# Variance of Departure Process in Two-Node Tandem Queue with Unreliable Servers and Blocking

Yang Woo Shin[1] and Dug Hee Moon[2]

[1]*Department of Statistics, Changwon National University, Changwon, Gyeongnam 51140, Korea*

[2]*School of Industrial Engineering and Naval Architecture, Changwon National University, Changwon, Gyeongnam 51140, Korea*

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Abstract: This paper provides an effective method for evaluating the second moments such as variance and covariance for the number of departures in two-node tandem queue with unreliable servers. The behavior of the system is described by a level dependent quasi-birth-and-death process and the departure process is modeled by a Markovian arrival process. Algorithms for the transient behavior, the variance and covariance structure for the output process and the time to the *n*th departure are developed. We show that the results can be applied to derive approximate formulae for the due-date performance and the distribution of the number of outputs in a time interval.

## 1 INTRODUCTION

There is an extensive literature for the analysis of manufacturing systems with finite buffers and unreliable servers. Most of the works related to the performance evaluation of manufacturing systems have been focused on analyzing the first order measures such as average production rates and average buffer levels in steady-state e.g. see the monographs (Buzacott and Shanthikumar, 1993; Gershwin, 1994), the survey papers (Dallery and Gershwin, 1992; Papadopoulos and Heavey, 1996; Li et al., 2009) and the references therein. The first order measures can be used to get information about the capabilities of a production system in the long run. However, there may be tremendous variability from a time period to period (Gershwin, 1994, Section 3.2; Tan, 1999a). Thus the second order measures such as the variance of the number of parts produced in a given time period and the inter-departure times and covariance between consecutive inter-departure times are also very useful to design and control production systems in a more effective way. The information about the time dependent second order measures can especially be useful to respond short-term and long-term requirements in an effective and timely way.

Studies on variance of the output process in a serial production line have been presented during the last decades, for a review of recent studies on the variance of the output for production systems, one can refer to the papers (Tan,2000; Tan, 2013; Lagershausena and Tan, 2015). For discrete material flow production systems with finite buffers, Tan (1999b, 2000) use a Markov reward model to calculate the variance of the number of parts produced in a given time period in a two-station production line with finite buffer capacities and deterministic processing times and geometrically distributed failure and repair times. Our approach to be developed in this paper is to model the output process by a Markovian arrival process (MAP) and to use the closed formulae for the transient behavior and the variance and covariance structure for the number of outputs during a period $(0,t]$ and the *n*th departure time in the literature.

This paper is aimed on providing an effective method for evaluating the second moments of the number of outputs and inter-departure times and investigating the effects of the system parameters to the second moments. The results can be applied to the practical problem such as due-time performance in manufacturing system and are basis on analyzing the long line. This paper concerns to the two-station system with finite buffer capacities. A model of a two-node system is simple, but it helps us to understand the behavior of the system and gives some insights of the more complicated system. The approach can also be used as building block for analyzing the more complex system with multiple nodes.



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This paper is organized as follows. In Section 2, the model is described in detail. The moment formulae for MAP are reviewed and algorithms for the performance measures are presented in Sections 3 and 4, respectively. In section 5, numerical results are presented. Concluding remarks are given in Section 6.

## 2 MODEL

We consider a tandem queueing network that consists of two service stations $S_1$ and $S_2$ and one buffer of finite size $b$ between them. Each station $S_i$ has an unreliable server $M_i$, $i = 1,2$. Assume the following system characteristics.

*BAS blocking mechanism* : Blocking after service (BAS) rule is adopted, that is, if the buffer is full upon a completion of service at the first station, the server $M_1$ is blocked and the customer is held at the station where it just completed its service until the station $S_2$ can accommodate it.

*Open and saturated system*: In many manufacturing system, it has been assumed that the first station the first station is never starved and the last station is never blocked. For potential applications of the method and results to developing approximation method of more complicated system, we assume that the server $M_1$ in the first station is never starved and it starts new service immediately after a service completion unless the server is blocked and the server $M_2$ in the second station is never blocked and the customer at $M_2$ leaves the system immediately after completing its service.

*ODF rule* : Each server is either up (operational) or under repair (broken-down) at any time. Operation dependent failure (ODF) is assumed. That is, a server can fail only while the server is working and a server never fails while the server is blocked or starved.

*Exponential distributions of service time, failure time and repair time*: We define the failure time by the operation time in units between two successive failures (from a repair to a failure). The failure time does not contain the time period while the server is being blocked, starved or repaired. Service time, failure time and repair time of $M_i$ are assumed to be of exponential with rates $\mu_i$, $\nu_i$ and $\eta_i$, respectively.

Let $X(t)$ be the number of customers in the buffer and at station $S_2$ and the customer blocked at station $S_1$. The state space of $X(t)$ is $\{0,1,\cdots,K\}$, where $K = b+2$. Let $J_i(t)$ be service phase of the server $M_i$ at time $t$ denote the states of $J(t)$ by

$$J_i(t) = \begin{cases} w, & M_i \text{ is working} \\ s, & M_i \text{ is starved} \\ b, & M_i \text{ is blocked} \\ f, & M_i \text{ is failed.} \end{cases}$$

The state space of the stochastic process $\mathbf{Z} = \{Z(t), t \geq 0\}$ with $Z(t) = (X(t), J_1(t), J_2(t))$ is

$$\mathcal{S} = \cup_{n=0}^{K} \mathcal{S}_n,$$

where

$$\begin{aligned} \mathcal{S}_0 &= \{(0,w,s),(0,f,s)\}, \\ \mathcal{S}_n &= \{(n,j_1,j_2) : j_1, j_2 \in \{w,f\}\},\ 1 \leq n \leq K-1, \\ \mathcal{S}_K &= \{(K,b,w),(K,b,f)\}. \end{aligned}$$

The stochastic process $\mathbf{Z} = \{Z(t), t \geq 0\}$ forms a Markov chain with generator of the form

$$Q = \begin{pmatrix} B_0 & A_0 & & & & \\ C_1 & B_1 & A_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & C_{K-1} & B_{K-1} & A_{K-1} \\ & & & C_K & B_K \end{pmatrix}.$$

The matrices $B_n$, $A_n$, $C_n$ are as follows:

$$B_n = \begin{pmatrix} * & \nu_2 & \nu_1 & 0 \\ \eta_2 & * & 0 & \nu_1 \\ \eta_1 & 0 & * & \nu_2 \\ 0 & \eta_1 & \eta_2 & * \end{pmatrix},\ 1 \leq n \leq K-1,$$

$$B_0 = \begin{pmatrix} * & \nu_1 \\ \eta_1 & * \end{pmatrix},\ B_K^* = \begin{pmatrix} * & \nu_2 \\ \eta_2 & * \end{pmatrix},$$

$$A_n = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ 1 \leq n \leq K-2,$$

$$A_0 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{K-1} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_n = \begin{pmatrix} \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ 2 \leq n \leq K-1,$$

$$C_1 = \begin{pmatrix} \mu_2 & 0 \\ 0 & 0 \\ 0 & \mu_2 \\ 0 & 0 \end{pmatrix},\ C_K = \begin{pmatrix} \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the diagonal entries of $B_n$ are determined by $Q\mathbf{e} = 0$ and $\mathbf{e}$ is a column vector of appropriate size whose elements are all 1.

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## 3 DEPARTURE PROCESS

Let $T$ be the first time until a customer leaves the system and

$$F_{zz'}(t) = P(Z(T) = z', T \le t \,|\, Z(0) = z),\ z, z' \in \mathcal{S}.$$

Then $T$ is the same as the absorbing time of a Markov chain with rate matrix of the form

$$Q_T = \begin{pmatrix} D_0 & D_1 \\ 0 & 0 \end{pmatrix},$$

where

$$D_0 = \begin{pmatrix} B_0 & A_0 & & & \\ & B_1 & A_1 & & \\ & & \ddots & \ddots & \\ & & & B_{K-1} & A_{K-1} \\ & & & & B_K \end{pmatrix},$$

$$D_1 = \begin{pmatrix} O_0 & & & & \\ C_1 & O_1 & & & \\ & C_2 & O_2 & & \\ & & \ddots & \ddots & \\ & & & C_K & O_K \end{pmatrix}.$$

The matrix $F(t) = (F_{zz'}(t))$ is given by

$$F(t) = \int_0^t \exp(D_0 u)\, du D_1,\ t \ge 0$$

which is the inter arrival time of a Markovian arrival process (MAP) with representation $MAP(D_0, D_0)$, see Lucantoni et al. (1990).

Let $N(t)$ be the number of customers that leave the system during an interval $(0,t]$ and $P(n,t) = (P_{zz'}(n,t))$ be the square matrix of size $|\mathcal{S}|$ whose $(z,z')$-component is

$$P_{zz'}(n,t) = P(N(t) = n, Z(t) = z' | Z(0) = z).$$

It follows from the Kolomogorov equations that

$$\frac{d}{dt}P(n,t) = P(n,t)D_0 + P(n-1,t)D_1,\ n \ge 1,\ t \ge 0 \tag{1}$$

and $P(0,0) = I$ the identity matrix. The matrix generating function $P^*(w,t) = \sum_{n=0}^{\infty} w^n P(n,t)$ is given by

$$P^*(w,t) = \exp[(D_0 + wD_1)t],\ |w| \le 1,\ t \ge 0.$$

For later use, define the following notation. Let $\boldsymbol{\pi} = (\pi(x), x \in \mathcal{S})$ be the stationary distribution of $Q$ and

$$\boldsymbol{\Pi} = \mathbf{e}\boldsymbol{\pi}, \quad \Psi = (\mathbf{e}\boldsymbol{\pi} - Q)^{-1}, \quad \lambda = \boldsymbol{\pi} D_1 \mathbf{e}$$
$$\mathbf{c} = \boldsymbol{\pi} D_1 \Psi, \quad \mathbf{d} = \Psi D_1 \mathbf{e}.$$

It can be easily seen that $\boldsymbol{\pi}\Psi = \boldsymbol{\pi}$, $\Psi\mathbf{e} = \mathbf{e}$ and $\mathbf{ce} = \lambda = \boldsymbol{\pi}\mathbf{d}$.

The following theorem can be found in (Neuts, 1989, Theorems 5.4.1 and 5.4.2; Artalejo et al, 2010).

**Theorem 3.1.** *In stationary state, that is,* $\pi(x) = P(Z(0) = x)$, *mean* $\mu(t) = E[N(t)]$, *variance* $\sigma^2(t) = \mathrm{Var}[N(t)]$ *and the covariance* $\mathrm{Cov}(t,u,v) = Cov[N(t), N(v) - N(u)]$ $(0 < t \le u < v)$ *are given as follows:*

$$\mu(t) = \lambda t,$$
$$\sigma^2(t) = \tilde{\sigma}^2(t) + 2\mathbf{c}[\exp(Qt) - \boldsymbol{\Pi}]\mathbf{d},$$
$$\mathrm{Cov}(t,u,v) = \boldsymbol{\pi} D_1 [I - \exp(Qt)] \exp[Q(u-t)] \times [I - \exp(Q(v-u))]\Psi\mathbf{d}.$$

*where*

$$\tilde{\sigma}^2(t) = 2(\lambda^2 - \mathbf{cd}) + (\lambda - 2\lambda^2 + 2\mathbf{c}D_1\mathbf{e})t. \tag{2}$$

*Remark 1.* It is well known that as $t \to \infty$

$$\exp(Qt) = \boldsymbol{\Pi} + O(t^{r-1}e^{-\eta t}), \tag{3}$$

where $-\eta$ is the real part of $\eta^*$, the non-zero eigen value of $Q$ with maximum real part, and $r$ is the multiplicity of $\eta^*$, see e.g. (Narayana and Neuts, 1992). It can be easily seen from Theorem 3.1 and (3) that $\mathrm{Cov}(t,u,v) \to 0$ as $u - t \to \infty$.

*Remark 2.* It can be seen from Theorem 3.1 that the variance rate is given by the closed formula

$$V = \lim_{t\to\infty} \frac{\mathrm{Var}[N(t)]}{t} = (\lambda - 2\lambda^2 + 2\mathbf{c}D_1\mathbf{e}).$$

Tan (1999b) use numerical result of the asymptotic variance rate $V$ to determine the variance $\sigma^2(t) \approx Vt$ for large $t$. We can see that $\tilde{\sigma}^2(t)$ provides more accurate approximation of $\sigma^2(t)$ than that of $Vt$ and it is easy to compute $\tilde{\sigma}^2(t)$.

Let $\xi_n$, $n = 0,1,2,\cdots$ be the $n$th transition time of $\boldsymbol{N}$ with $\xi_0 = 0$ and set $\tau_n = \xi_n - \xi_{n-1}$ and $Z_n = Z(\xi_n + 0)$, $n = 1,2,\cdots$ with $Z_0 = Z(0)$. The transition probability matrix of $\{Z_n, n = 0,1,2,\cdots\}$ is $P = (-D_0)^{-1}D_1$ and the stationary distribution $\boldsymbol{p}$ of $P$ is given by

$$\boldsymbol{p} = \frac{1}{\lambda}\boldsymbol{\pi} D_1.$$

The following theorem can be found in Artalejo et al. (2010).

**Theorem 3.2.** *Assume that* $Z(0)$ *has a distribution* $\mathbf{a} = (a(x), x \in \mathcal{S})$ *with* $a(x) = P(Z(0) = x)$. *The mean, variance and covariance of* $\tau_n$ *are given as follows:*

$$\mathbb{E}[\tau_n] = \mathbf{a}P^{n-1}\mathbf{d}_0,$$
$$\mathrm{Var}[\tau_n] = 2\mathbf{a}P^{n-1}(-D_0)^{-2}\mathbf{e} - \left(\mathbf{a}P^{n-1}\mathbf{d}_0\right)^2,$$
$$\mathrm{Cov}(\tau_k, \tau_n) = \mathbf{a}\mathcal{X}_{k,n}\mathbf{e} - (\mathbb{E}[\tau_k])(\mathbb{E}[\tau_n]),\ 1 \le k < n,$$

*where*

$$\mathbf{d}_0 = (-D_0)^{-1}\mathbf{e},$$
$$\mathcal{X}_{k,n} = P^{k-1}(-D_0)^{-1}P^{n-k}(-D_0)^{-1}P,\ 1 \le k < n.$$

*Remark 3.* Since $\lim_{n\to\infty} P^n = \mathbf{e}\boldsymbol{p}$, it can be easily seen that for each $k = 1,2,\cdots$,

$$\lim_{n\to\infty} \mathrm{Cov}(\tau_k,\tau_n) = 0.$$

*Remark 4.* The mean and variance of $\xi_n$ are as follows

$$\mathbb{E}[\xi_n] = \sum_{i=1}^{n} \mathbb{E}[\tau_i],$$

$$\mathrm{Var}[\xi_n] = \sum_{i=1}^{n} \mathrm{Var}[\tau_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathrm{Cov}(\tau_i,\tau_j).$$

*Remark 5.* Assuming $\mathbf{a} = \boldsymbol{p}$, the mean, variance and covariance of $\tau_n$ are as follows:

$$\mathbb{E}[\tau_n] = \frac{1}{\lambda},$$

$$\mathrm{Var}[\tau_n] = \frac{2}{\lambda}\boldsymbol{\pi}\mathbf{d}_0 - \frac{1}{\lambda^2},$$

$$\mathrm{Cov}(\tau_k,\tau_n) = \frac{1}{\lambda}\boldsymbol{\pi}P^n\mathbf{d}_0 - \frac{1}{\lambda^2}, n = 1,2,\cdots.$$

# 4 ALGORITHMS

It is necessary to $\boldsymbol{\pi}$, $\Psi = (\mathbf{e}\boldsymbol{\pi} - Q)^{-1}$ and $\exp(Qt)$ for the variance, covariance of $N(t)$ and $\tau_n$. In this section, some algorithms for computing $\boldsymbol{\pi}$, $\Psi = (\mathbf{e}\boldsymbol{\pi} - Q)^{-1}$ and $\exp(Qt)$ are presented.

*1. Algorithm for stationary distribution* $\boldsymbol{\pi}$ *of* $Q$. Here, we present an algorithm for stationary distribution $\boldsymbol{\pi}$ of $Q$. Write $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_K)$, where $\boldsymbol{\pi}_i$ is the vector of size $l_i$, $0 \le i \le K$. Let $R_1, \cdots, R_K$ be the matrices satisfies the following matrix equations

$$A_{n-1} + R_n B_n + R_n R_{n+1} C_{n+1} = 0,\ 1 \le n \le K-1,$$
$$A_{K-1} + R_K B_K = 0.$$

The solutions of the equation are given as follows:

$$R_K = A_{K-1}(-B_K)^{-1},$$
$$R_n = A_{n-1}[-(B_n + R_{n+1}C_{n+1})]^{-1}, n = K-1,\cdots,1.$$

Then the stationary distribution $\boldsymbol{\pi}$ of $Q$ is given as follows

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R_1 \cdots R_n,\ n = 1,2,\cdots,K$$

with

$$\boldsymbol{\pi}_0[B_0 + R_1 C_1] = 0$$

and normalizing condition

$$\boldsymbol{\pi}_0\left(\mathbf{e} + \sum_{n=1}^{K} R_1\cdots R_n \mathbf{e}\right) = 1.$$

Once $\boldsymbol{\pi}$ is obtained, $\boldsymbol{p} = (\boldsymbol{p}_0, \boldsymbol{p}_1, \cdots, \boldsymbol{p}_K)$ can be calculated by

$$\boldsymbol{p}_k = \begin{cases} \frac{1}{\lambda}\boldsymbol{\pi}_{k+1}C_{k+1}, & k = 0,1,\cdots,K-1, \\ 0, & k = K. \end{cases}$$

*2. Algorithm for* $(-D_0)^{-1}$. It can be seen from the structure of $D_0$ that $(-D_0)^{-1}$ is of the form

$$(-D_0)^{-1} = \begin{pmatrix} X(0,0) & X(0,1) & \cdots & X(0,K) \\ & X(1,1) & \cdots & X(1,K) \\ O & & \ddots & \vdots \\ & & & X(K,K) \end{pmatrix}.$$

The block components $X(i,j)$, $0 \le i \le j \le K$ are calculated following the algorithm in (Shin, 2009) as follows :

(1) Compute
$$G_n = A_{n-1}(-B_n)^{-1},\ n = K, K-1, N-2, \cdots, 1$$
and $G_0 = (-B_0)^{-1}$.

(2) Compute $X(n,k)$, $0 \le n \le K$, $k = n, n+1, \cdots, K$ as follows: For $n = 0,1,\cdots,K$, set $X(n,n) = (-B_n)^{-1}$ and
$$X(n,k) = X(n,k-1)G_k,\ k = n+1, n+2, \cdots, K.$$

*3. Algorithm for* $\Psi = (\mathbf{e}\boldsymbol{\pi} - Q)^{-1}$. Let $E_n = (1,\cdots,1)^T$ $(0 \le n \le K)$ be the $l_n$-dimensional column vector whose components are all one and $E_1^* = (1,\cdots,1)^T$ be the $(\sum_{i=1}^{K} l_i)$-dimensional column vector, where $l_i$ is the number of elements of $\mathcal{S}_i$. Let $\boldsymbol{\pi}_1^* = (\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_K)$ and $\Pi[i,j] = E_i\boldsymbol{\pi}_j$, $0 \le i,j \le K$. Denote the $(i,j)$ block matrix of a matrix $A$ corresponding to $(i,j)$ block of $Q$ by $A[i,j]$, $0 \le i,j \le K$. Write the matrix $Q$ in the block form

$$Q = \begin{pmatrix} B_0 & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix},\ \mathbf{e}\boldsymbol{\pi} - Q = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

where

$$A_{00} = E_0\boldsymbol{\pi}_0 - B_0, \quad A_{01} = E_0\boldsymbol{\pi}_1^* - Q_{01},$$
$$A_{10} = E_1^*\boldsymbol{\pi}_0 - Q_{10}, \quad A_{11} = E_1^*\boldsymbol{\pi}_1^* - Q_{11}.$$

Then the block matrix form of $\Psi$ is given by (e.g. (Horn and Johnson, 1985, page 18))

$$\Psi = \begin{pmatrix} A_{00}^{*-1} & -A_{00}^{-1}A_{01}A_{11}^{*-1} \\ -A_{11}^{*-1}A_{10}A_{00}^{-1} & A_{11}^{*-1} \end{pmatrix},$$

where

$$A_{00}^* = A_{00} - A_{01}A_{11}^{-1}A_{10},$$
$$A_{11}^* = A_{11} - A_{10}A_{00}^{-1}A_{01}.$$

The matrix $\Psi$ is calculated by the following step:

(1) Calculate $(-B_0)^{-1}$ using the ordinary algorithm.

(2) Since $Q_{11}$ is block tridiagonal matrix, one can use the algorithm in (Shin, 2009) for $(-Q_{11})^{-1}$.

(3) For $A_{11}^{-1}$ and $A_{00}^{-1}$, one can use the following formula (see Horn and Johnson(1985, page 19))
$$A_{11}^{-1} = (-Q_{11})^{-1} - \frac{1}{1+\boldsymbol{\pi}_1^*\mathbf{q}_{11}}\mathbf{q}_{11}\boldsymbol{\pi}_1^*(-Q_{11})^{-1},$$
where $\mathbf{q}_{11} = (-Q_{11})^{-1}E_1^*$. The inverse matrix $A_{00}^{-1}$ is calculated by usual method.

(4) Calculate $A_{11}^{*-1}$ by the formula,

$$\begin{aligned} A_{11}^{*-1} &= (A_{11}+A_{10}(-A_{00}^{-1})A_{01})^{-1} \\ &= A_{11}^{-1}+A_{11}^{-1}A_{10}A_{00}^{*-1}A_{01}A_{11}^{-1}, \end{aligned}$$

where $A_{00}^{*-1}$ is calculated by usual method.

(5) The $(i,j)$ block $\Psi[i,j]$ of $\Psi$ is

$$\Psi[i,j] = \begin{cases} A_{00}^{*-1}, & i=0, j=0, \\ -A_{00}^{-1}(A_{01}A_{11}^{*-1})[j], & i=0, 1\le j\le K, \\ -(A_{11}^{*-1}A_{10})[i]A_{00}^{-1}, & 1\le i\le K, j=0, \\ A_{11}^{*-1}[i,j], & 1\le i,j\le K. \end{cases}$$

*4. Calculation of* $\exp(Qt)=\sum_{n=0}^{\infty}\frac{t^n}{n!}Q^n$. We use the uniformization technique. Let

$$q=\max_{z\in\mathcal{S}}([-Q]_{zz})$$

and $\Theta = I+\frac{1}{q}Q$. Then

$$\exp(Qt)=Q_M(t)+\mathcal{E}^{(M)}(t),$$

where

$$Q_M(t)=\sum_{n=0}^{M}e^{-qt}\frac{(qt)^n}{n!}\Theta^n,$$

$$\mathcal{E}^{(M)}(t)=\sum_{n=M+1}^{\infty}e^{-qt}\frac{(qt)^n}{n!}\Theta^n.$$

For given $\varepsilon>0$, let $M(\varepsilon)$ be the positive integer such that

$$1-\sum_{n=0}^{M(\varepsilon)}e^{-qt}\frac{(qt)^n}{n!}<\varepsilon.$$

For large $t$, the following addition formula is useful. First, take an integer $n_0$ such that $t_0=\frac{qt}{n_0}$ is moderate with $\mathcal{E}^{(M)}(t_0)\mathbf{e}<\varepsilon_0\mathbf{e}$. Note that

$$\begin{aligned} \exp(Qt) &= [\exp(Qt_0)]^{n_0} = \left[Q_M(t_0)+\mathcal{E}^{(M)}(t_0)\right]^{n_0} \\ &= [Q_M(t_0)]^{n_0}+\mathcal{E}^{(M)}(t_0,n_0), \end{aligned}$$

where

$$\mathcal{E}^{(M)}(t_0,n_0)=\sum_{k=1}^{n_0}\binom{n_0}{k}\left[\mathcal{E}^{(M)}(t_0)\right]^{n_0-k}[Q_M(t_0)]^k.$$

Since $Q_M(t_0)\mathbf{e}<(1-\varepsilon_0)\mathbf{e}$ and $\mathcal{E}^{(M)}(t_0)\mathbf{e}<\varepsilon_0\mathbf{e}$, it can be seen that

$$\mathcal{E}^{(M)}(t_0,n_0)\mathbf{e}<(1-(1-\varepsilon_0)^{n_0})\mathbf{e}.$$

*5. Calculation of* $P(n,t)$. Let $\Theta_0=I+\frac{1}{q}D_0$. Applying the uniformization technique to $P(n,t)$ and using the the Kolmogorov equation (1), it can be seen that

$$P(k,t)=\sum_{n=0}^{\infty}e^{-qt}\frac{(qt)^n}{n!}K_k^{(n)},\ k\ge 1,$$

where $\{K_k^{(n)}\}$ satisfies the followings: for $k=1,2,\cdots$,

$$K_k^{(n+1)}=\frac{1}{q}K_{k-1}^{(n)}D_1+K_k^{(n)}\Theta_0,\ n=0,1,2,\cdots \quad (4)$$

with $K_k^{(0)}=0$, $k\ge 1$ and $K_0^{(0)}=I$ and

$$K_0^{(n+1)}=K_0^{(n)}\Theta_0,\ n\ge 0.$$

The recursive formula (4) is also given in (Lucantoni, 1991). Let

$$\mathcal{E}_k^{(M)}=P(k,t)-\sum_{n=0}^{M}e^{-qt}\frac{(qt)^n}{n!}K_k^{(n)},\ k\ge 0.$$

Note that

$$\exp(Qt)=\sum_{k=0}^{\infty}P(k,t)=\sum_{n=0}^{\infty}e^{-qt}\frac{(qt)^n}{n!}\sum_{k=0}^{\infty}K_k^{(n)}.$$

It can be seen from

$$\Theta^n=\sum_{k=0}^{\infty}K_k^{(n)},\ n\ge 0$$

that $K_k^{(n)}\mathbf{e}<\mathbf{e}$ and hence

$$\mathcal{E}_k^{(M(\varepsilon))}\mathbf{e}<\varepsilon\mathbf{e},\ k\ge 0.$$

# 5 NUMERICAL RESULTS

We apply the algorithms in section 4 to the system with two-node tandem queue with a finite buffer and server breakdown. We consider the system with service rates $\mu_1=\mu_2=1.0$, failure rates $\nu_1=0.1$, $\nu_2=0.04$ and repair rates $\eta_1=0.5$, $\eta_2=0.2$. The isolated efficiency of each server is the same as $\frac{\eta_i}{\nu_i+\eta_i}=0.833$. In this section, we assume that the system is in stationary state.

*1. Speed of convergence to stationary state.* We investigate how fast the distribution of the process $\boldsymbol{Z}$ converges to the stationary distribution $\boldsymbol{\pi}$. It can be seen from (3) that $\Delta_Q(t)=\log||\exp(Qt)-\Pi||$ is almost linear for large $t$. We also have seen that the speed of convergence of $\exp(Qt)$ decreases as buffer size increases. For example, the time $t_s(b):=\min\{t>0:\Delta_Q(t)<-5\}$ are $t_s(3)=74$, $t_s(5)=99$, $t_s(7)=134$.

*2. Variance of* $N(t)$. Figures 1 and 2 show variance $\sigma^2(t)$ and the difference $\Delta_{\sigma^2}(t)=\sigma^2(t)-\tilde{\sigma}^2(t)$. Figure 1 exhibits that $\sigma^2(t)$ increases almost linearly as $t$ increases as expected in the formula in Theorem 3.1. It can be seen from fig. 2 that $\tilde{\sigma}^2(t)$ can be used instead of $\sigma^2(t)$ for $t\ge t_s$. In fact, it follows from Theorem 3.1 and (3) that for $t_s$ with $\Delta_Q(t_s)<\log_{10}\varepsilon$,

$$|\Delta_{\sigma^2}(t)|<2|\mathbf{cd}|\varepsilon,\ t\ge t_s.$$

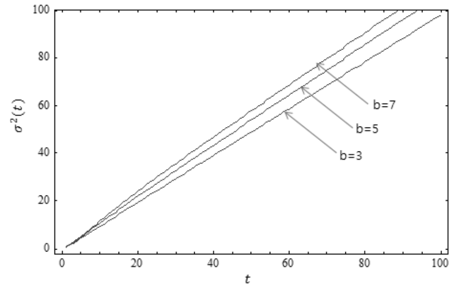


Figure 1: The variance $\sigma^2(t)$.

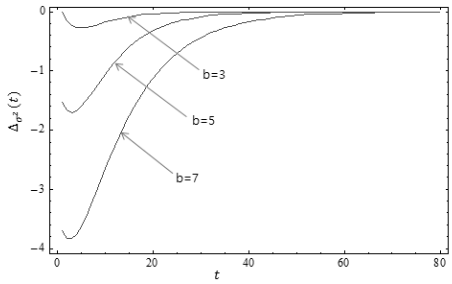


Figure 2: Differences $\Delta_{\sigma^2}(t) = \sigma^2(t) - \tilde{\sigma}^2(t)$.

Indeed, for $b = 5$, $\Delta_{\sigma^2}(64) < 10^{-3}$, $\Delta_{\sigma^2}(81) < 10^{-4}$ and $\Delta_{\sigma^2}(99) < 10^{-5}$.

*3. Covariance of $N(t)$.* Figure 3 depicts the $\mathrm{Cov}[t] = \mathrm{Cov}[N(t), N(2t) - N(t)]$ and $\lim_{t\to\infty} \mathrm{Cov}[t] = \boldsymbol{\pi}\mathbf{d} - \lambda^2$. Figure 3 shows that $\mathrm{Cov}[t]$ is positive for $b = 3$ and negative for $b = 5$.

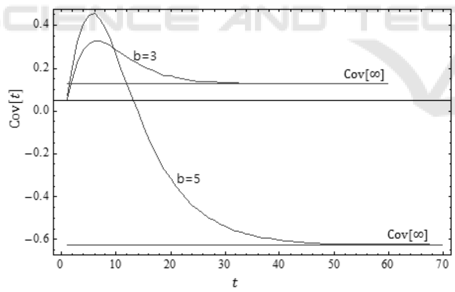


Figure 3: Covariance $\mathrm{Cov}[N(t), N(2t) - N(t)]$ for $b = 3, 5$.

*5. Distribution of $N(t)$.* The distribution of $N(t)$ is depicted in figure 4 for $t = 30$, $t = 50$, $t = 70$. The figures show that the distribution of $N(t)$ visually resembles the normal distribution. The the pair (skewness, kurtosis) of $N(t)$ are $(-0.2599, 0.5054)$, $(-0.2393, 0.1739)$ and $(-0.2141, 0.1486)$ for $t = 30$, $t = 50$ and $t = 70$, respectively. Here, we approximate the distribution of $N(t)$ in stationary state with the normal distribution $N(\mu(t), \sigma^2(t))$ with mean $\mu(t) = \lambda t$ and variance $\sigma^2(t)$ as Tan (1999b), that is,

$$P(N(t) \geq n) \approx 1 - \Phi\left(\frac{n - 0.5 - \lambda t}{\sqrt{\sigma^2(t)}}\right), \qquad (5)$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\, dy$ is the distribution function of the standard normal distribution and $n - 0.5$ is used for correction of the approximation of discrete random variable using continuous distribution and $\tilde{\sigma}^2(t)$ can be used as an approximation of $\sigma^2(t)$ for large $t$.

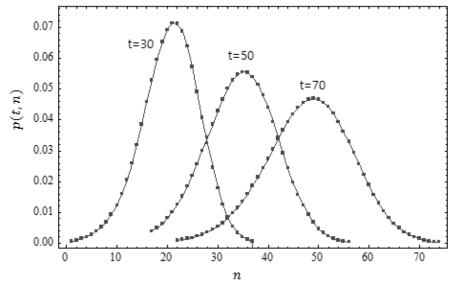


Figure 4: Plot of $p(n,t) = P(N(t) = n)]$ for $b = 5$.

The approximation errors $\Delta_N(t)$ between $P(N(t) \geq n)$ and normal approximation are depicted in figure 5 for $t = 30$, $t = 50$, $t = 70$ and $b = 5$. The maximal error of approximation occurs at the mean $\lambda t$ for each case. Figure shows that the accuracy increases as $t$ increases.

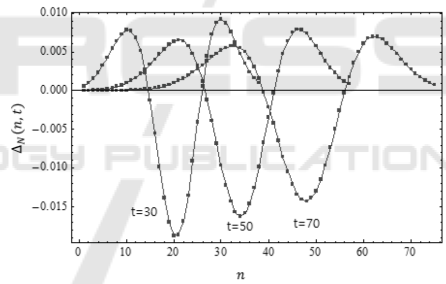


Figure 5: Error of normal approximation for $P(N(t) \geq n)$.

*6. Due time performance.* The due-time performance of a production line can be measured by a probability

$$p = P(N(t^*) \geq n^*)$$

of meeting a customer's order $n^*$ on time $t^*$. Some numerical results for $t^*$ for given $n^*$ and $p$ are listed in Table 1.

Table 1: Due time $t^*$

| | $p$ | | | | | |
|---|---|---|---|---|---|---|
| $n^*$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.99 |
| 50 | 72 | 76 | 79 | 84 | 91 | 109 |
| 70 | 101 | 105 | 110 | 115 | 123 | 143 |
| 100 | 145 | 150 | 155 | 161 | 170 | 193 |

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## 6 CONCLUSIONS

We have provided an algorithm for the transient behavior, the variance and covariance structure for the output process and inter-departure time in two-node tandem queue. Some numerical results are presented. We also showed that the results can be applied to derive approximate formulae for the due-date performance and the distribution of the number of outputs in a time interval.

The algorithm is based on the Markovian arrival process (MAP) which gives closed formula for variance and asymptotic variance. This is a different point from the other methods in the literature for variance of departure process. The algorithm requires only the inversions of the block matrices of size 4 in the computing process. Thus the computational complexity of the algorithm dose not severely depend on the buffer size of the system. The approach using MAP can be easily applied to the system with more general service, failure and repair time than exponential case.

Although the method developed in this paper is quite efficiently, it will be limited to apply the method to the system with multiple nodes due to the rapid increase of the number of states when the number of stations and the buffer capacities increase. Therefore developing approximation methods to estimate the second moment measures in multiple node system are required. There are many approximation methods for throughput in a complicated system, for example, decomposition method and aggregation method (Dallery and Gershwin, 1992; Li et al., 2009) that use the the two-node system. The method of analyzing the two-node system can be used as a building block of analyzing the more complex system.

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