bmatrix}\\ &-\bar{B}_{21}K_p\begin{bmatrix}\alpha_1^{i-\delta_1}\beta_1^{j-\sigma_1}\\ \vdots\\ \alpha_n^{i-\delta_n}\beta_n^{j-\sigma_n}\end{bmatrix}\end{aligned} \tag{21}$$

Yet, equations (20) and (21) can be recasted as

$$\begin{aligned}&\left\{K_p\begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ & & \alpha_n\end{bmatrix} - \bar{A}_{11}K_p - \bar{B}_{11}K_p\right.\\ &\left.\begin{bmatrix}\alpha_1^{-\delta_1}\beta_1^{-\sigma_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\beta_n^{-\sigma_n}\end{bmatrix}\right\}\\ &\times\begin{bmatrix}\alpha_1^{i}\beta_1^j\\ \vdots\\ \alpha_n^{i}\beta_n^j\end{bmatrix}\\ &= \left\{\bar{A}_{12}K_q - \bar{B}_{12}K_q\right.\\ &\left.\times\begin{bmatrix}\hat{\alpha}_1^{-\hat{\delta}_1}\hat{\beta}_1^{-\hat{\sigma}_1} & & 0\\ & \ddots & \\ 0 & & \hat{\alpha}_n^{-\hat{\delta}_n}\hat{\beta}_n^{-\hat{\sigma}_n}\end{bmatrix}\right\}\\ &\times\begin{bmatrix}\hat{\alpha}_1^{i}\hat{\beta}_1^j\\ \vdots\\ \hat{\alpha}_n^{i}\hat{\beta}_n^j\end{bmatrix}\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\left\{\mathrm{K_q}\begin{bmatrix}\hat\beta_1 & & 0\\ & \ddots & \\ & & \hat\beta_n\end{bmatrix} - \bar{\mathrm{A}}_{22}\mathrm{K_q} - \bar{\mathrm{B}}_{22}\mathrm{K_q}\right.\\ &\left.\begin{bmatrix}\hat\alpha_1^{-\hat\delta_1}\hat\beta_1^{-\hat\sigma_1} & & 0\\ & \ddots & \\ 0 & & \hat\alpha_n^{-\hat\delta_n}\hat\beta_n^{-\hat\sigma_n}\end{bmatrix}\right\}\\ &\times\begin{bmatrix}\hat\alpha_1^i\hat\beta_1^j\\ \vdots\\ \hat\alpha_n^i\hat\beta_n^j\end{bmatrix}\\ &=\left\{\bar{\mathrm{A}}_{21}\mathrm{K_p} - \bar{\mathrm{B}}_{21}\mathrm{K_p}\right.\\ &\left.\begin{bmatrix}\alpha_1^{-\delta_1}\beta_1^{-\sigma_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\beta_n^{-\sigma_n}\end{bmatrix}\right\}\\ &\times\begin{bmatrix}\alpha_1^i\beta_1^j\\ \vdots\\ \alpha_n^i\beta_n^j\end{bmatrix}\end{aligned}\tag{23}$$

respectively. Note that these equations have terms depending only on α's and β's on one side of the equality, and $\hat\alpha$'s and $\hat\beta$'s on the other side. Furthermore this equality has to hold for all values that the indices assume. Since, in general, the pairs of α's and β's, and the corresponding pairs $\hat\alpha$'s and $\hat\beta$'s are not equal, these equalities are valid only if the following expressions are valid.

$$\begin{aligned}&\mathrm{K_p}\begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ & & \alpha_n\end{bmatrix} - \bar{\mathrm{A}}_{11}\mathrm{K_p}\\ &-\bar{\mathrm{B}}_{11}\mathrm{K_p}\begin{bmatrix}\alpha_1^{-\delta_1}\beta_1^{-\sigma_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\beta_n^{-\sigma_n}\end{bmatrix} = 0\end{aligned}\tag{24}$$

$$\begin{aligned}&\bar{\mathrm{A}}_{12}\mathrm{K_q} - \bar{\mathrm{B}}_{12}\mathrm{K_q}\\ &\times\begin{bmatrix}\hat\alpha_1^{-\hat\delta_1}\hat\beta_1^{-\hat\sigma_1} & & 0\\ & \ddots & \\ 0 & & \hat\alpha_n^{-\hat\delta_n}\hat\beta_n^{-\hat\sigma_n}\end{bmatrix} = 0\end{aligned}\tag{25}$$

$$\begin{aligned}&\mathrm{K_q}\begin{bmatrix}\hat\beta_1 & & 0\\ & \ddots & \\ & & \hat\beta_n\end{bmatrix} - \bar{\mathrm{A}}_{22}\mathrm{K_q} - \bar{\mathrm{B}}_{22}\mathrm{K_q}\\ &\begin{bmatrix}\hat\alpha_1^{-\hat\delta_1}\hat\beta_1^{-\hat\sigma_1} & & 0\\ & \ddots & \\ 0 & & \hat\alpha_n^{-\hat\delta_n}\hat\beta_n^{-\hat\sigma_n}\end{bmatrix} = 0\end{aligned}\tag{26}$$

$$\begin{aligned}&\bar{\mathrm{A}}_{21}\mathrm{K_p}\\ &-\bar{\mathrm{B}}_{21}\mathrm{K_p}\begin{bmatrix}\alpha_1^{-\delta_1}\beta_1^{-\sigma_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\beta_n^{-\sigma_n}\end{bmatrix} = 0\end{aligned}\tag{27}$$

Hence, (24) and (27) yield

$$\begin{aligned}&\mathrm{K_p}\begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ & & \alpha_n\end{bmatrix}\\ &-\bar{\mathrm{A}}_{11}\mathrm{K_p} - \bar{\mathrm{B}}_{11}\mathrm{K_p}(\mathrm{B}_{21}\mathrm{K_p})^{-1}\bar{\mathrm{A}}_{21}\mathrm{K_p} = 0\end{aligned}\tag{28}$$

taking into account the hypothesis of the theorem (28) translates into

$$\begin{bmatrix}\alpha_1 & & 0\\ & \ddots & \\ 0 & & \alpha_n\end{bmatrix} = \bar{\mathrm{A}}_{11} + \bar{\mathrm{B}}_{11}\bar{\mathrm{B}}_{21}^{-1}\bar{\mathrm{A}}_{21}\tag{29}$$

As for $\beta_\star$ ($\star = 1,\cdots,n$), they are computed from (27) , which with the hypothesis of the theorem lead to

$$\begin{aligned}&\begin{bmatrix}\beta_1^{\sigma_1} & & 0\\ & \ddots & \\ 0 & & \beta_n^{\sigma_n}\end{bmatrix} =\\ &\bar{\mathrm{A}}_{21}^{-1}\bar{\mathrm{B}}_{21}\begin{bmatrix}\alpha_1^{-\delta_1} & & 0\\ & \ddots & \\ 0 & & \alpha_n^{-\delta_n}\end{bmatrix}\end{aligned}\tag{30}$$

Similarly, from (26) and (25)

$$\begin{aligned}&\mathrm{K_q}\begin{bmatrix}\hat\beta_1 & & 0\\ & \ddots & \\ 0 & & \hat\beta_n\end{bmatrix}\\ &-\bar{\mathrm{A}}_{22}\mathrm{K_q} - \bar{\mathrm{B}}_{22}\mathrm{K_q}(\mathrm{B}_{12}\mathrm{K_q})^{-1}\bar{\mathrm{A}}_{12}\mathrm{K_q} = 0\end{aligned}\tag{31}$$

and it turns out that the hypothesis of the theorem brings up

$$\begin{bmatrix}\hat\beta_1 & & 0\\ & \ddots & \\ & & \hat\beta_n\end{bmatrix} = \bar{\mathrm{A}}_{22} + \bar{\mathrm{B}}_{22}\bar{\mathrm{B}}_{12}^{-1}\bar{\mathrm{A}}_{12}\tag{32}$$

In the same way, $\hat\alpha_\star$, $\star = 1,\cdots,n$ are computed from (25) according to

$$\begin{aligned}&\begin{bmatrix}\hat\alpha_1^{\hat\delta_1} & & 0\\ & \ddots & \\ 0 & & \hat\alpha_n^{\hat\sigma_n}\end{bmatrix} =\\ &\begin{bmatrix}\hat\beta_1^{-\hat\sigma_1} & & 0\\ & \ddots & \\ 0 & & \hat\beta_n^{-\hat\sigma_n}\end{bmatrix}\bar{\mathrm{A}}_{12}^{-1}\bar{\mathrm{B}}_{12}\end{aligned}\tag{33}$$

Finally, (29), (30), (32), and (33) establish the claim of the theorem. □

In practical terms, the key to solving the problem comes down to the determination of the diagonal matrices. The following remark provides a directive to roughly evaluate the diagonal matrices.

Remark 3.3. *It is clear from (15) and (16) that the diagonal entries of $\bar{A}_{21}^{-1}\bar{B}_{21}$ as well as $\bar{A}_{12}^{-1}\bar{B}_{12}$ must have absolute values smaller than unit in order to be able to establish Lagrange solutions. Thus, in practice, begin focusing on these diagonal matrices trying to set the product at relatively small values; then set each of the matrices individually. Once these matrices are defined, handle (29) and (32) to define the remaining diagonal matrices. The controller matrices are computed only after an estimate of these diagonal matrices are obtained.*

In the next section, a numerical example is presented to show how to make the calculations.

4 NUMERICAL EXAMPLE

Let the 2-D control system be given by

$$\begin{bmatrix} x_1(i+1,j) \\ x_2(i+1,j) \\ \hat{x}_1(i,j+1) \\ \hat{x}_2(i,j+1) \end{bmatrix} = \begin{bmatrix} 0.66 & 0.19 & 0.29 & 0.13 \\ 0.15 & 0.27 & 0.53 & 0.35 \\ 0.83 & 0.43 & 0.31 & 0.55 \\ 0.44 & 0.17 & 0.29 & 0.57 \end{bmatrix} \begin{bmatrix} x_1(i,j) \\ x_2(i,j) \\ \hat{x}_1(i,j) \\ \hat{x}_2(i,j) \end{bmatrix} + \begin{bmatrix} 0.10 & 0.23 & 0.47 & 0.15 \\ 0.20 & 0.37 & 0.17 & 0.25 \\ 0.51 & 0.67 & 0.71 & 0.15 \\ 0.35 & 0.11 & 0.31 & 0.13 \end{bmatrix} \times \begin{bmatrix} x_1(i-\delta_1, j-\sigma_1) \\ x_2(i-\delta_2, j-\sigma_2) \\ \hat{x}_1(i-\hat{\delta}_1, j-\hat{\sigma}_1) \\ \hat{x}_2(i-\hat{\delta}_2, j-\hat{\sigma}_2) \end{bmatrix} + \begin{bmatrix} 0.70 & 0.13 & 0.17 & 0.11 \\ 0.05 & 0.80 & 0.07 & 0.15 \\ 0.11 & 0.23 & 0.75 & 0.17 \\ 0.15 & 0.13 & 0.37 & 0.90 \end{bmatrix} \begin{bmatrix} u_1(i,j) \\ u_2(i,j) \\ \hat{u}_1(i,j) \\ \hat{u}_2(i,j) \end{bmatrix} \quad (34)$$

from which the composing matrices read

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0.66 & 0.19 \\ 0.15 & 0.27 \end{bmatrix} \\ A_{12} &= \begin{bmatrix} 0.29 & 0.13 \\ 0.53 & 0.35 \end{bmatrix} \\ A_{21} &= \begin{bmatrix} 0.83 & 0.43 \\ 0.44 & 0.17 \end{bmatrix} \\ A_{22} &= \begin{bmatrix} 0.31 & 0.55 \\ 0.29 & 0.57 \end{bmatrix} \\ B_{11} &= \begin{bmatrix} 0.10 & 0.23 \\ 0.20 & 0.37 \end{bmatrix} \\ B_{12} &= \begin{bmatrix} 0.47 & 0.15 \\ 0.17 & 0.25 \end{bmatrix} \\ B_{21} &= \begin{bmatrix} 0.51 & 0.67 \\ 0.35 & 0.11 \end{bmatrix} \\ B_{22} &= \begin{bmatrix} 0.71 & 0.15 \\ 0.31 & 0.13 \end{bmatrix} \\ C_{11} &= \begin{bmatrix} 0.70 & 0.13 \\ 0.05 & 0.80 \end{bmatrix} \\ C_{12} &= \begin{bmatrix} 0.17 & 0.11 \\ 0.07 & 0.15 \end{bmatrix} \\ C_{21} &= \begin{bmatrix} 0.11 & 0.23 \\ 0.15 & 0.13 \end{bmatrix} \\ C_{22} &= \begin{bmatrix} 0.75 & 0.17 \\ 0.37 & 0.90 \end{bmatrix} \end{aligned} \quad (35)$$

Here $\bar{A}_{21}^{-1}\bar{B}_{21}$ as well as $\bar{A}_{12}^{-1}\bar{B}_{12}$ are required to satisfy

$$\bar{A}_{21}^{-1}\bar{B}_{21} = \begin{bmatrix} 0.0667 & 0.000 \\ 0.0000 & 0.1667 \end{bmatrix} \quad (36)$$

as well as

$$\bar{A}_{12}^{-1}\bar{B}_{12} = \begin{bmatrix} 0.0723 & 0.000 \\ 0.0000 & 0.2704 \end{bmatrix} \quad (37)$$

From these and keeping in mind the equations (29) and (32), the diagonal matrices are set as

$$\begin{aligned} \bar{A}_{11} &= \begin{bmatrix} 0.11 & 0.00 \\ 0.00 & 0.67 \end{bmatrix} \\ \bar{A}_{12} &= \begin{bmatrix} 6.50 & 0.00 \\ 0.00 & 2.70 \end{bmatrix} \\ \bar{A}_{21} &= \begin{bmatrix} 1.50 & 0.00 \\ 0.00 & 3.00 \end{bmatrix} \\ \bar{A}_{22} &= \begin{bmatrix} 0.51 & 0.00 \\ 0.00 & 0.47 \end{bmatrix} \\ \bar{B}_{11} &= \begin{bmatrix} 0.05 & 0.00 \\ 0.00 & 0.01 \end{bmatrix} \\ \bar{B}_{12} &= \begin{bmatrix} 0.47 & 0.00 \\ 0.00 & 0.73 \end{bmatrix} \\ \bar{B}_{21} &= \begin{bmatrix} 0.10 & 0.00 \\ 0.00 & 0.50 \end{bmatrix} \\ \bar{B}_{22} &= \begin{bmatrix} 0.03 & 0.00 \\ 0.00 & 0.07 \end{bmatrix} \end{aligned} \quad (38)$$

Thus (7) gives the controller matrices

$$\begin{aligned} F_{11} &= \begin{bmatrix} 0.9422 & 0.5315 \\ 0.0816 & 0.0607 \end{bmatrix} \\ F_{21} &= \begin{bmatrix} -1.2451 & 1.3361 \\ 0.8320 & -3.7911 \end{bmatrix} \\ F_{22} &= \begin{bmatrix} 0.4778 & 1.6626 \\ 1.5606 & -0.1926 \end{bmatrix} \\ F_{12} &= \begin{bmatrix} -9.4028 & 0.3825 \\ 0.9157 & -3.0708 \end{bmatrix} \end{aligned} \tag{39}$$

along with

$$\begin{aligned} G_{11} &= \begin{bmatrix} -0.1045 & 0.1509 \\ 0.1804 & 0.5314 \end{bmatrix} \\ G_{21} &= \begin{bmatrix} 0.4637 & 0.9148 \\ 0.1896 & -0.9113 \end{bmatrix} \\ G_{22} &= \begin{bmatrix} 0.8978 & 0.3677 \\ -0.0055 & -0.0334 \end{bmatrix} \\ G_{12} &= \begin{bmatrix} -0.2451 & 0.2494 \\ 0.1503 & -0.6415 \end{bmatrix} \end{aligned} \tag{40}$$

Now taking into account expressions in (29) and (30)

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.00 \\ 0.00 & 0.73 \end{bmatrix} \tag{41}$$

and

$$\begin{aligned} \begin{bmatrix} \beta_1^{\sigma_1} & 0 \\ 0 & \beta_2^{\sigma_2} \end{bmatrix} &= \begin{bmatrix} 0.0667 & 0.0000 \\ 0.0000 & 0.1667 \end{bmatrix} \\ &\times \begin{bmatrix} (0.76)^{-\delta_1} & 0.00 \\ 0.00 & (0.73)^{-\delta_2} \end{bmatrix} \end{aligned} \tag{42}$$

are computed. Note that as far as the diagonal values of the resulting matrix on the right hand side of (42) have absolute values less than unit, β's compose Lagrange solutions to the control system. On the other hand (32) and (33) reduce to

$$\begin{bmatrix} \hat{\beta}_1 & 0 \\ & \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 0.9249 & 0.0000 \\ 0.0000 & 0.7289 \end{bmatrix} \tag{43}$$

and

$$\begin{aligned} \begin{bmatrix} \hat{\alpha}_1^{-\hat{\delta}_1} & 0 \\ 0 & \hat{\alpha}_2^{-\hat{\sigma}_2} \end{bmatrix} &= \begin{bmatrix} 0.0723 & 0.0000 \\ 0.0000 & 0.2704 \end{bmatrix} \\ &\times \begin{bmatrix} (0.9249)^{\hat{\sigma}_1} & 0.0000 \\ 0.0000 & (0.7289)^{\hat{\sigma}_2} \end{bmatrix} \end{aligned} \tag{44}$$

Analogous comments to (42) hold here for the existence of Lagrange solutions.

If (41), (42), (43) and (44) are not Lagrange, (36) and (37) are set to different values; in general to smaller values, and the computations are carried out all over again.

5 FINAL REMARKS

This paper was concerned with the design of a feedback controller such that the overall system is asymptotically stable. The key point was the imposition of a very specific Lagrange solution to then find out the condition for the existence of such a system.

The results showed that a solution can be found if the controller can diagonalize the matrix blocks of the matrices composing the model, which is represented by the set of partial difference equations. A numerical example was presented to show how the mechanics of the calculations work..

REFERENCES

Anderson, B. D. O. and Jury, E. I. (1974). Stability of multidimensional recursive filters. In *IEEE Trans. Circuits and Syst.Proc. IEEE*, number 21, pages 300–304.

Attasi, S. (1973). Systemes lineaires homogenes a deux indices. In *Rapport Laboria*, number 31.

Bose, N. K. (1982). *Applied multdimensional systems theory*. England: Van Nostradand Reinhold Co.

Cheng, S. S. (2003). *Partial difference equations*. London: Taylor & Francis.

Du, C. and Xie, L. (2002). *H-infinity control and filtering of two-dimensional systems*. Berlin, Germany: Springer Verlag.

Elaydi, S. (2005). *An introduction to difference equations*. USA: Springer Science.

Fornasini, E. and Marchesini, G. (1978). Doubly indexed dynamical systems: State space models and structural properties. In *Math. Syst. Th.*, number 12, pages 59–72.

Fornasini, E. and Marchesini, G. (1980). Stability analysis of 2-d systems. In *IEEE Trans. Circ. and Syst.*, volume CAS, pages 1210–1217.

Givone, D. D. and Roesser, R. P. (1972). Multidimensional linear iterative circuits - general properties. In *IEEE Trans. Comp.*, volume C, pages 1067–1073.

Izuta, G. (2007). Stability and disturbance attenuation of 2-d discrete delayed systems via memory state feedback controller. *Int. J. Gen. Systems*, 36(3):263–280.

Izuta, G. (2010). Stability analysis of 2-d discrete systems on the basis of lagrange solutions and doubly similarity transformed systems. In *Proc. 35th annual conf. IEEE IES*, Porto, Portugal.

Izuta, G. (2012). Networked 2-d linear discrete feedback control systems design on the basis of observer controller. In *Proc. 16th Int. Conf. on System Theory, Control and Computing IEEE CS*, Sinaia, Romania.

Izuta, G. (2014a). Controller design for 2-d discrete linear networked systems. In *Proc. The Int. Electrical Eng. Congress*, Pataya, Thailand.

Izuta, G. (2014b). On the asymptotic stability analysis of a certain type of discrete-time 3-d linear systems. In

Proc. 11th Int. Conf. on Informatics in Control, Automation and Robotics (ICINCO 2014), Vienna, Austria.

Jerri, A. J. (1996). *Linear difference equations with discrete transform methods*. Netherlands: Kluwer Acad. Pub.

Kaczorek, T. (1985). *Two-dimensional linear systems*. Berlin: Springer Verlag.

Lim, J. S. (1990). *Two-dimensional linear signal and image processing*. New Jersey, USA: Prentice Hall.

Pazke, W., Lam, J., Galkowski, K., and Xu, S. (2004). Robust stability and stabilisation of 2d discrete state-delayed systems. In *Systems and Control*, volume 51, pages 277–291.

Piekarski, M. S. (1977). Algebraic characterization of matrices whose multivariable characteristic polynomial is hurwitzian. In *Proc. Int. Symp. Operator Theory*, pages 121–126.

Rogers, E., Krzysztof, G., and Owens, D. H. (2007). *Control Systems Theory and Applications for Linear Repetitive Processes*. Springer, London.

Tzafestas, S. G. (1986). *Multidimensional systems - Techniques and Applications*. New York: Marcel Dekker.

Zerz, E. (2000). *Topics in Multidimensional Linear Systems Theory*. Springer.