Relaxation and Optimization of Impulsive Hybrid Systems
Inspired by Impact Mechanics

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Abstract: The paper compresses some results on modeling and optimization in a class of hybrid systems with control switches of dynamics. The study is motivated by widespread physical phenomena of impulsive nature, faced in contact dynamics, such as unilateral contacts of rigid bodies and impactively blockable degrees of freedom. The developed modeling approach is based on a representation of hybrid events as impulsive control actions produced by distributions or Borel measures under constraints on states before and after the action. Basically, such systems are described by measure differential equations with states of bounded variation, and the relations between the trajectory and the control measure are given by a specific mixed condition of a complementarity type. The main goal of the study is to describe the closure of the tube of solutions to the addressed system. For this, we design an approximation of the hybrid property, and develop a specific singular time-spatial transformation of the original system. A convexification of the transformed system then defines – after the inverse transform – the closed set of generalized, limit solutions. The main result concerns the asymptotic behavior of these generalized solutions, stating that the hybrid property is preserved after the relaxation.

1 INTRODUCTION

Control systems with affine impulses present one of the most popular objects of research in the modern control theory (Arutyunov et al., 2011; Bressan and Rampazzo, 1993; Bressan and Rampazzo, 1994; Dykhta, 1990; Dykhta, 1997; Dykhta and Samsonyuk, 2000; Dykhta and Samsonyuk, 2001; Gurman, 1991; Gurman, 1997a; Gurman, 1997b; Gurman, 2004; Krotov, 1960; Krotov, 1961b; Krotov, 1961a; Krotov, 1989; Miller, 2011; Pereira and Vinter, 1986; Vinter and Pereira, 1988; Rishel, 1965; Warga, 1965; Warga, 1972; Zavalishchin and Sesekin, 1997). From the mathematical viewpoint, such a system is the result of a trajectory relaxation (below we give a formal definition of this notion) of a control-affine ordinary system of the form

$$\dot{x} \doteq \frac{dx(t)}{dt} = f(t,x) + G(t,x)\,u \tag{1}$$

driven by the input signal $u = u(\cdot)$ being a Lebesgue (or, possibly, Borel) measurable and summable (L_1) function. Here, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are state and control vectors, respectively; $f:\ \mathbb{R}^n \to \mathbb{R}^n$, $G:\ \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given vector and matrix functions.

Aimed at studying the system evolution from an initial state $x(0) = x_0 \in \mathbb{R}^n$ over a certain finite time period $\mathcal{T} \doteq [0,T]$, one can meaningfully suppose an a priori bound $M > 0$ on the "total control action" available during $\mathcal{T}$. In other words, one can impose the condition

$$\|u\|_{L_1} \doteq \int_{\mathcal{T}} |u|\,dt \leq M. \tag{2}$$

Under standard regularity assumptions on the functions f and G, this implies a uniform bound on the total variations of Carathéodory solutions $x = x[u](\cdot)$ under all controls satisfying (2). At the same time, one can design a sequence of Carathéodory solutions that tends to a discontinuous function, which is not admitted by (1). This fact entails that the tube of solutions to the system is not compact in the space C of continuous functions. As a consequence, one fails to guarantee the existence of a solution to an optimal control problem stated for system (1), (2). Still, the mentioned uniform estimate on the total variation of trajectories enables us to design a compactification of the trajectory tube (or reachable sets) in a certain topology of the space BV of functions with bounded variation. If the function G does not depend on x, the extension is given by the measure differential equation

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$$dx = f(t,x)\,dt + G(t)\,\mu(dt). \tag{3}$$

Literally, this is done by multiplying equation (1) by dt, and further replacing the term $u\,dt$ with a formal differential dU of a, possibly, discontinuous function U of bounded variation. The latter defines the so-called differential, or a Lebesgue-Stieltjes measure $\mu(dt) \doteq dU(t)$, which is a first order distribution, possibly, of the Dirac ("δ-") type.

If G does depend on x (*for brevity, in what follows we get rid of the explicit dependence of functions f and G on the time variable*), the operation $G(x)d\mu$ is formally incorrect, as it is, in general, the product of a discontinuous function — the composition $G \circ x$ — and a Dirac distribution concentrated at discontinuity points of $G \circ x$. In this case, a correct trajectory relaxation is provided by a space-time extension (Gurman, 1997a; Gurman, 1997b; Miller, 2011; Warga, 1965), based on the principle of *separation of motions*. By tradition, the relaxed dynamics is also written in the form of the measure differential equation:

$$dx = f(x)\,dt + G(x)\,\mu(dt), \tag{4}$$

which now is just a formal, conceptual object.

Let us stress that equations (3), (4) describe deterministic processes. The readers familiar with stochastic differential equations can note a certain similarity. However, stochastic processes are generically driven not by measures, but by rough paths generated by functions of unbounded variation (in fact, of bounded p-variation with a certain $p > 1$ (Lyons et al., 2007)).

1.1 Mixed Constrained Measure Differential Equations: Complementarity Problem

In (Goncharova and Staritsyn, 2012) we first find the following complementarity problem for the measure differential equation (4):

$$x(t^-) \in \mathcal{Z}_- \text{ and } x(t) \in \mathcal{Z}_+ \quad |\mu|\text{-a.e.} \tag{5}$$

Here, $x(t^-)$ denotes the left one-sided limit of a function x at a point t (throughout the paper we operate with *right continuous* functions; this is just a technical assumption and does not lead to loss of reasonable generality); "$|\mu|$-a.e." means "almost everywhere with respect to the total variation of the measure μ"; $\mathcal{Z}_\pm \subseteq \mathbb{R}^n$ are given *closed* (not necessarily bounded) subsets of the state space. Condition (5) establishes the hybrid property, i.e., it *prescribes* possible system configurations before and after switches of state. In convention of hybrid systems theory (Branicky et al., 1998; Haddad et al., 2006; Teel et al., 2012; van der Schaft and Schumacher, 2001), the sets $\mathcal{Z}_\pm$ are called the *jump permitting* and *jump destination* sets, respectively. Without loss of generality we can suppose that $\mathcal{Z}_\pm$ are defined as $\mathcal{Z}_\pm = \{x \in \mathbb{R}^n |\ W_\pm(x) = 0\}$, where $W_\pm$ are certain *nonnegative continuous* functions $\mathbb{R}^n \to \mathbb{R}$. Then one can rewrite (5) in the equivalent form:

$$\int_{\mathcal{T}} \left[W_-(x) + W_+(x)\right]\, |\mu|_c(dt) = 0, \text{ and} \tag{6}$$

$$W_-\big(x(\tau^-)\big) + W_+\big(x(\tau)\big) = 0 \quad \forall \tau \in D_{|\mu|}. \tag{7}$$

For absolutely continuous measures μ, the condition (5) just expresses the orthogonality in L_2 between the control u and the composition $(W_+ + W_-) \circ x$.

Conditions of the sort (5) were originally introduced based on methodological considerations — as a "natural form" of mixed constraints in problems of impulsive control — and were promptly recognized as a useful mathematical framework for a representation of a class of hybrid systems and for modeling certain effects in impact mechanics to be discussed in the following Section. Since system (4), (5) performs the hybrid feature produced by impulsive control, we call this class of models *impulsive hybrid systems*.

1.2 Mechanical Inspiration. Motivating Examples

Affine impulses naturally appear in the framework of impact mechanics and contact dynamics as a mathematical formalization of elastic collision with unilateral constraints and dry friction (Acary et al., 2011; Brogliato, 2016; Glocker, 2001b; Glocker, 2001a; Kozlov and Treshchëv, 1991; Moreau, 1966; Moreau, 1979a; Moreau, 1979b; Yunt and Glocker, 2007; Yunt, 2011).

As is well known, in the smooth case the rigid body dynamics of generalized coordinates $(q, \dot q)$ can be described by the classical Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = F$$

with the Lagrangian $L = K - P$, the total kinetic energy $K = K(q, \dot q)$ and the total potential energy $P = P(q)$. Here the term F comprises all external forces acting on the mechanical system, in particular, control forces.

If the dynamics is non-smooth (say, admits collisions), the equations of motion are better written in the form

$$\mathbf{M}(q)\,\ddot q - \mathbf{h}(q, \dot q) = \mathbf{f} + \mathbf{B}(q)\,u. \tag{8}$$

Here, the forces are composed of the uncontrolled *external* force $\mathbf{f}$ and the control force u, the (symmetric, positive definite) generalized mass matrix $\mathbf{M} = \mathbf{M}(q)$

is defined as $\mathbf{M} = \partial^2_{\dot{q}\dot{q}}K$, the function $\mathbf{h} = \mathbf{h}(q,\dot{q})$ collects finite smooth forces (spring, damper, centripetal, gyroscopical, coriolis etc.) and is determined by $\mathbf{h} = \partial^2_{\dot{q}q}K - \partial_q L$, and $\mathbf{B} = \mathbf{B}(q)$ is a linear mapping that represents directions of control forces.

J.J. Moreau proposed to generalize the equations of motion (8) up to the measure-driven system

$$\mathbf{M}(q)\, d(\dot{q}) - \mathbf{h}(q,\dot{q})\, dt = \pi(dt) + \mathbf{B}(q)\mu(dt). \quad (9)$$

Here, measures π and μ formalize the generalized uncontrolled and control external forces, respectively.

Note that, compared to the previous Section, here the impulsive behavior is *a priori postulated* rather than thought of as an idealized mathematical phenomenon, originated by a sequence of smooth models.

As an illustration, we consider the following toy example.

Example 1. “Klapstos” strike in billiard. The model presents two “balls” (simplified to material points of unit mass) moving without friction in the line. The “klapstos” problem is to strike a ball x by applying a force $\rho \geq 0$ (the billiard cue) such that x collides with another ball y, and the balls exchange their momenta at the instant of collision: the incoming ball stops at the contact position, while the other ball, being in rest up to this moment, acquires a translational velocity and starts moving. The simplified dynamics takes the form

$$d(\dot{x}) = (\rho - \mu)(dt), \quad d(\dot{y}) = k\,\mu(dt),$$

where $k \in (0,1)$ is the coefficient of restitution, and $\mu \geq 0$ is the complementary contact force. Assumed that

$$(x,y)(0) = (x_0,y_0), \quad (\dot{x},\dot{y})(0) = (0,0),$$

and $y_0 > x_0$, the maneuver can be formalized as the condition

$$x(t^-) = y_0, \quad \dot{x}(t) = 0 \quad \mu\text{-a.e.}$$

In general, the force μ could also play the part of control; this case refers us to the framework of *impulsive control in the phase of unilateral contact* (Bentsman et al., 2012).

We can mark out two (to some extent, standard) approaches to modeling of contact dynamics: the so-called regularized and non-smooth approaches. They are well known, we refer to (Brogliato, 2016; Glocker, 2001b; Moreau, 1966) and the bibliography therein. Yet another approach (Bentsman et al., 2007; Bentsman et al., 2007; Bentsman et al., 2008; Bentsman et al., 2012; Miller and Bentsman, 2006) is based on active singularities, and it is mostly close in the methodological sense to the one we develop in the present study. Note that these approaches do not give a solution, if impulse forces of impacts act simultaneously with dry friction (Pfeiffer and Glocker, 1996). Such phenomena are known as Painlevé paradoxes (Painlevé, 1895; Stewart, 2000). Successful attempts to overcome this problem are made in (Miller et al., 2016; Stewart, 1997).

The classical approaches operate with forces or torques as driving signals. However, in practical applications these physical quantities are technically hardly measurable signals. Sometimes, one is compelled to design control signals based on the observation of their influence on the system or relying on some average characteristics of control forces. In (Bressan and Rampazzo, 1993; Bressan and Wang, 2009; Bressan and Rampazzo, 2010; Bressan et al., 2013) it is proposed to control Lagrangian systems directly by a part of state coordinates. This lets one avoid a straightforward computation of driving forces, since their effects are already accounted by frictionless holonomic constraints provided by the control coordinates. Our approach — based on a complementarity problem for the measure differential equation — enjoys the same advantages: condition (5) contains the complete information about the system behavior prior to and after applying control forces. This information lets us avoid calculation of effortfully observable signals.

A pretty eloquent example of real-life models, where control forces produce such a *prescribed influence on the system*, is given by Lagrangian systems with impactively blockable degrees of freedom. The following example is aimed to illustrate the idea.

Example 2. A double pendulum with a blockable joint. The model describes planar motion of a double pendulum with links of unit length and mass, actuated by gravity. Two degrees of freedom of the system are due to angular positions $\varphi = (\varphi_1, \varphi_2)$ of the links. The total kinetic energy is

$$K = \dot{\varphi}_1^2 + 1/2\,\dot{\varphi}_2^2 + \varphi_1\varphi_2\cos\Delta\varphi,$$

and the total potential energy is

$$P = -g\,(2\cos\varphi_1 + \cos\varphi_2),$$

where $\Delta\varphi = \varphi_1 - \varphi_2$, and g is the acceleration of gravity.

Denote by $\omega = (\omega_1, \omega_2) \doteq (\dot{\varphi}_1, \dot{\varphi}_2)$ the vector of angular velocities; the states are now (φ, ω). Suppose that one can control the pendulum by instantaneously blocking/releasing the joint between the two links. Such blocking is provided by an impulsive force, performed by a scalar signed Borel measure μ. In the inertial reference system, the equations of motion then should be written as

$$\dot{\varphi} = \omega,$$
$$\begin{pmatrix} 1 & 1/2\cos\Delta\varphi \\ \cos\Delta\varphi & 1 \end{pmatrix} \begin{pmatrix} d\omega_1 \\ d\omega_2 \end{pmatrix} +$$
$$\begin{pmatrix} 1/2\sin\Delta\varphi\ \omega_2^2 \\ -\sin\Delta\varphi\ \omega_1^2 \end{pmatrix} dt + g \begin{pmatrix} \sin\varphi_1 \\ \sin\varphi_2 \end{pmatrix} dt =$$
$$\begin{pmatrix} 0 \\ \mu(dt) \end{pmatrix},$$

while the blocking condition takes the form (5) with $\mathcal{Z}_- = \mathbb{R}^4$, and $\mathcal{Z}_+ = \{x = (x_1, \ldots, x_4) \in \mathbb{R}^4 : x_3 = x_4\}$, i.e., in our terms,

$$\omega_1(t) = \omega_2(t) \quad |\mu|\text{-a.e.}$$

The latter condition postulates the configuration $\mathcal{Z}_+ = \{(\varphi_1, \varphi_2, \omega_1, \omega_2) \in \mathbb{R}^4 : \ \omega_1 = \omega_2\}$ after any jump of the trajectory component ω_2.

1.3 Notations and Basic Mathematical Background

By $\mathbb{N}$ we denote the set of positive integers, and $\mathbb{R}^n$ is the n-dimensional arithmetic space with the Manhattan norm $|\cdot| \doteq \|\cdot\|_1$, $\mathbb{R} = \mathbb{R}^1$ is the set of real numbers, and $\mathbb{R}^n_+$ is the cone of vectors with nonnegative components.

Given a finite interval $\mathcal{T} \doteq [0,T] \subset \mathbb{R}_+$, let $C = C(\mathcal{T}, \mathbb{R}^n)$ denote the Banach space of n-dimensional continuous functions on $\mathcal{T}$ with the topology of uniform convergence $\rightrightarrows$, and $AC = W^{1,1} \subset C$ denote the set of absolutely continuous vector functions; $L^1 = L^1(\mathcal{T}, \mathbb{R}^n)$ is the Lebesgue quotient space of summable functions.

The dual C^* of C is known to be the space of vector-valued signed Borel measures, i.e., countably additive set functions $\mu : \mathcal{B} \to \mathbb{R}^n$ defined on the Borel sigma-algebra $\mathcal{B} = \mathcal{B}_{\mathcal{T}}$ of subsets of the interval $\mathcal{T}$. Among all measures we single out the usual Lebesgue measure on $\mathbb{R}$, denoted by λ. A classical fact from Analysis claims that any Borel measure admits a unique Lebesgue-Stieltjes extension, and the set of extended measures is isomorphic to the space $BV = BV^+(\mathcal{T}, \mathbb{R}^n)$ of right continuous vector functions with bounded variation on $[0,T)$. The extended measures are called the Lebesgue-Stieltjes measures. In what follows, the term "measure" is understood in the extended sense.

Given a measure μ, we denote by $|\mu|$ its total variation, by $\text{supp}\mu$ its support, i.e., the minimal closed subset of $\mathcal{T}$ such that $\mu(\text{supp}\mu) = \mu(\mathcal{T})$; by D_μ the set $\{\tau \in \mathcal{T} |\ |\mu|(\{\tau\}) > 0\}$ of its atoms, and by $F_\mu \in BV$ its distribution function defined as $F_\mu(t) = \mu([0,t])$, $F_\mu(0^-) = 0$. Any measure μ can be thought of as the distributional derivative or "differential" of its distribution function, i.e., $\mu(dt) = dF_\mu(t)$. Given $\mu, \nu \in C^*$, $\mu \preceq \nu$ indicates that $\mu(A) \leq \nu(A)$ for any $A \in \mathcal{B}$; $\frac{d\mu}{d\nu}$ stands for the Radon-Nikodym derivative of μ with respect to ν. If $\frac{d\mu}{d\nu} = 0$, the measures μ and ν are said to be *mutually singular*. A Lebesgue-Stieltjes measure admits a *unique Lebesgue decomposition* into the sum of the absolutely continuous μ_{ac}, singular (with respect to λ) continuous μ_{sc}, and singular discrete components μ_d: $\mu = \mu_c + \mu_d \doteq \mu_{ac} + \mu_{sc} + \mu_d$. If μ is absolutely continuous, then $\frac{d\mu}{d\lambda} = \dot{F}_\mu$. All the notions, raised in the context of measures, can be adopted to functions of bounded variation.

1.4 Weak* Topology of BV. Relaxation of System's Trajectory Tube

The weak* topology performs one of the two regular ways to introduce a "coarse" topological structure in the dual X^* of a linear space X. Sometimes, especially in Probability Theory, the weak* topology is called simply the *weak topology*. In Analysis these two notions are distinguished: The weak topology on X^* corresponds to the following notion of convergence of a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset X^*$ to a point $\phi \in X^*$: $\psi(\phi_n) \to \psi(\phi)$ for all $\psi \in X^{**}$. On the other hand, X can be embedded into its double dual X^{**} by the mapping $x \mapsto T_x$, where $T_x(\phi) \doteq \phi(x)$. Thus $T : X \to X^{**}$ is an injective linear mapping (not necessarily surjective). The weak* topology on X^* is the weak topology induced by the image of T, $T(X) \subset X^{**}$. A sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset X^*$ converges to a point $\phi \in X^*$ in the weak* topology iff $\phi_n(x) \to \phi(x)$ for all $x \in X$. Note that the weak* topology of X^* is weaker than the weak topology of this space, and furthermore, it is the weakest topology such that the maps $T_x : X^* \to \mathbb{R}$ are continuous.

In this paper we deal with BV-functions. By the Helley's selection principle, a sequence of functions $\{x_n\}_{n\in\mathbb{N}} \subset BV$ is pre-compact in BV in the topology of pointwise convergence iff $\{x_n\}_{n\in\mathbb{N}}$ is uniformly bounded and has uniformly bounded variation. Note that the compactness in the topology of pointwise convergence implies the compactness in the weak* topology.

The weak* topology of BV (i.e., of C^*) can be specified as follows: A sequence $\{x_n\}_{n\in\mathbb{N}} \subset BV$ converges to a function $x \in BV$ in the weak* topology of BV (we will write $x_n \rightharpoonup x$) iff $x_n(t) \to x(t)$ for all continuity points t of x, and at the boundary points $t = 0$, and $t = T$.

By a *trajectory relaxation* of a control dynamical system we mean a closure of the tube of its solutions (in the accepted sense) in a certain weak topology.

Our study will operate with a specific impulsive solution concept of a measure differential equation. And the problem of its trajectory relaxation will be stated in the weak* topology of BV.

1.5 Notion of Impulsive Control Input, and Solution Concept for Measure-Driven Dynamical Systems

We accept the basic assumptions (H): the functions f and G are uniformly Lipschitz continuous.

Similarly to (Arutyunov et al., 2011; Karamzin et al., 2015), by an impulsive control we mean a collection

$$\vartheta \doteq (\mu, \nu, \{u_\tau\}_{\tau \in D_\nu}).$$

Here,

- $\mu \in C^*(\mathcal{T}, \mathbb{R}^n)$ and $\nu \in C^*(\mathcal{T}, \mathbb{R})$ are measures with
$$|\mu| \preceq \nu, \ |\mu|_c = \nu_c, \text{ and } \nu(\mathcal{T}) \leq M. \tag{10}$$
- $\{u_\tau\}_{\tau \in D_\nu}$ is a family of Borel measurable functions
$$u_\tau : \mathcal{T}_\tau \doteq [0, T_\tau] \to \mathbb{R}^m,$$
$T_\tau \doteq \nu(\{\tau\})$, parameterized by atoms of the measure ν and meeting the constraints
$$|u_\tau(\theta)| = 1 \quad \lambda\text{-a.e. on } \mathcal{T}_\tau, \tag{11}$$
$$\int_{\mathcal{T}_\tau} u_\tau(\theta)\, d\theta = \mu(\{\tau\}) \tag{12}$$

for all $\tau \in D_\nu$.

The "attached" controls u_τ are jump-driving parameters (see the comment below). Since control measures μ are signed, the limit of a sequence $|\mu_n|$ of total variations of weakly* converging measures $\mu_n \rightharpoonup \mu$ may not coincide with $|\mu|$. The nonnegative measure ν appears here to overcome this problem. We set $|\vartheta| \doteq \nu$ and formally regard $|\vartheta|$ as the total variation of impulsive control.

Let Θ denote the set of admissible controls, i.e., all collections ϑ satisfying (10)–(12).

Given $\vartheta = (\mu, \nu, \{u_\tau\}) \in \Theta$, and $x_0 \in \mathbb{R}^n$, by a solution to (4) under the control input ϑ with the initial condition $x(0^-) = x_0$ we mean a function $x \in BV^+(\mathcal{T}, \mathbb{R}^n)$ meeting the following integral relation for each $t \in \mathcal{T}$:

$$x(t) = x_0 + \int_0^t f(x)\, d\theta + \int_0^t G(x)\, \mu_c(d\theta) + \sum_{\tau \in D_\nu} \left[x(\tau) - x(\tau^-)\right]. \tag{13}$$

The integration with respect to the measure μ_c is understood in the Lebesgue-Stieltjes sense; jump exit points $x(\tau)$ of a function x at the instants $\tau \in D_\nu$ of impulses are defined as $x(\tau) = \varkappa_\tau(T_\tau)$, where $\varkappa_\tau$ is a Carathéodory solution of the *limit control system*

$$\frac{d}{d\varsigma} \varkappa(\varsigma) = G(\varkappa(\varsigma))\, u_\tau(\varsigma), \quad \varkappa(0) = x(\tau^-) \tag{14}$$

on the time interval $\mathcal{T}_\tau \doteq [0, T_\tau]$.

Assumptions (H) guarantee the existence and uniqueness of a solution $x[\vartheta]$ to (13), (14) for any input $\vartheta \in \Theta$ (see, e.g., (Miller, 2011)).

The employed concept of impulsive solution is dictated by the fact (generic, if we deal with vector-valued measures) that the result of impulsive action – i.e., the value of the jump of a state – depends on the way of approximation of Dirac distributions by ordinary controls. In other words, the mapping "control measure $\mapsto$ trajectory" is, typically, multivalued, and the selection of this map is implemented by the choice of an attached control u_τ. The input-output map is single-valued when the matrix function G enjoys the so-called Frobenius correctness property (Rampazzo, 1999), which corresponds to the commutativity of the vector fields defined by the columns G_i of the matrix G. In this case, the terms ν and $\{u_\tau\}$ in the definition of impulsive control can be omitted, and the solution concept can be simplified (Miller, 2011).

1.6 Final Model Statement

In view of the given notion of impulsive control, we have to rewrite conditions (5) in the correct form:

$$x(t^-) \in \mathcal{Z}_- \text{ and } x(t) \in \mathcal{Z}_+ \quad |\vartheta|\text{-a.e.} \tag{15}$$

A pair $\sigma \doteq (x, \vartheta)$ with $\vartheta \in \Theta$ and $x = x[\vartheta]$ is called an *impulsive control process*. Note that ordinary control processes (x, u) of system (1), (2) are embedded into the measure-driven system (4) (or (13), (14)) by setting $\mu = u\, d\lambda$.

By Σ we denote the set of all impulsive control processes satisfying the complementarity conditions (5) and the following endpoint constraints:

$$x(0^-) = x_0, \quad x(T) \in \Omega, \tag{16}$$

where $x_0 \in \mathbb{R}^n$ is a fixed initial position, and $\Omega \subseteq \mathbb{R}^n$ is a given closed set representing a desired terminal configuration of the modeled object. We are to assume that $\Sigma \neq \emptyset$.

Our final goal is to design a proper relaxation of the set Σ in connection with related optimization problems.

2 TRAJECTORY RELAXATION OF IMPULSIVE HYBRID SYSTEMS

The study is arranged as follows:

- First, we give a correct definition of ε-approximation (perturbation) of solutions to system (13)–(16).
- As a basic tool for further analysis, we develop a specific singular transformation of the impulsive hybrid system, and establish the equivalence result between the original and transformed models.
- By passing to the weak* limit in BV, the set of perturbed solutions will be extended, and this extension will be found to be greater than Σ. Technically, this will be done by a convexification of the transformed system with a consequent inverse (discontinuous) reparameterization of time.
- Finally, we investigate the asymptotic behavior of the hybrid property (15), and discover its limit form.

2.1 Approximation of Impulsive Hybrid Systems

We start with the following important

Definition 1. Given $\varepsilon > 0$, an impulsive control process $\sigma = (x = x[\vartheta], \vartheta = (\mu, \nu, \{u_\tau\}))$, $\vartheta \in \Theta$, is said to be an ε-*approximate solution of the complementary system (13)–(15)*, if there exists another process $\widetilde{\sigma} \doteq (\widetilde{x} \doteq x[\widetilde{\vartheta}], \widetilde{\vartheta} \doteq (\widetilde{\mu}, \widetilde{\nu}, \{\widetilde{u}_\tau\}))$ such that the following relations hold:

1. $(\widetilde{x}, F_{\widetilde{\nu}})$ belongs to an ε-neighborhood of (x, F_ν) in the weak* topology of BV, i.e.,
$$\big\|(x, F_\nu)(t) - (\widetilde{x}, F_{\widetilde{\nu}})(t)\big\| \leq \varepsilon \text{ for } t \in \big((0,T) \setminus D_\nu\big) \cup \{0\} \cup \{T\}. \tag{17}$$
2. Processes meet the following "ordering" condition:
$$\int_{\mathcal{T}} Q(F_\nu, F_{\widetilde{\nu}})\, d\nu_c + \int_{\mathcal{T}} Q(F_\nu, F_{\widetilde{\nu}})\, d\widetilde{\nu}_c + \sum_{\tau \in D_\nu} Q(F_\nu(\tau), F_{\widetilde{\nu}}(\tau))\, \nu(\{\tau\}) + \sum_{\tau \in D_{\widetilde{\nu}}} Q(F_\nu(\tau), F_{\widetilde{\nu}}(\tau))\, \widetilde{\nu}(\{\tau\}) \leq \varepsilon. \tag{18}$$
Here, $Q = Q(\eta_+, \eta_-)$ is an arbitrary fixed *continuous nonnegative* function $\mathbb{R}^2_+ \to \mathbb{R}$ vanishing only on the set $\{(\eta_+, \eta_-) \in \mathbb{R}^2_+ :\ \eta_- \leq \eta_+\}$.
3. The perturbed version of the complementarity condition (15) is satisfied:
$$\int_{\mathcal{T}} W_-(\widetilde{x})\, d\nu + \int_{\mathcal{T}} W_+(x)\, d\widetilde{\nu} \leq \varepsilon. \tag{19}$$

We remark on conditions (18), (19): inequality (18) is a relaxed version of the pointwise state constraint
$$F_{\widetilde{\nu}} \leq F_\nu. \tag{20}$$
In the asymptotic respect, (20) serves to distinguish jumps $\mathcal{Z}_- \to \mathcal{Z}_+$ compared to $\mathcal{Z}_+ \to \mathcal{Z}_+$. If $\mathcal{Z}_- = \mathcal{Z}_+$, this demand is ambiguous, and (18) can be dropped.

Relation (19) establishes ε-complementarity of a process with respect to its small weak* perturbation (rather than with respect to itself). As we will see below, this is an important invention.

Let us stress that the part of ε-solutions can be played by ordinary control processes defined by (1), (2) (see the forthcoming Example 3). In this respect, Definition 1 solves the problem of a correct approximation of the impulsive hybrid behavior by a regular, ordinary system.

Example 3. On the time interval $[0,1]$, consider the following model:
$$dx = d\mu,\ \ x(0^-) = 0,$$
$$\mu \succeq 0,\ \ \mu([0,1]) \leq 1,$$
$$x(t) = 1 \quad \mu\text{-a.e.}$$
(any control action should steer the state to the set $\mathcal{Z}_+ = \{1\}$).

Note that, for any ordinary control $d\mu = u\,dt$ with $\|u\|_{L_1} = 1$, the complementarity condition does not hold, even approximately:
$$\int_{[0,1]} (1-x)\, u\, dt = x(1) - \frac{x^2(1)}{2} = 1/2.$$
On the other hand, consider the following two families of ordinary control processes:
$$\sigma_\varepsilon(t) \doteq (x,u)_\varepsilon = \begin{cases} (\frac{1}{\varepsilon}t, \frac{1}{\varepsilon}), & t \in [0,\varepsilon), \\ (1,0), & t \in [\varepsilon, 1], \end{cases}$$
$$\widetilde{\sigma}_\varepsilon(t) \doteq (\widetilde{x},\widetilde{u})_\varepsilon = \begin{cases} (0,0), & t \in [0,\varepsilon), \\ (\frac{1}{\varepsilon}t - 1, \frac{1}{\varepsilon}), & t \in [\varepsilon, 2\varepsilon), \\ (1,0), & t \in [2\varepsilon, 1]. \end{cases}$$
The pair $(\sigma_\varepsilon, \widetilde{\sigma}_\varepsilon)$ satisfies the assumptions of Definition 1, in particular, (19) collapses into
$$\int_{[0,1]} (1 - x_\varepsilon)\, \tilde{u}_\varepsilon\, dt = 0.$$

Let $\overline{\mathbb{X}}$ denote the set of functions $x \in BV^+(\mathcal{T}, \mathbb{R}^n)$ such that there exists a sequence $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ of impulsive control processes $\sigma_\varepsilon = (x, \vartheta)_\varepsilon$ with the following properties:

- for any $\varepsilon > 0$, σ_ε satisfies constraints (16) with accuracy to within ε, i.e., $|x_\varepsilon(0) - x_0| \le \varepsilon$ and $x_\varepsilon(T) \in U_\varepsilon(\Omega)$, where U_ε stands for the ε-neighborhood of a set;
- σ_ε is an approximate ε-solution of (4), (15) in the sense of Definition 1, and
- $x_\varepsilon \rightharpoonup x$ as $\varepsilon \to 0$.

The next step is to design a constructive representation of $\overline{\mathbb{X}}$.

3 TRAJECTORY EXTENSION: SEPARATION OF MOTIONS AND DISCONTINUOUS SPACE-TIME TRANSFORM

In this Section, we develop an equivalent transformation of the impulsive hybrid system to an ordinary control system with bounded inputs and absolutely continuous states. The main technical trick is a specific Lipschitzian parameterization of the time variable t named the *discontinuous time change* (Bressan and Rampazzo, 1994; Miller, 2011; Warga, 1965; Warga, 1972).

The idea originates in the general approach of "separation of motions". When operating with system (13)–(15) we implicitly separate the following essences:

- *Separation in time* – "slow" motions versus "fast" dynamics. Informally speaking, the continuous component of a solution to the measure differential equation (4) (performing slow motions) and solutions to the limit system (14) "live" in different time scales.
- *Separation in space* – the left and right one-sided limits of an impulsive solution play the parts of different (independent) states. Here one can return to Definition 1 and observe that it operates with a couple of control processes. It is not hard to see that – asymptotically – $\tilde{x}$ corresponds to the left one-sided limit of a generalized solution, while x describes the right one.

The following time-space transformation puts all the separated objects in a common scale: the instants of impulses will be extended into intervals and the dimension of the state space will increase.

On a time interval $\mathcal{S} \doteq [0,S]$, $S \geqslant T$, we consider the following auxiliary ordinary control system:

$$\frac{d}{ds}\xi = \alpha,\ \frac{d}{ds}y_\pm = \alpha f(y_\pm) + G(y_\pm)\,\beta_\pm, \quad (21)$$

$$\tfrac{d}{ds}\eta_\pm = |\beta_\pm|, \quad (22)$$

$$\begin{aligned}\frac{d}{ds}\zeta &= \alpha\big(\Delta^\pm\eta + |\Delta^\pm y|\big)\\ &+ \big(|\beta_+| + |\beta_-|\big)\,Q(\eta_+,\eta_-)\\ &+ |\beta_+|\,W_-(y_-) + |\beta_-|\,W_+(y_+),\end{aligned} \quad (23)$$

$$y_\pm(0) = x_0, \quad (\xi,\eta_+,\eta_-,\zeta)(0) = 0 \in \mathbb{R}^4, \quad (24)$$

$$\xi(S) = T,\ \Delta^\pm(y,\eta)(S) = 0 \in \mathbb{R}^{n+1},\ \zeta(S) = 0, \quad (25)$$

$$y_+(S) \in \Omega, \quad \eta_+(S) \le M, \quad (26)$$

$$(\alpha,\beta_+,\beta_-) \in \mathbf{U}. \quad (27)$$

Here, s is a new "extended" time; the control set $\mathbf{U} \doteq \mathbf{U}(S)$ is formed by Borel measurable control functions $\mathbf{u} \doteq (\alpha,\beta_+,\beta_-)$, where $\alpha,\beta_\pm : \mathcal{S} \to \mathbb{R}$ are such that $\alpha(s) \ge 0$ and $\alpha(s) + |\beta_+(s)| + |\beta_-(s)| = 1$ λ-a.e. over the interval $\mathcal{S}$.

States of the reduced system are presented by $\mathbf{x} \doteq (\xi, y \doteq (y_+,y_-), \eta \doteq (\eta_+,\eta_-), \zeta) \in \mathbb{R}^{2n+4}$, where $y_\pm(s) \in \mathbb{R}^n$, and $\xi(s),\eta_\pm(s),\zeta(s) \in \mathbb{R}^+$.

The operation $\Delta^\pm$ applied to a vector c of the structure (c_+,c_-), $c_\pm \in \mathbb{R}^r$, returns the vector $c_+ - c_-$, and $\Delta^\mp = -\Delta^\pm$.

By $\mathbf{x}[\mathbf{u}]$ we denote a Carathéodory solution of system (21)–(24) on the interval $\mathcal{S}$, under control $\mathbf{u} \in \mathbf{U}$.

Given an impulsive control $\vartheta \in \Theta$, we denote $\hat{\mu} \doteq \lambda + 2|\vartheta|$ and introduce a strictly increasing function

$$\Upsilon :\, \mathcal{T} \to [0,\hat{\mu}(\mathcal{T})]$$

as follows:

$$\Upsilon(t) = F_{\hat{\mu}}(t),\ t \in \mathcal{T}.$$

Since Υ is strictly monotone, there exists its inverse $[0,\hat{\mu}(\mathcal{T})] \to \mathcal{T}$, which we denote by υ.

The following theorem, proved in (Goncharova and Staritsyn, 2015), claims that systems (13)–(16) and (21)–(27) are equivalent to each other.

Theorem 1 1) Let $\vartheta \in \Theta$ be such that the solution $x = x[\vartheta]$ of (13), (14) meets conditions (15) and (16). Then, there exist a real $S \ge T$ and a control $\mathbf{u} \in \mathbf{U}(S)$ such that the solution $\mathbf{x} = \mathbf{x}[\mathbf{u}]$ of control system (21)–(24) satisfies the terminal constraints (25), (26), and

$$y_- \circ \Upsilon = y_+ \circ \Upsilon = x, \quad \upsilon = \xi. \quad (28)$$

2) Assume that $S \ge T$ and $\mathbf{u} \in \mathbf{U}(S)$ are chosen such that the related solution $\mathbf{x} \doteq (\xi,y,\eta,\zeta)[\mathbf{u}]$ of system (21)–(24) satisfies constraints (25), (26). Define the function $x \in BV^+(\mathcal{T},\mathbb{R}^n)$ by the composition

$$x = y_+ \circ \Xi \ \text{ on } \mathcal{T}, \quad (29)$$

where $\Xi:\ \mathcal{T}\to\mathcal{S}$ is given by

$$\Xi(t)=\inf\{s\in\mathcal{S}:\ \xi(s)>t\},\ t\in[0,T),\qquad \Xi(T)=S. \tag{30}$$

Then, x satisfies (13), (14) together with constraints (15), (16).

Note that the function Ξ defined in (30) is strictly monotone increasing, and right continuous. Furthermore, it is a pseudo-inverse of ξ, that is, the composition $\xi\circ\Xi$ coincides with the identity mapping Id on $\mathcal{T}$, while $\Xi\circ\xi=\mathrm{Id}$ for continuity points $t=\xi(s)$ of Ξ (Miller, 2011).

The proof of Theorem 1 is based on the following formulas for the direct and inverse transforms.

Direct transformation. Given $\vartheta=(\mu,\nu,\{u_\tau\})\in\Theta$, put $S\doteq\Upsilon(T)$, and introduce the control functions

$$\alpha(s)\doteq\begin{cases}(m_1\circ\upsilon)(s), & \upsilon(s)\in\operatorname{supp}\nu_{ac}\\ 0, & \text{otherwise;}\end{cases} \tag{31}$$

$$\beta_\pm(s)\doteq\begin{cases}(u_\tau\circ\theta_{\tau\pm})(s), & \exists\tau\in D_\nu,\\ & \text{s.t. } s\in\mathcal{S}_{\tau\pm},\\ (m_2\circ\upsilon)(s)\alpha(s), & \upsilon(s)\in\operatorname{supp}\nu_{ac},\\ (m_3\circ\upsilon)(s), & \upsilon(s)\in\operatorname{supp}\nu_{sc},\\ 0, & \text{otherwise.}\end{cases} \tag{32}$$

Here, $m_1\doteq\dfrac{d\lambda}{d\hat\mu}$, $m_2\doteq\dfrac{d\mu_{ac}}{d\lambda}$, $m_3\doteq\dfrac{d\mu_{sc}}{d\hat\mu}$ (the derivatives are regarded in the Radon-Nikodym sense); $\mathcal{S}_{\tau+}\doteq\Upsilon(\tau^-)+\nu(\{\tau\})$, $\mathcal{S}_{\tau-}\doteq[\Upsilon(\tau^-),\Upsilon(\tau)]\setminus\mathcal{S}_{\tau+}$; $\theta_{\tau+}(s)\doteq s-\Upsilon(\tau^-)$, $s\in\Upsilon(\tau^-)+[0,\nu(\{\tau\})]$, and $\theta_{\tau-}(s)\doteq s-\theta_{\tau+}(s)$, $s\in(\Upsilon(\tau^-)+\nu(\{\tau\}),\Upsilon(\tau)]$.

Inverse transformation. Given $\mathbf{u}=(\alpha,\beta_+\beta_-)\in\mathbf{U}(S)$, define a desired impulsive control $\vartheta=(\mu,\nu,\{u_\tau\})\in\Theta$ as follows:

$$(\mu,\nu)\doteq d(F_\mu,F_\nu), \tag{33}$$

$$F_\nu(0^-)-0,\ \ F_\nu\doteq\eta_+\circ\Xi, \tag{34}$$

$$F_\mu(0^-)=0,\ \ F_\mu(t)=\int_0^{\Xi(t)}\beta_+(s)\,ds,\ t\in\mathcal{T}, \tag{35}$$

$$u_\tau=\beta_+\circ s_\tau, \tag{36}$$

where $s_\tau(\theta)\doteq\inf\{S\in[\Xi(\tau^-),\Xi(\tau)]:\ \theta_\tau(s)>\theta\}$, and $\theta_\tau(s)=\eta_+(s)-\eta_+(\Xi(\tau^-))$, $s\in[\Xi(\tau^-),\Xi(\tau)]$.

Now we proceed to the convexification of the reduced system. Let F denote the right-hand side of system (21)–(24). Consider the relaxed dynamics

$$\frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s}\in\overline{\mathrm{co}}\left\{F\left(\mathbf{x},\mathbf{u}\right)\middle|\ \mathbf{u}\in\mathbf{U}\right\} \tag{37}$$

with the initial state $\mathbf{x}(0)$ defined by (24). Here, $\overline{\mathrm{co}}A$ denotes the closed convex hull of a set A.

Theorem 2. The set $\overline{\mathbb{X}}$ coincides with the trajectory tube of (37), (24)–(26), up to a discontinuous time change. Namely:

1) For any $x\in\overline{\mathbb{X}}$, there exist $S\geq T$ and a solution $\mathbf{x}=(\xi,y_+,y_-,\eta_+,\eta_-,\zeta)$ of the terminally constrained differential inclusion (37), (24)–(26), such that relations (28) hold.

2) Let $\mathbf{x}=(\xi,y_+,y_-,\eta_+,\eta_-,\zeta)$ be a solution to the Cauchy problem for differential inclusion (37) on a time interval $\mathcal{S}=[0,S]$, $S\geq T$, such that conditions (25), (26) hold. Define x by (29). Then, $x\in\overline{\mathbb{X}}$.

This theorem is a straightforward generalization of a similar result proved in (Goncharova and Staritsyn, 2017) for absolutely continuous approximations $(x_\varepsilon,\widetilde{x}_\varepsilon)_{\varepsilon>0}$. The generalization consists in admitting all approximations, not only absolutely continuous. As the proof is not essentially different, but quite lengthy, it is omitted here because of the lack of space.

Note that Theorem 2 does not claim that a limit solution satisfies the measure-driven system (13), (14). On the other hand, we can prove that limit solutions somehow preserve the complementarity relations (15).

4 LIMIT BEHAVIOR OF IMPULSIVE HYBRID PROCESSES

Our goal now is to show that any limit solution $x\in\overline{\mathbb{X}}$, even not satisfying the impulsive hybrid system, enjoys the hybrid property (15).

We will need the following simple assertion.

Lemma 1. Let $\{\phi_\varepsilon\}_{\varepsilon>0}\subset C$ and $\{\psi_\varepsilon\}_{\varepsilon>0}\subset AC$ be two sequences of functions $\mathcal{S}\to\mathbb{R}$ such that $\{|\dot\psi_\varepsilon|\}_{\varepsilon>0}$ is uniformly essentially bounded by a certain constant independent of ε, $(\phi_\varepsilon,\psi_\varepsilon)\rightrightarrows(\phi,\psi)$ as $\varepsilon\to0$, and $\psi\in AC$. Then,

$$\int_{\mathcal{S}}\dot\psi_\varepsilon\,\phi_\varepsilon\,dt\to\int_{\mathcal{S}}\dot\psi\,\phi\,dt\ \text{as }\varepsilon\to0.$$

Proof: Consider the difference

$$\int_{\mathcal{S}}\left(\dot\psi\,\phi-\dot\psi_\varepsilon\,\phi_\varepsilon\right)dt=$$

$$\int_{\mathcal{S}}\left(\dot\psi-\dot\psi_\varepsilon\right)\phi\,dt+\int_{\mathcal{S}}\dot\psi_\varepsilon\left(\phi-\phi_\varepsilon\right)dt=I_{1_\varepsilon}+I_{2_\varepsilon}.$$

Thanks to the uniform boundedness of derivatives $\{\dot\psi_\varepsilon\}$, the strong convergence $\psi_\varepsilon\rightrightarrows\psi$ implies the weak convergence of the sequence $\dot\psi_\varepsilon$ to $\dot\psi$. As $\varepsilon\to0$, this entails the convergence $I_{1_\varepsilon}\to0$.

In its turn, the convergence $I_{2_\varepsilon}\to0$ follows from the uniform convergence $\phi_\varepsilon\rightrightarrows\phi$, and, again, the uniform estimation on $\dot\psi_\varepsilon$.

Theorem 3. Let $x\in\overline{\mathbb{X}}$. Consider an approximating sequence of impulsive processes $\sigma_\varepsilon=(x_\varepsilon,\vartheta_\varepsilon)$,

$\vartheta_\varepsilon \doteq (\mu,\nu,\{u_\tau\})_\varepsilon$, $x_\varepsilon \doteq x[\vartheta_\varepsilon]$, proposed by the definition of the set $\overline{\mathbb{X}}$. Then, the sequence ν_ε converges in the weak* topology of C^* to a nonnegative measure ν such that $\nu(\mathcal{T}) \le M$, and x satisfies (15) with the measure ν.

Proof: Consider the monotone increasing map Υ_ε:

$$\Upsilon_\varepsilon(t) \doteq t + F_{\nu_\varepsilon}(t) + F_{\widetilde{\nu}_\varepsilon}(t), \quad t \in \mathcal{T},$$

and let υ_ε denote its inverse. Set $S_\varepsilon \doteq \Upsilon_\varepsilon(T)$. On the time intervals $\mathcal{S}_\varepsilon \doteq [0,S_\varepsilon]$, we can define a control $\mathbf{u}_\varepsilon$ by formulas similar to (31), (32):

$$\alpha_\varepsilon \doteq \begin{cases} m_1^\varepsilon \circ \upsilon_\varepsilon, & \upsilon_\varepsilon \in \operatorname{supp}(\nu_{\varepsilon_{ac}} + \widetilde{\nu}_{\varepsilon_{ac}}) \\ 0, & \text{otherwise;} \end{cases}$$

$$\beta_{+\varepsilon} \doteq \begin{cases} \frac{u_\tau}{|u_\tau|+|\widetilde{u}_\tau|} \circ \theta_{\varepsilon_\tau}, & \exists \tau \in D_{\nu_\varepsilon}, \\ & \text{s.t. } s \in \mathcal{S}_{\varepsilon_\tau}, \\ m_{2+}^\varepsilon \circ \upsilon_\varepsilon\, \alpha_\varepsilon, & \upsilon_\varepsilon \in \operatorname{supp} \nu_{\varepsilon_{ac}}, \\ m_{3+}^\varepsilon \circ \upsilon_\varepsilon, & \upsilon_\varepsilon \in \operatorname{supp} \nu_{\varepsilon_{sc}}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_{-\varepsilon} \doteq \begin{cases} \frac{\widetilde{u}_\tau}{|u_\tau|+|\widetilde{u}_\tau|} \circ \theta_{\varepsilon_\tau}, & \exists \tau \in D_{\widetilde{\nu}_\varepsilon}, \\ & \text{s.t. } s \in \mathcal{S}_{\varepsilon_\tau}, \\ m_{2-}^\varepsilon \circ \upsilon_\varepsilon\, \alpha_\varepsilon, & \upsilon_\varepsilon \in \operatorname{supp} \widetilde{\nu}_{\varepsilon_{ac}}, \\ m_{3-}^\varepsilon \circ \upsilon_\varepsilon, & \upsilon_\varepsilon(s) \in \operatorname{supp} \widetilde{\nu}_{\varepsilon_{sc}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $m_1^\varepsilon \doteq \dfrac{d\lambda}{d(\lambda+\nu_\varepsilon+\widetilde{\nu}_\varepsilon)}$, $m_{2+}^\varepsilon \doteq \dfrac{d\mu_{\varepsilon_{ac}}}{d\lambda}$, $m_{3+}^\varepsilon \doteq \dfrac{d\mu_{\varepsilon_{sc}}}{d(\lambda+\nu_\varepsilon+\widetilde{\nu}_\varepsilon)}$, $m_{2-}^\varepsilon \doteq \dfrac{d\widetilde{\mu}_{\varepsilon_{ac}}}{d\lambda}$, $m_{3-}^\varepsilon \doteq \dfrac{d\widetilde{\mu}_{\varepsilon_{sc}}}{d(\lambda+\nu_\varepsilon+\widetilde{\nu}_\varepsilon)}$, $\theta_{\varepsilon_\tau}(s) \doteq s + \Upsilon_\varepsilon(\tau^-)$, $s \in \mathcal{S}_{\varepsilon_\tau}$; $\mathcal{S}_{\varepsilon_\tau} \doteq [\Upsilon_\varepsilon(\tau^-), \Upsilon_\varepsilon(\tau)]$.

As is easily checked, the defined control satisfies $\alpha_\varepsilon(s) \ge 0$ and $\alpha_\varepsilon + |\beta_{+\varepsilon}(s)| + |\beta_{-\varepsilon}(s)| = 1$, and the respective solution $\mathbf{x}_\varepsilon \doteq (\xi, y_+, y_-, \eta_+, \eta_-, \zeta)_\varepsilon$ to the transformed system (21)–(24) meets terminal conditions (25), (26) with accuracy to within ε. Furthermore, by a change of variable under the sign of the Lebesgue-Stieltjes integral it holds $(y_+, y_-, \eta_+, \eta_-)_\varepsilon \circ \Upsilon_\varepsilon = (x_\varepsilon, \widetilde{x}_\varepsilon, F_{\nu_\varepsilon}, F_{\widetilde{\nu}_\varepsilon})$.

Consider the extension of solutions $\mathbf{x}_\varepsilon$ to the common interval $\mathcal{S} \doteq [0, S \doteq \sup\{S_\varepsilon : \varepsilon > 0\}]$ by constant values: $\mathbf{x}_\varepsilon(s) = \mathbf{x}_\varepsilon(S_\varepsilon)$ on $(S_\varepsilon, S]$. For the extended functions, we keep the same notation $\mathbf{x}_\varepsilon$.

By assumptions (H) and usual arguments based on the Gronwall's inequality, the sequence $\{\mathbf{x}_\varepsilon\}_{\varepsilon>0}$ is equicontinuous and uniformly bounded. Then, by the Arzelá-Ascoli selection principle, there is a uniformly converging subsequence. Let $\mathbf{x} \doteq (\xi, y_+, y_-, \eta_+, \eta_-, \zeta)$ be the limit point.

Introduce the inverse time changes $\Xi_\varepsilon : \Xi_\varepsilon(t) \doteq \inf\{s \in \mathcal{S} : \ \xi_\varepsilon(s) > t\}$, $\Xi_\varepsilon(T) = S$, and $\Xi : \Xi(t) \doteq \inf\{s \in \mathcal{S} : \ \xi(s) > t\}$, $\Xi(T) = S$ (note that $\Xi_\varepsilon = \Upsilon_\varepsilon$ for all $\varepsilon > 0$). Thanks to the equalities $\Xi_\varepsilon = \Upsilon_\varepsilon$, we get $y_{+_\varepsilon} \circ \Xi_\varepsilon = x_\varepsilon$. At the same time, by arguments similar to (Miller, 2011, Theorem 2.13), one easily derives that $y_{+_\varepsilon} \circ \Xi_\varepsilon \rightharpoonup y_+ \circ \Xi$. On the other hand, $x_\varepsilon \rightharpoonup x$ by the assumption of the theorem. Thus,

$$y_+ \circ \Xi = x. \tag{38}$$

Analogously, setting $\eta_+ \circ \Xi = F_\nu$, we obtain $\nu_\varepsilon \rightharpoonup \nu$. Recall that $\mathbf{x}_\varepsilon$ meets all the terminal conditions with accuracy to within ε. The estimate $\eta_{+_\varepsilon}(S) \le M + \varepsilon$ implies that $\eta_+(S) \le M$. Thus, $\nu(\mathcal{T}) \le M$, as desired.

Consider the estimation $\zeta_\varepsilon(S) \le \varepsilon$. Thanks to the non-negativity of the velocities $\frac{d}{ds}\zeta_\varepsilon$ and the initial condition $\zeta_\varepsilon(0) = 0$, the considered estimation can be decomposed into the following system of integral equalities:

$$J_{1_\varepsilon} \doteq \int_{\mathcal{S}} \dot{\xi}_\varepsilon \big(\Delta^\pm \eta_\varepsilon + |\Delta^\pm y_\varepsilon|\big)\, ds \le \varepsilon, \tag{39}$$

$$J_{2_\varepsilon} \doteq \int_{\mathcal{S}} \big(\dot{\eta}_{+_\varepsilon} + \dot{\eta}_{-_\varepsilon}\big)\, Q(\eta_\varepsilon)\, ds \le \varepsilon, \tag{40}$$

$$J_{3_\varepsilon} \doteq \int_{\mathcal{S}} \big(\dot{\eta}_{+\varepsilon} W_-(y_{-_\varepsilon}) + \dot{\eta}_{-_\varepsilon} W_+(y_{+\varepsilon})\big)\, ds \le \varepsilon \tag{41}$$

Note that $\mathbf{x}$ is a solution to differential inclusion (37), (24) on $\mathcal{S}$ (Aubin and Cellina, 1984), and, therefore, it is absolutely continuous. The compositions $W_\pm \circ y_\pm$ and $Q \circ (\eta_+, \eta_-)$ are continuous. Then, by Lemma 1

$$J_{1\varepsilon} \to J_1 \doteq \int_{\mathcal{S}} \dot{\xi}\big(\Delta^\pm \eta + |\Delta^\pm y|\big)\, ds,$$

$$J_{2\varepsilon} \to J_2 \doteq \int_{\mathcal{S}} \big(\dot{\eta}_+ + \dot{\eta}_-\big)\, Q(\eta)\, ds,$$

$$J_{3\varepsilon} \to J_3 \doteq \int_{\mathcal{S}} \big(\dot{\eta}_+ W_-(y_-) + \dot{\eta}_- W_+(y_+)\big)\, ds$$

as $\varepsilon \to 0$, which, together with (39)–(41), establishes $J_1 = J_2 = J_3 = 0$.

Let us analyze these relations:

(i) The equality $J_1 = 0$ implies $\Delta^\pm(y,\eta) = 0$ λ-a.e. over $\operatorname{supp}\dot{\xi}$. Thanks to the continuity of (y,η), this equality, in fact, holds *for all* $s \in \mathcal{S} \setminus \bigcup_\tau [\Xi(\tau^-), \Xi(\tau)]$. Furthermore,

$$0 = J_1 = \int_0^{\Xi(T)} \big(\Delta^\pm \eta + |\Delta^\pm y|\big)\, d\xi(s) =$$

$$\int_{\mathcal{T}} \big(\Delta^\pm \eta + |\Delta^\pm y|\big) \circ \Xi\, dt,$$

which yields $\Delta^\pm(y,\eta) \circ \Xi = 0$ λ-a.e. over $\mathcal{T}$. By continuity of the latter composition over $\mathcal{T} \setminus D_{d\Xi}$ and (25), we then obtain: $(y,\eta)_+ \circ \Xi = (y,\eta)_- \circ \Xi$ for all $t \in \mathcal{T}$.

(ii) Note that, thanks to the definition of the function Q, the relation $J_2 = 0$ together with dynamics (22) and the endpoint conditions $\eta_+(0) = \eta_-(0) = 0$ and

$\eta_+(S) = \eta_-(S)$ imply the pointwise relation $\eta_- \leq \eta_+$ on $\mathcal{S}$. This is clear, since the functions $\eta_\pm$ have no growth points beyond the domain, where $\eta_-(s) \leq \eta_+(s)$.

(iii) Now we can show that $y_+(\Xi(\tau^-)) \in \mathcal{Z}_-$, and $y_+(\Xi(\tau)) \in \mathcal{Z}_+$, which completes the proof of the statement. To do this, let us focus on the remaining equality: $J_3 = 0$. The integral here is separated into the series of integrals with nonnegative integrands. The first term gives:

$$0 = \int_{\mathcal{S}\setminus\cup_\tau[\Xi(\tau^-),\Xi(\tau)]} W_\pm(y_\pm)\, d\eta_\mp(s) = \int_{\mathcal{T}} W_\pm(y_+ \circ \Xi)\, d\nu_c(t),$$

which yields the continuous part of the desired complementarity (5). Consider the remaining conditions

$$0 = \int_{\Xi(\tau^-)}^{\Xi(\tau)} |\beta_+|\, W_-(y_-)\, ds =$$
$$\int_{\Xi(\tau^-)}^{\Xi(\tau)} |\beta_-| W_+(y_+)\, ds, \quad \tau \in D_{d\Xi}.$$

We now claim that $y_-(\Xi(\tau)) \in \mathcal{Z}_-$, and $y_+(\Xi(\tau)) \in \mathcal{Z}_+$ for all discontinuity points of Ξ, and, due to observation (i), this would finally imply the hybrid property (15). Assume, ad absurdum, that, say, $y_-(\Xi(\tau)) \notin \mathcal{Z}_-$ for some $\tau \in D_{d\Xi}$. Then, by continuity, $W^X_{\mathcal{Z}_-}(y_-(s)) > 0$ on an interval $\hat{\mathcal{S}} = [\Xi(\tau^-), \hat{s})$. This immediately implies that $\beta_+ = 0$ λ-a.e. on $\hat{\mathcal{S}}$, and therefore $\dot\eta_+ = 0$, $\dot\eta_- = (1 - |\beta_+|) = 1$ λ-a.e. on $\hat{\mathcal{S}}$, i.e., η_+ stays in rest, while η_- increases. Since $\eta_+(\Xi(\tau^-)) = \eta_-(\Xi(\tau^-))$ by (i), this fact contradicts estimation (ii). Inclusions $y_+(\Xi(\tau)) \in \mathcal{Z}_+$ are validated by similar arguments.

Due to (38), observing that discontinuity points of the function $x \doteq y_+ \circ \Xi$ are concentrated within the set of jump points of the function Ξ, we immediately obtain the discrete part of inclusions (5), as desired.

In other words, the hybrid property is preserved under the designed relaxation.

5 OPTIMAL CONTROL

Consider the following optimal impulsive control problems:

(P) Minimize $I(\sigma) = \varphi(x(T))$ over $\sigma \in \Sigma$;

$(\overline{P})$ Minimize $\varphi(x(T))$ over $x \in \overline{\mathbb{X}}$.

Here, φ is a given terminal cost being a lower semicontinuous function $\mathbb{R}^n \to \mathbb{R}$.

The ordinary counterparts or these problems are

(RP) Minimize $\varphi(y_+(S))$ subject to (21)–(27);

$(\overline{RP})$ Minimize $\varphi(y_+(S))$ subject to (37), (24)–(26).

As a theoretical application of the developed model transformation, we establish the equivalence between problems (P) and (RP) (respectively, between $(\overline{P})$ and $(\overline{RP})$), which would open the possibility to treat the impulsive, non-regular models by ordinary analytical methods or existing software.

Theorem 4. 1) For problems (P) and (RP), the existence of a solution to one of them implies the existence of a solution to the other, and $\inf(P) = \inf(RP)$.

2) Solutions to problems $(\overline{P})$ and $(\overline{RP})$ do exist, furthermore, $\min(\overline{P}) = \min(\overline{RP})$.

Proof: We do the first assertion, as the second one is its corollary. Consider a minimizing sequence for $\{\sigma_k \doteq (x_k, \vartheta_k)\} \subset \Sigma$ of problem (P). Assume, ad absurdum, that $\inf(P) \doteq \lim_{k\to\infty} \varphi(x_k(T)) > \inf(RP)$. Then, there should exist $\hat{S} \geq T$ and $\hat{\mathbf{u}} \in \mathbf{U}(\hat{S})$ such that the respective solution $\hat{\mathbf{x}} = (\hat\xi, \hat{y}_+, \hat{y}_-, \hat\eta_+, \hat\eta_-, \hat\zeta, \hat\imath)$ of system (21)–(24) satisfies rightpoint conditions (25), (26), and $\inf(P) > \varphi(\hat{y}_+(\hat{S}))$. Thanks to Theorem 1, we can define a control $\hat\vartheta \in \Theta$ such that $\hat{x} = x[\hat\vartheta]$ solves (13), (14) and is related with $\mathbf{x}_k$ by the inverse transform (29). Then, the inequality $\lim_{k\to\infty} \varphi(x_k(T)) > \varphi(\hat{x}(T))$ contradicts the definition of the sequence x_k. Thus, $\inf(P) \leq \inf(RP)$. By discarding the dual hypothesis, we obtain the inverse inequality $\inf(RP) \leq \inf(P)$. Next, by contradiction, we prove that a minimizer for one problem is a minimizer for the other one, up to direct or inverse transform.

6 CONCLUSIONS

The addressed model statement comprises a large variety of reasonable hybrid systems arising in mechanics. In our opinion, the developed approach can be efficient in modeling and analysis of biomechanical systems, where impulsive effects such as "almost instantaneous" blocking/releasing degrees of freedom, associated with joints of limbs of biological beings, are, sometimes, not really meaningfully described in terms of friction or unilateral contact.

A challenging option for further study is the extension of the obtained results to systems with quadratic impulses, representing fast vibration of invisibly small amplitude (Bressan and Rampazzo, 1994).

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