Peaks Emergence Conditions in Free Movement Trajectories of Linear Stable Systems

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Abstract: The paper considers a asymptotically stable linear system with real eigenvalues of state matrix. It was found that a peak in free movement trajectories arises. Geometric interpretation of peaks emergence was presented through eigenspace. Quantitative estimate of the peak was obtained by using the condition number of matrix of eigenvectors.

1 INTRODUCTION

The problem statement is to determine the eigenvectors influence on free movement of asymptotically stable continuous linear MIMO system with real spectrum. It will be shown that specific disposition of eigenvectors allows to peaking effect (peak) emergence. It means that the norm of state vector growths up and exceeds the norm of initial conditions during some time and then converges to zero. Necessary conditions of peak emergence are the goal of research of current article.

2 GEOMETRIC INTERPRETATION OF PEAKS IN FREE MOVEMENT TRAJECTORIES THROUGH EIGENSPACE

Consider the linear system that is described as

$$\dot{x}(t) = Fx(t);\ x(0) = x(t)\big|_{t=0}, \tag{1}$$

where $x(0), x(t)$ are vectors of initial and current states of the system respectively; F is the state matrix with eigenvalues $\lambda_i < 0; i = \overline{1,n}$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and eigenvectors $\{\xi_i : F\xi_i = \lambda_i \xi_i; i = \overline{1,n}\}$; $x(0), x(k) \in R^n; F \in R^{n\times n}$.

The solution ((Andreev, 1976), (Gantmaher, 2004), (Moler at al., 2003)) of the system (1) is

$$x(t) = e^{Ft}x(0). \tag{2}$$

The vector $x(0)$ can be decomposed into the sum of eigenvectors $x(0) = \sum_{i=1}^{n} \gamma_i \xi_i$. Taking into account properties of matrix exponential the solution (2) can be write as follows

$$x(t) = \sum_{i=1}^{n} \gamma_i e^{\lambda_i t} \xi_i\ ,. \tag{3}$$

where $\|\xi_i\| = 1; i = \overline{1,n}$: $\|*\|$ is the Euclidean norm on R^n.

Definition 1. The system (1) has the peak in the case if there is a vector $x(0) : \|x(0)\| = 1$ such that for some value $t > 0$ the solution of the system satisfies the condition $\|x(t)\| > 1$ (in general case $\|x(0)\| = a$, where $a > 0$ - const).

Let us formulate a statement and let us prove it by using geometric representations.

Statement 1. Necessary conditions of peaks emergence in free movement trajectories of the system (1) are:

1. There is at least one pair of eigenvectors (ξ_i, ξ_j) such that the angle between them is greater than $\pi/2$ in the subspace spanned by those eigenvectors;

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2. There are eigenvalues λ_l, λ_j associated with eigenvectors ξ_l, ξ_j such that $|\lambda_l| >> |\lambda_j|$.

Let us prove of the statement 1 by geometric way. Let consider the linear span (subspace) $L\{\xi_l, \xi_j\}$ of the vectors (ξ_l, ξ_j) which dispose at an obtuse angle (see Fig. 1).

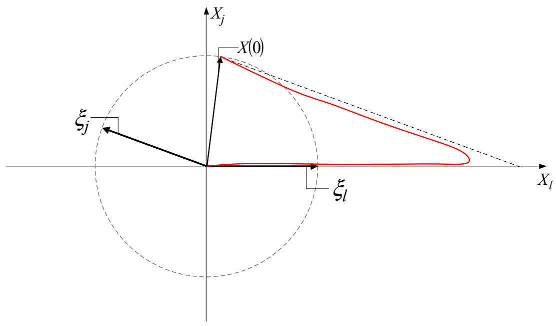

Figure 1.

Suppose the initial condition vector $x(0)$ of the system (1) belongs to the span $x(0) \in L\{\xi_l, \xi_j\}$ and has the unit norm $\|x(0)\| = 1$. Then the vector $x(0)$ can be represented in the form

$$x(0) = \gamma_j \xi_j + \gamma_l \xi_l . \tag{4}$$

Now suppose the vector $x(0)$ is a bisector of the angle between vectors ξ_l, ξ_j; then following relations are true: $\gamma_j = \gamma_l, |\gamma_j| > 1, |\gamma_l| > 1$.

Taking into account (3) we can write the movement of system (1) $x(t) = x(x(0), t)$ in following form

$$x(t) = x(x(0), t) = \gamma_j e^{\lambda_j t} + \gamma_l e^{\lambda_l t} . \tag{5}$$

If in (5) $|\lambda_l| >> |\lambda_j|$ and the system (1) is stable, then from time $t = t_{III} = 3|\lambda_l^{-1}| \approx 0$ following conditions become true: $\gamma_l e^{\lambda_l t} \cong 0; x(t) = x(x(0), t) \cong \gamma_j e^{\lambda_j t}$ and the norm of the vector $x(t)$ is $\|x(t)\| \cong |\gamma_j| e^{\lambda_j t}$. The statement 1 is proved.

Note 1. It is obvious that there are no peaks in in free movement trajectories of the system (1) if any of following conditions holds:

1. The angle between vectors ξ_l, ξ_j is equal to $\pi/2$ for any combinations of λ_l, λ_j.

2. The vector $x(0)$ is a bisector of the acute angle between vectors ξ_l, ξ_j.

3. The vector $x(0)$ is inside the obtuse angle between vectors ξ_l, ξ_j but not its bisector and one of two following cases is realized: $\{\gamma_l \to 1, \gamma_j \to 0\}$ or $\{\gamma_l \to 0, \gamma_j \to 1\}$ for any combinations of eigenvalues λ_l, λ_j.

Let's illustrate the validity of the statement 1 on the example 1.

Example 1. Let the state matrix F of the system (1) has eigenvectors $\xi_1 = [1 \quad 0]^T; \xi_2 = [-0.9987 \quad 0.05]^T$ such that they have unit norm and condition 1 of statement 1 is fulfilled. Let the state matrix F has the spectrum $\sigma\{F\} = \{\lambda_i = \arg[det(\lambda I - F) = 0] : \lambda_1 = -1; \lambda_2 = -50\}$ such that the condition 2 of statement 1 is fulfilled.

Using the eigenspace and the spectrum, we have

$$F = M\Lambda M^{-1} = [\xi_1 \quad \xi_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}[\xi_1 \quad \xi_2]^{-1} =$$
$$= \begin{bmatrix} 1 & -0.9987 \\ 0 & 0.05 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -50 \end{bmatrix}\begin{bmatrix} 1 & -0.9987 \\ 0 & 0.05 \end{bmatrix}^{-1} =$$
$$= \begin{bmatrix} -1 & 978.726 \\ 0 & -50 \end{bmatrix},$$

where M - the matrix of eigenvectors.

Let the initial condition vector $x(0) = [0.0255 \quad 0.9997]^T$ be a bisector of the angle between ξ_1, ξ_2 and has the unit norm $\|x(0)\| = 1$. Decompose the vector $x(0)$ into eigenvectors of the matrix F : $x(0) = 19.9935\xi_1 + 19.994\xi_2$. Now, we can write the free movement (5) of the system (1) with the state matrix F in the following form

$$x(t) = x(x(0), t) = \exp(Ft)x(0) = \gamma_1 e^{\lambda_1 t} + \gamma_2 e^{\lambda_2 t} =$$
$$= 19.9935 e^{-t} + 19.994 e^{-50t} .$$

It is obvious that the component $x_{\xi_2}(t) = 19.994\xi_2 e^{-50t}$ of the free movement is close to zero at the time $t = \lambda_2^{-1} \ln(\varepsilon)\big|_{\varepsilon=0.05} = 0.0599$. At the same time the component $x_{\xi_1}(t) = 19.9935\xi_1 e^{-t}$ of the free movement is equal to $x_{\xi_1}(t) = 19.9935\xi_1 e^{-0.0599} = 18.8311\xi_1$. Clearly, there is a peak $\max_t \|x(t)\|$ of the norm $\|x(t)\|$ in the free movement of the constructed two-dimensional system of type (1). The peak takes on the value $\max_t \|x(t)\| = 17.8324$.

Let us confirm this result by observing the free movement norm $\|x(t)\|$. It is computed using the following formula $\|x(t)\| = \|\exp(Ft)x(0)\|$. The obtained curve is shown on Fig. 2.a (curve 1). The curve confirms correctness of estimation of peak of free movement trajectories obtained through the geometrical interpretation. Fig.2.a and fig.2.b demonstrate norms $\|x(t)\|$ of the system with same eigenvectors but with following spectra: $\sigma\{F\} = \{\lambda_1 = -1; \lambda_2 = -25\}$ (curve 2), $\sigma\{F\} = \{\lambda_1 = -1; \lambda_2 = -10\}$ (curve 3), $\sigma\{F\} = \{\lambda_1 = -1; \lambda_2 = -5\}$ (curve 4).

Moreover, fig. 2.a illustrates processes in norm, and fig. 2.b does the same in phase space spanned by eigenvectors.

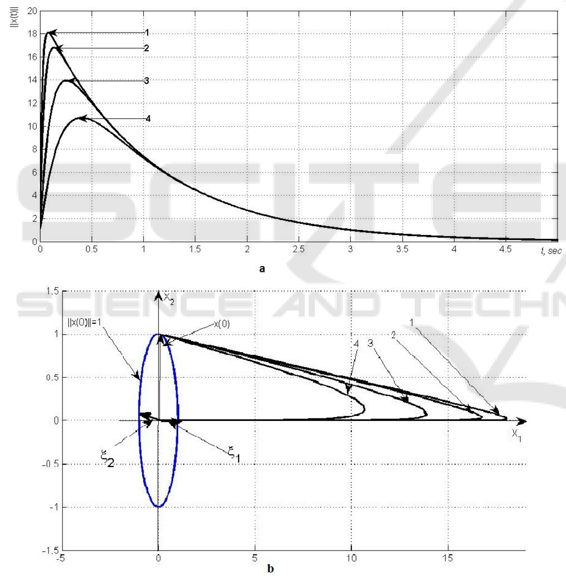

Figure 2: Example of peaks.

3 ALGEBRAIC INTERPRETATION. CONDITION NUMBER AS A QUANTITATIVE ESTIMATION OF PEAKS

Consider the solution (2) of system (1) in order to estimate the norm of possible peaks. If in (2) we turn to norms ((Andreev, 1976), (Gantmaher, 2004), (Moler at al., 2003), (Lancaster at al., 1985), (Golub at al., 1976)), we get

$$\|x(t)\| = \|\exp(Ft)x(0)\| \le \|\exp(Ft)\| \cdot \|x(0)\|. \qquad (6)$$

Recall that the system (1) satisfies conditions:

$$\sigma\{F\} = \left\{ \begin{array}{l} \lambda_i = \arg(\det(\lambda I - F) = 0);\ \lambda_i < 0; \\ Jm(\lambda_i) = 0; \lambda_i \ne \lambda_j \text{ при } i \ne j \end{array} \right\} \qquad (7)$$

The matrix F can be represented in the form

$$F = M \Lambda M^{-1}, \qquad (8)$$

where $M = row\{\xi_i; i = \overline{1,n}\}$ is matrix composed of eigenvectors of matrix F such that the following condition is true: $F\xi_i = \lambda_i \xi_i$; $\Lambda = diag\{\lambda_i; i = \overline{1,n}\}$ is diagonal matrix of eigenvalues. It is common knowledge ((Gantmaher, 2004), (Lancaster at al., 1985)) that the representation (8) holds for a matrix function $f\{(*)\}$ of a matrix $(*)$: $f(F) = Mf(\Lambda)M^{-1}$. If the matrix function is the matrix exponential $f(F) = \exp(Ft)$; then we can write

$$\begin{aligned} \exp(Ft) &= M \exp(\Lambda) M^{-1} = \\ &= M diag\{e^{\lambda_i t}; i = \overline{1,n}\} M^{-1}. \end{aligned} \qquad (9)$$

Substituting (9) in (6), we get

$$\begin{aligned} \|x(t)\| &= \|\exp(Ft)x(0)\| \le \|\exp(Ft)\| \cdot \|x(0)\| = \\ &= \left\| M diag\{e^{\lambda_i t}; i = \overline{1,n}\} M^{-1} \right\| \cdot \|x(0)\|. \end{aligned} \qquad (10)$$

Let us form inequality using (10) to obtain upper estimate of $\|x(t)\|$

$$\begin{aligned} \|x(t)\| &\le \left\| M diag\{e^{\lambda_i t}; i = \overline{1,n}\} M^{-1} \right\| \cdot \|x(0)\| \le \\ &\le \|M\| \cdot \left\| diag\{e^{\lambda_i t}; i = \overline{1,n}\} \right\| \cdot \left\| M^{-1} \right\| \cdot \|x(0)\|, \end{aligned} \qquad (11)$$

where $\|M\| \cdot \|M^{-1}\|$ is equal to condition number $C\{M\}$ ((Golub, 1996), (Wilkinson, 1984, 1984), (Zhang at al., 2014)): $C\{M\} = \|M\| \cdot \|M^{-1}\|$

$\left\| diag\{e^{\lambda_i t}; i = \overline{1,n}\} \right\| = e^{\lambda_M t}$, where λ_M is maximum eigenvalue of matrix F and it determines stability index η (Andreev, 1976) of the system (1) in the form $\eta = |\lambda_M|$. The condition number $C\{M\}$ takes minimal value if the matrix $M = row\{\xi_i; i = \overline{1,n}\}$ is composed of vectors with unit norm. Then we can write

$$\|x(t)\| \le roof\|x(t)\| = C\{\widetilde{M}\}e^{\lambda_M t}\|x(0)\|, \qquad (12)$$

where $\widetilde{M}$ is modified matrix of eigenvectors of matrix F such that it is composed of eigenvectors with unit norm: $\widetilde{M} = M \cdot diag\{(\|\xi_i\|_2)^{-1}; i = \overline{1,n}\}$.

Example 2. Consider the system from example (1)

$$\dot{x}(t) = Fx(t);\ x(0) = x(t)|_{t=0},$$

where the state matrix is $F = \begin{bmatrix} -1 & 978.726 \\ 0 & -50 \end{bmatrix}$; the modified matrix of eigenvectors is $\widetilde{M} = [\widetilde{\xi}_1 \quad \widetilde{\xi}_2] = \begin{bmatrix} 1 & -0.9987 \\ 0 & 0.05 \end{bmatrix}$ with condition number $C\{\widetilde{M}\}$. Using (12) we get $\|x(t)\| \le 39.973e^{-t}\|x(0)\|$. Fig. 3 illustrates curves from the fig. 2 (curves 1-4) and the estimate $\|x(t)\| \le C\{\widetilde{M}\}e^{-t}\|x(0)\|$.

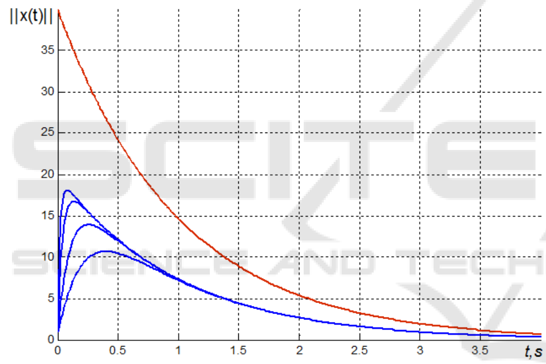

Figure 3: Quantitative estimation of peaks.

4 CONCLUSIONS

Linear asymptotically stable systems with a simple real spectrum of state matrix were studied. Necessary conditions for emergence of peaks in free movement trajectories of those systems were found. It has been established that peaks arise by certain initial conditions in the case that the structure of eigenvectors is close to collinear. Quantitative estimation of peaks such as upper estimate of the state vector norm was found through the condition number of the modified matrix of eigenvectors.

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