Testing Fuzzy Hypotheses with Fuzzy Data and Defuzzification of the Fuzzy p-value by the Signed Distance Method

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Keywords: Fuzzy P-Value, Fuzzy Statistics, Fuzzy Hypotheses, Fuzzy Data, One-Sided and Two-Sided Tests, Defuzzification, Signed Distance Method.

Abstract: We extend the classical approach of hypothesis testing to the fuzzy environment. We propose a method based on fuzziness of data and on fuzziness of hypotheses at the same time. The fuzzy p-value with its α-cuts is provided and we show how to defuzzify it by the signed distance method. We illustrate our method by numerical applications where we treat a one and a two sided test. For the one-sided test, applying our method to the same data and performing tests on the same significance level, we compare the defuzzified p-values between different cases of null and alternative hypotheses.

1 INTRODUCTION AND MOTIVATION

The so-called classical approach of the statistical inference is the most used one in statistics. Its extension to the fuzzy environment was of a big discussion in many research papers in the last decade. Several testing methods and approaches in the fuzzy context were treated. We mention for example (Filzmoser and Viertl, 2004), (Parchami et al., 2010), (Grzegorzewski, 2000) and many others. For instance, (Grzegorzewski, 2000) proposed a fuzzy test based on confidence intervals. This test leads to a fuzzy decision, which provides a degree of conviction, meaning a degree of acceptability of the null and alternative hypotheses. From another side, (Filzmoser and Viertl, 2004) extended the classical test to a fuzzy test asserting that the fuzziness is a matter of data. They proposed a fuzzy p-value and a “three decision” procedure where no rejection of both null and alternative hypotheses is considered. (Parchami et al., 2010) reasoned similarly to (Filzmoser and Viertl, 2004) but assumed that the fuzziness is coming from the hypotheses instead of the data.

On the other hand, defuzzifying a fuzzy p-value could be in several situations important to make a decision. (Grzegorzewski, 2001) presented different defuzzification operators to defuzzify his proposed fuzzy p-value. In the same way, (Berkachy and Donzé, 2017), based on the work of (Grzegorzewski, 2000), described how one can defuzzify a fuzzy decision by the so-called signed distance defuzzification method. This method was basically used for instance by (Berkachy and Donzé, 2016) in the context of evaluating linguistic questionnaires.

In this paper, we reconsider the tests' procedures described by (Filzmoser and Viertl, 2004) and (Parchami et al., 2010). We propose an inference method based on both fuzzy data and fuzzy hypotheses. We also put our attention on fuzzy p-value with its α-cuts, and show how to apply the signed distance method to defuzzify this fuzzy p-value. We know that while the defuzzification of a fuzzy set reduces some informations of it, defuzzifying a fuzzy p-value can be useful in several cases of decision making. At last, we give two numerical examples of a one-sided and a two-sided tests. On the same occasion, we play with a set of different fuzzy null and alternative hypotheses in order to be able to compare and understand the differences between cases.

To summarize, we present in Section 2 some useful definitions and notations. The Section 3 is devoted to a brief presentation of the signed distance defuzzification method. In Section 4, we recall briefly the classical testing approach, and we describe the procedure of testing fuzzy hypotheses with fuzzy data. We finally end in Section 5 with two numerical examples.

Berkachy R. and DonzÃ´ L.
Testing Fuzzy Hypotheses with Fuzzy Data and Defuzzification of the Fuzzy p-value by the Signed Distance Method.
DOI: 10.5220/0006500602550264
In *Proceedings of the 9th International Joint Conference on Computational Intelligence (IJCCI 2017)*, pages 255-264
ISBN: 978-989-758-274-5

2 DEFINITIONS AND NOTATIONS

Let us recall some fundamental definitions and notations useful in further sections.

Definition 2.1 (Fuzzy Set)**.**
If A is a collection of objects denoted generically by x then a fuzzy set $\tilde{X}$ in A is a set of ordered pairs:

$$\tilde{X} \quad = \quad \{(x,\mu_{\tilde{X}}(x)) : x \in A\}, \tag{1}$$

where $\mu_{\tilde{X}}(x)$ is the membership function of x in $\tilde{X}$ which maps A to the closed interval [0,1] that characterizes the degree of membership of x in $\tilde{X}$.

Definition 2.2 (Fuzzy Number)**.**
A fuzzy number $\tilde{X}$ is a convex and normalized fuzzy set on $\mathbb{R}$, such that its membership function is continuous and its support is bounded.

Definition 2.3 (α-cut of a Fuzzy Number)**.**
The α-cut of a fuzzy number $\tilde{X}$ is a non-fuzzy set defined as:

$$\tilde{X}_\alpha \quad = \quad \{x \in \mathbb{R} : \mu_{\tilde{X}}(x) \geqslant \alpha\}. \tag{2}$$

The fuzzy number $\tilde{X}$ can be represented by the family set $\{\tilde{X}_\alpha : \alpha \in [0,1]\}$ of its α-cuts.

The α-cut of a fuzzy number $\tilde{X}$ is the closed interval $[\tilde{X}^L_\alpha, \tilde{X}^R_\alpha]$. $\tilde{X}^L_\alpha = \inf\{x \in \mathbb{R} : \mu_{\tilde{X}}(x) \geqslant \alpha)\}$ is its left α-cut and $\tilde{X}^R_\alpha = \sup\{x \in \mathbb{R} : \mu_{\tilde{X}}(x) \geqslant \alpha)\}$ its right one. We note that the α-cut of a fuzzy number $\tilde{X}$ is a union of finite compact and bounded intervals. Furthermore, the least-upper bound property generalized to ordered sets and the extension principle induces the following expression of the membership function of $\tilde{X}$ (see (Viertl, 2011)):

$$\mu_{\tilde{X}}(x) = \max\{\alpha I_{\tilde{X}_\alpha}(x) : \alpha \in [0,1]\}, \tag{3}$$

where $I_{\tilde{X}_\alpha}(x)$ is the following indicator function:

$$\begin{aligned} I_{\tilde{X}_\alpha}(x) &= I_{\{x \in \mathbb{R}; \mu_{\tilde{X}} \geq \alpha\}}(x) \\ &= \begin{cases} 1 & \text{if} \quad \mu_{\tilde{X}}(x) \geq \alpha, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

Definition 2.4 (Triangular Fuzzy Number)**.**
A triangular fuzzy number $\tilde{X}$ is a fuzzy number with membership function given as follows:

$$\mu_{\tilde{X}}(x) = \begin{cases} \frac{x-u}{v-u} & \textit{if} \quad u < x \leq v, \\ \frac{x-w}{v-w} & \textit{if} \quad v < x \leq w, \\ 0 & \textit{elsewhere}. \end{cases} \tag{5}$$

It is common to represent a triangular fuzzy number by the tuple of three values u, v and w, i.e. $\tilde{X} = (u,v,w)$, where $u < v < w \in \mathbb{R}$. For a triangular fuzzy number, the left and right α-cuts $\tilde{X}^L_\alpha$ and $\tilde{X}^R_\alpha$ are given respectively by

$$\begin{cases} \tilde{X}^L_\alpha = u + (v-u)\alpha, \\ \tilde{X}^R_\alpha = w - (w-v)\alpha. \end{cases} \tag{6}$$

3 THE SIGNED DISTANCE DEFUZZIFICATION METHOD

The signed distance defuzzification method was described mainly by (Yao and Wu, 2000) and (Lin and Lee, 2010). (Berkachy and Donzé, 2016) use it extensively in the context of evaluating linguistic questionnaires. The method appears to have nice properties and will be implemented in our test procedure for the defuzzification of the fuzzy p-values. Let us define briefly this measure.

Definition 3.1. *The signed distance $d_0(e,0)$ measured from 0 for a real value e in $\mathbb{R}$ is e.*

Definition 3.2. *Let $\tilde{X}$ be a fuzzy set on $\mathbb{R}$, such as $\tilde{X} = \{(x,\mu_{\tilde{X}}(x)) | x \in \mathbb{R}\}$ where $\mu_{\tilde{X}}(x)$ is the membership function of x in $\tilde{X}$. Suppose that the α-cuts $\tilde{X}^L_\alpha$ and $\tilde{X}^R_\alpha$ exist, and as a function of α are integrable for $\alpha \in [0,1]$. The signed distance of $\tilde{X}$ measured from the fuzzy origin $\tilde{0}$ is:*

$$d(\tilde{X},\tilde{0}) = \frac{1}{2}\int_0^1 [\tilde{X}^L_\alpha + \tilde{X}^R_\alpha] \mathrm{d}\alpha. \tag{7}$$

4 TESTING FUZZY HYPOTHESES WITH FUZZY DATA

(Filzmoser and Viertl, 2004) and (Parchami et al., 2010) were ones of many that treated the problem of testing hypotheses in the fuzzy environment. They introduced as instance the concept of fuzzy p-value. Inspired by their methods, we propose a hypotheses testing method based on fuzzy hypotheses and fuzzy data at the same time. The signed distance will be applied in order to defuzzify the fuzzy p-value. But, first, let us recall the main ideas of the classical approach.

4.1 Testing Hypotheses in the Classical Approach

We consider a population described by a probability distribution P_θ depending on the parameter θ, and belonging to a family of distributions $\mathbb{P} = \{P_\theta : \theta \in \Theta\}$. Testing hypotheses on a parameter θ in the classical approach consists on considering a null hypothesis denoted by H_0, H_0: $\theta \in \Theta_{H_0}$ and an alternative one denoted by H_1, H_1: $\theta \in \Theta_{H_1}$. Θ_{H_0} and Θ_{H_1} are subsets of Θ such that $\Theta_{H_0} \cap \Theta_{H_1} = \emptyset$. A test statistic is a function of a random sample $Y_1, \ldots, Y_n$ used in testing the null hypothesis against the alternative one. We call T such a test statistic, where T: $\mathbb{R}^n \mapsto \mathbb{R}$. For this test, two decisions are often treated: not reject the null hypothesis H_0 or reject the null hypothesis H_0. However, the Neyman-Pearson testing approach (Neyman and Pearson, 1933) could consider the possibility of having a three decision procedure where a third case appears: both the null and alternative hypotheses are neither rejected or not rejected.

The hypothesis testing dilemma is reduced to a decision problem based on the test statistic T, where the space of possible values of T is decomposed into a rejection region R and its complement R^c. Three forms of R are possible depending on the alternative hypotheses H_1.

Let us suppose the following three tests:

$$1. \quad H_0 : \theta \geq \theta_0 \text{ vs. } H_1 : \theta < \theta_0; \tag{8}$$

$$2. \quad H_0 : \theta \leq \theta_0 \text{ vs. } H_1 : \theta > \theta_0; \tag{9}$$

$$3. \quad H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta \neq \theta_0; \tag{10}$$

where θ is the parameter to test and θ_0 a particular value of this parameter.

Then, we would reject the null hypothesis H_0 if respectively:

$$1. \quad T \leq t_l \quad \text{(one-sided test);} \tag{11}$$

$$2. \quad T \geq t_r \quad \text{(one-sided test);} \tag{12}$$

$$3. \quad T \notin (t_a, t_b) \quad \text{(two-sided test);} \tag{13}$$

where t_l, t_r, t_a and t_b are quantiles of the distribution of T.

From another side, we denote by δ the significance level of the test. The quantiles of the distribution t_l, t_r, t_a, and t_b are found such that the following probabilities hold:

$$1. \quad P(T \leq t_l) = \delta, \tag{14}$$

$$2. \quad P(T \geq t_r) = \delta, \tag{15}$$

$$3. \quad P(T \leq t_a) = P(T \geq t_b) = \frac{\delta}{2}. \tag{16}$$

By this method, we decide to reject the null hypothesis if the value of the test statistic $t = T(y_1, \ldots, y_n)$ falls into the rejection region R.

A pratical way to take a decision is to calculate the p-value in order to decide whether we reject or don't the null hypothesis H_0. Notice that a p-value depends on different elements as, for instance, the sample and its distribution, the boundary of the null hypothesis, the distribution of the test statistic and its observed value. We define a p-value $p_{\theta*}$ as function of the boundary $\theta*$ of the null-hypothesis. This p-value for the three cases (11), (12) and (13) can be written respectively as follows:

$$1. \quad p_{\theta^*} = P_{\theta^*}(T \leq t), \tag{17}$$

$$2. \quad p_{\theta^*} = P_{\theta^*}(T \geq t), \tag{18}$$

$$3. \quad p_{\theta^*} = 2\min[P_{\theta^*}(T \leq t), P_{\theta^*}(T \geq t)], \tag{19}$$

where "P_{θ^*}" means that the probability distribution depends on the boundary θ^*.

The decision is made by comparing the p-value to the significance level δ: If the p-value is smaller than δ, we reject the null hypothesis H_0. Otherwise, we don't reject it.

4.2 Fuzzy Hypotheses

In their paper, (Filzmoser and Viertl, 2004) discussed the test of hypotheses in the case of fuzzy data. However, (Parchami et al., 2010) asserted that the fuzziness is rather coming from the hypothesis. In the following, we are treating a case inspired by the above papers but where both data and hypotheses are fuzzy. First, let us define a fuzzy hypothesis.

Definition 4.1 (Fuzzy Hypothesis)**.**
A fuzzy hypothesis $\tilde{H}$ on the parameter θ, denoted as "$\tilde{H} : \theta$ is H", is a fuzzy subset of the parameter space Θ with its corresponding membership function $\mu_{\tilde{H}}$.

Remark 4.1. *A given fuzzy hypothesis $\tilde{H}$ reduces to a crisp hypothesis H when the membership function $\mu_{\tilde{H}} = I_\Theta$.*

It is common practice to postulate as membership functions of a fuzzy left one-sided hypothesis (an increasing function) or of a fuzzy right one-sided hypothesis (a decreasing function) respectively the following functions:

$$\mu_{\tilde{H}^{OL}}(x) = \begin{cases} 0 & \text{if } x < u; \\ \frac{x-u}{v-u} & \text{if } u \leq x < v; \\ 1 & \text{if } x \geq v. \end{cases} \tag{20}$$

$$\mu_{\tilde{H}^{OR}}(x) = \begin{cases} 1 & \text{if } x \leq u; \\ \frac{x-v}{u-v} & \text{if } u < x \leq v; \\ 0 & \text{if } x > v; \end{cases} \tag{21}$$

In that case, we simply note these fuzzy hypotheses as $\tilde{H}^{OL} = (u,v)$ and $\tilde{H}^{OR} = (u,v)$.

A fuzzy two-sided hypothesis $\tilde{H}^T$ is generally treated as a triangular fuzzy number, i.e. $\tilde{H}^T = (u,v,w)$, with membership function (5).

Consider now a crisp random sample $Y_1,\ldots,Y_n$ with probability distribution P_θ and a corresponding fuzzy random sample $\tilde{X} = (\tilde{X}_1,\ldots,\tilde{X}_n)$ where $\tilde{X}_i$ is a fuzzy number as described in **Definition 2.2**. We denote by $\mu_{\tilde{X}}$ the membership function of $\tilde{X}$, where $\mu_{\tilde{X}}$: $\mathbb{R}^n \to [0,1]$. We suppose that it exists a given value of $\mathbf{x}$, $\mathbf{x}$ considered as a n-dimensional vector, for which $\mu_{\tilde{X}}(\mathbf{x})$ reaches 1, and that the α-cuts of $\mu_{\tilde{X}}$ build a closed compact and convex subset of $\mathbb{R}^n$. On the other hand, consider furthermore a real valued function ϕ such as ϕ: $\mathbb{R}^n \to \mathbb{R}$. Denote by the fuzzy number $\tilde{Z}$ the result of applying the function ϕ to the fuzzy random sample, i.e. $\tilde{Z} = \phi(\tilde{X}_1,\ldots,\tilde{X}_n)$. Then, by the extension principle (Zadeh, 1965), the membership function $\mu_{\tilde{Z}}$ of $\tilde{Z}$ is written in the following manner:

$$\mu_{\tilde{Z}}(z) = \begin{cases} \sup\{\mu_{\tilde{X}}(\mathbf{x}) : \phi(\mathbf{x}) = z\} & \text{if } \exists \mathbf{x} : \phi(\mathbf{x}) = z, \\ 0 & \text{if } \nexists \mathbf{x} : \phi(\mathbf{x}) = z, \end{cases} \tag{22}$$

for all z $\in \mathbb{R}$. Furthermore, the α-cuts of $\tilde{Z}$ are given by:

$$\tilde{Z}_\alpha = [\min_{\mathbf{x}\in\tilde{X}_\alpha} \phi(\mathbf{x}), \max_{\mathbf{x}\in\tilde{X}_\alpha} \phi(\mathbf{x})], \tag{23}$$

for all $\alpha \in (0,1]$ (Viertl, 2011).

Finally, we have to define the fuzzy boundaries of fuzzy hypotheses.

Definition 4.2 (Boundary of a Hypothesis). *The boundary $\tilde{H}^*$ of a hypothesis $\tilde{H} : \theta$ is H, is a fuzzy subset of Θ, with membership function μ_{H^*}.*

As instance, the fuzzy boundaries corresponding to the tests (8), (9) and (10) are given respectively by:

1. $\tilde{H}^* = H \quad \text{if} \quad \theta \leq \theta_0, \quad 0 \quad \text{otherwise},$
   ($\tilde{H}$ is left one-sided and $\mu_{\tilde{H}^{OL}}$ increasing);
2. $\tilde{H}^* = H \quad \text{if} \quad \theta \geq \theta_0, \quad 0 \quad \text{otherwise},$
   ($\tilde{H}$ is right one-sided and $\mu_{\tilde{H}^{OR}}$ decreasing);
3. $\tilde{H}^* = H, (\tilde{H}$ is two-sided).

4.3 Fuzzy p-value

Considering hypotheses as fuzzy let us as well see p-values as fuzzy ones, and in this case taking α-cuts of the fuzzy p-values can help to evaluate the results of the tests. Taking in consideration the three possible rejection regions as defined in (11), (12) and (13), the following proposition shows how to calculate the corresponding α-cuts.

Proposition 4.1. *Given a test procedure based on fuzziness of data and hypotheses. Considering the three rejection regions* (11), (12) *and* (13), *the α-cuts of the fuzzy p-value $\tilde{p}$ are given by:*

1. $\tilde{p}_\alpha = [P_{\theta_R}(T \leq \tilde{t}^L_\alpha), P_{\theta_L}(T \leq \tilde{t}^R_\alpha)];$ (24)
2. $\tilde{p}_\alpha = [P_{\theta_L}(T \geq \tilde{t}^R_\alpha), P_{\theta_R}(T \geq \tilde{t}^L_\alpha)];$ (25)
3. $\tilde{p}_\alpha = \begin{cases} [2P_{\theta_R}(T \leq \tilde{t}^L_\alpha), 2P_{\theta_L}(T \leq \tilde{t}^R_\alpha)] & \text{if} \quad A_l > A_r, \\ [2P_{\theta_L}(T \geq \tilde{t}^R_\alpha), 2P_{\theta_R}(T \geq \tilde{t}^L_\alpha)] & \text{if} \quad A_l \leq A_r; \end{cases}$ (26)

for all $\alpha \in (0,1]$, where $\tilde{t}^L_\alpha$ and $\tilde{t}^R_\alpha$ are the left and right α-cuts of $\tilde{t} = \phi(\tilde{X}_1,\ldots,\tilde{X}_n)$, θ_L and θ_R are the α-cuts of the boundary of $\tilde{H}_0$, A_l is the area under the membership function $\mu_{\tilde{t}}$ of the fuzzy number $\tilde{t}$ on the left side of the median, and A_r is the one on the right side. In this case, one has to decide on which side the median is located based on the biggest amount of fuzziness.

Proof 4.1. *Since we are treating the case of both fuzzy data and fuzzy hypotheses, the proof of the Proposition 4.1 will be done in three steps:*

1. *According to the above sections, the resulting fuzzy value $\tilde{t} = T(\tilde{X}_1,\ldots,\tilde{X}_n)$ is fuzzy with its membership function $\mu_{\tilde{t}}$. We denote by $supp(\mu_{\tilde{t}})$, the support of $\mu_{\tilde{t}}$ given by $supp(\mu_{\tilde{t}}) = \{x \in \mathbb{R} : \mu_{\tilde{t}}(x) > 0\}$. Using the extension principle, (Filzmoser and Viertl, 2004) wrote the (precise) p-value p for $\tilde{T}$ for a one-sided test respectively to cases (8) and (9) as follows:*

   1. $p = P(T \leq t = \max supp(\mu_{\tilde{t}}));$ (27)
   2. $p = P(T \geq t = \min supp(\mu_{\tilde{t}})).$ (28)

Our purpose at this moment is to write the α*-cuts of the fuzzy p-value* $\tilde{p}$*. Hence, we know that* $\mu_{\tilde{t}}$ *is a membership function and all its* α*-cuts are compact and closed on* $\mathbb{R}$*, then we can define the* α*-cuts of the p-value* $\tilde{p}_D$ *related to (27) and (28) by*

$$1.\quad \tilde{p}_{FV_\alpha} = [P(T \leq \tilde{t}^L_\alpha), P(T \leq \tilde{t}^R_\alpha)]; \tag{29}$$

$$2.\quad \tilde{p}_{FV_\alpha} = [P(T \geq \tilde{t}^R_\alpha), P(T \geq \tilde{t}^L_\alpha)]. \tag{30}$$

The same procedure can be easily written for the two-sided test.

2. *From another side, we want to extend these formulas to the case of fuzzy hypothesis. (Parchami et al., 2010) related the fuzziness to the hypothesis and presented the* α*-cuts of the fuzzy p-value in the following form:*

$$1.\quad \tilde{p}_{PA_\alpha} = [P_{\theta_R}(T \leq t), P_{\theta_L}(T \leq t)]; \tag{31}$$

$$2.\quad \tilde{p}_{PA_\alpha} = [P_{\theta_L}(T \geq t), P_{\theta_R}(T \geq t)]; \tag{32}$$

$$3.\quad \tilde{p}_{PA_\alpha} = \begin{cases} [2P_{\theta_R}(T \leq t), 2P_{\theta_L}(T \leq t)] \text{ if } A_l > A_r, \\ [2P_{\theta_L}(T \geq t), 2P_{\theta_R}(T \geq t)] \text{ if } A_l \leq A_r; \end{cases} \tag{33}$$

Referring to the Equations (17), (18), (29), (30), to the Definition 4.1 and to the fuzzy p-value discussed by (Parchami et al., 2010), we get $\tilde{p}^R_\alpha$ *and* $\tilde{p}^R_\alpha$*, the left and right* α*-cuts of the fuzzy p-values based on fuzziness of data and hypotheses given by Proposition 4.1.*

3. *We finally have to be sure that the properties of the membership function are fulfilled: the facts that* $\mu_{\tilde{t}}$ *and* $\mu_{\tilde{H}_0}$ *are membership functions and the probabilities are restricted to* $[0,1]$ *induce that the resulting membership functions of* $\tilde{p}$ *are between* 0 *and* 1 *and reach* 1 *for a given value. Furthermore, the* α*-cuts of each case form a closed finite interval and thus they are compact and convex subsets of* $\mathbb{R}$ *for all* $\alpha \in (0,1]$*.*

For the decision making, (Filzmoser and Viertl, 2004) asserted that a three-decision problem is adopted based on the left and right α-cuts of $\tilde{p}$. For a test with a significance level δ, the decisions are made by the following rules:

- $\tilde{p}^R_\alpha < \delta$: reject the null hypothesis;
- $\tilde{p}^L_\alpha > \delta$: not reject the null hypothesis;
- $\delta \in [\tilde{p}^L_\alpha, \tilde{p}^R_\alpha]$: both null and alternative hypothesis are neither rejected or not.

4.4 Defuzzification of the Fuzzy p-value by the Signed Distance

As mentioned above, the signed distance method is an attractive one to defuzzify fuzzy numbers. Thus, we intend to apply this operator in order to defuzzify the fuzzy p-value and understand whether the decision made with resulting p-value is similar to the one in the classical and fuzzy approaches. The idea is to consider the α-cuts of the fuzzy p-values found in the previous section. We use the equation (7) to defuzzify the fuzzy p-value given in equations (24), (25) and (26). The defuzzified p-values are written as follows:

$$1.\ d(\tilde{p}, \tilde{0}) = \frac{1}{2}\int_0^1 (P_{\theta_R}(T \leq \tilde{t}^L_\alpha) + P_{\theta_L}(T \leq \tilde{t}^R_\alpha))\mathrm{d}\alpha; \tag{34}$$

$$2.\ d(\tilde{p}, \tilde{0}) = \frac{1}{2}\int_0^1 (P_{\theta_L}(T \geq \tilde{t}^R_\alpha) + P_{\theta_R}(T \geq \tilde{t}^L_\alpha))\mathrm{d}\alpha; \tag{35}$$

$$3.\ d(\tilde{p}, \tilde{0}) = \begin{cases} \frac{1}{2}\int_0^1 (2P_{\theta_R}(T \geq \tilde{t}^L_\alpha) + 2P_{\theta_L}(T \geq \\ \quad \tilde{t}^R_\alpha))\mathrm{d}\alpha, \qquad \text{if} \quad A_l > A_r, \\ \frac{1}{2}\int_0^1 (2P_{\theta_L}(T \leq \tilde{t}^R_\alpha) + 2P_{\theta_R}(T \leq \\ \quad \tilde{t}^L_\alpha))\mathrm{d}\alpha, \qquad \text{if} \quad A_l \leq A_r. \end{cases} \tag{36}$$

To interpret, the decision of this testing problem using the defuzzified p-values is similar to the classical approach where two main decisions are taken into account:

- $d(\tilde{p}, \tilde{0}) < \delta$: reject the null hypothesis;
- $d(\tilde{p}, \tilde{0}) > \delta$: not reject the null hypothesis;
- $d(\tilde{p}, \tilde{0}) = \delta$ (a rare case): one should decide whether to reject or not the null hypothesis.

The decision of no rejecting H_0 or H_1 doesn't occur in this case since the p-value is now on crisp. We remember that in the fuzzy sense, the no-decision case is considered. Thus, if one has to defuzzify the p-value, this case will no longer be possible. Not detecting the no-decision region might be a disadvantage of defuzzifying the fuzzy p-value.

5 NUMERICAL APPLICATIONS

In this section, we propose two numerical applications to help the reader to understand the reasoning. The first one treats a one-sided test as (9), and the second one a two-sided test as (10).

Example 5.1. *Consider a random sample of size* $n = 9$ *from a normal distribution with an unknown mean and a standard deviation of* 1, $\mathcal{N}(\mu, 1)$. *The aim is to test on the significance level* $\delta = 0.05$ *the following hypotheses:*

$$\tilde{H}_0: \quad \mu \text{ is approximately } 15,$$
$$\tilde{H}_1: \quad \mu \text{ is approximately bigger than } 15.$$

Suppose that not only the hypotheses are fuzzy but the sample as well. Let us assume that the fuzzy null hypothesis is given by the triangular fuzzy number $\tilde{H}_0^T = (14.8, 15, 15.2)$ *and the alternative one by* $\tilde{H}_1^{OR} = (15, 16)$. *These hypotheses are shown in Figure 2. We write the* α*-cuts of* $\tilde{H}_0^T$ *as:*

$$(\tilde{H}_0^T)_\alpha = \begin{cases} (\tilde{H}_0^T)_\alpha^L = 14.8 + 0.2\alpha; \\ (\tilde{H}_0^T)_\alpha^R = 15.2 - 0.2\alpha. \end{cases} \tag{37}$$

We assume furthermore that the membership function of the observed fuzzy sample mean $\overline{\tilde{X}}$ *is given by the following:*

$$\mu_{\overline{\tilde{X}}}(x) = \begin{cases} 2x - 31 & \text{if} \quad 15.5 < x \leq 16; \\ -2x + 33 & \text{if} \quad 16 < x \leq 16.5; \\ 0 & \text{otherwise}; \end{cases} \tag{38}$$

with the corresponding α*-cuts:*

$$\overline{\tilde{X}}_\alpha = \begin{cases} \overline{\tilde{X}}_\alpha^L = 15.5 + 0.5\alpha; \\ \overline{\tilde{X}}_\alpha^R = 16.5 - 0.5\alpha. \end{cases} \tag{39}$$

The rejection region in this one-sided test is given by equation (12). *Hence, the expression of the corresponding* α*-cuts* $\tilde{p}_\alpha$ *of the fuzzy p-value are given by equation* (25). *Combining all the previous informations,* $\tilde{p}_\alpha$ *is as follows:*

$$\tilde{p}_\alpha = [\int_{\theta_1(\alpha)}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) \mathrm{d}u, \int_{\theta_2(\alpha)}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) \mathrm{d}u], \tag{40}$$

where $\theta_1(\alpha)$ *et* $\theta_2(\alpha)$ *are the following functions of* α *based on Equations 37 and 39:*

$$\theta_1(\alpha) = \frac{\overline{\tilde{X}}_\alpha^R - (\tilde{H}_0^T)_\alpha^L}{\sigma / \sqrt{n}} = 5.1 - 2.1 \times \alpha$$

and

$$\theta_2(\alpha) = \frac{\overline{\tilde{X}}_\alpha^L - (\tilde{H}_0^T)_\alpha^R}{\sigma / \sqrt{n}} = 0.9 + 2.1 \times \alpha.^1$$

The membership function induced by the fuzzy p-value $\tilde{p}$ *is shown in Figure 3. Since the fuzzy p-value and the significance level overlap, we cannot make any decision visually. In order to make one, we defuzzify the fuzzy p-value. Applying the signed distance method (equation* (35)*), we get the following result:*

$$\begin{aligned} d(\tilde{p}, \tilde{0}) &= \frac{1}{2} \int_0^1 (P_{\theta_L}(T \geq \tilde{t}_\alpha^R) + P_{\theta_R}(T \geq \tilde{t}_\alpha^L)) \mathrm{d}\alpha \\ &= \frac{1}{2} \int_0^1 (\int_{5.1 - 2.1 \times \alpha}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) \mathrm{d}u + \\ &\qquad \int_{0.9 + 2.1 \times \alpha}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) \mathrm{d}u) \mathrm{d}\alpha \\ &= 0.0239. \end{aligned}$$

We remark that the defuzzified p-value (0.0239) is smaller than the significance level (0.05). In this case, the decision is to reject the null hypothesis at the level $\delta = 0.05$.

Furthermore, we are interested in understanding the influence of the form of the fuzzy hypotheses on the decision of the test. Let us view some variations of the fuzzy hypotheses. Table 1 shows the tests where we change the null or the alternative fuzzy hypotheses of Example 5.1. Figure 1 displays the membership functions of the fuzzy p-values obtained from all these tests. We can see that although the fuzzy alternative hypothesis determines the rejection region, then it does not influence anymore the decision of the test. Hence, the defuzzified p-value is sensitive to the fuzzy null hypothesis. From another side, if one has to compare between defuzzified p-values of the tests simulated versus the fuzzy p-values illustrated in Figure 1, we can say that the highest p-value corresponds to the largest spreaded fuzzy p-value. It is the opposite for the lowest one. Therefore, we can say that the defuzzified p-value can be a relevant indicator of fuzziness of the null hypothesis in order to make the most convenient decision.

Example 5.2. *Consider again a random sample of size* $n = 49$ *from a normal distribution with an unknown mean and a standard deviation of* 1, $\mathcal{N}(\mu, 1)$.

[1] The sum $S_{\tilde{X}_1, \tilde{X}_2}$ of two fuzzy numbers $\tilde{X}_1$ and $\tilde{X}_2$ with their corresponding α-cuts $\tilde{X}_{1\alpha} = [\tilde{X}_{1\alpha}^L, \tilde{X}_{1\alpha}^R]$ and $\tilde{X}_{2\alpha} = [\tilde{X}_{2\alpha}^L, \tilde{X}_{2\alpha}^R]$ is written by:

$$S_{\tilde{X}_1, \tilde{X}_2} = \tilde{X}_{1\alpha} + \tilde{X}_{2\alpha} = [\tilde{X}_{1\alpha}^L + \tilde{X}_{2\alpha}^L, \tilde{X}_{1\alpha}^R + \tilde{X}_{2\alpha}^R],$$

and their difference $D_{\tilde{X}_1, \tilde{X}_2}$ given by:

$$D_{\tilde{X}_1, \tilde{X}_2} = \tilde{X}_{1\alpha} - \tilde{X}_{2\alpha} = [\tilde{X}_{1\alpha}^L - \tilde{X}_{2\alpha}^R, \tilde{X}_{1\alpha}^R - \tilde{X}_{2\alpha}^L].$$

Table 1: Testing different hypotheses on the significance level $\delta = 0.05$ - Example 5.1.

Test	Fuzzy null hypothesis	Fuzzy alternative hypothesis	Defuzzified p-value	Decision
1	$\tilde{H}_0^T = (14.8, 15.0, 15.2)$	$\tilde{H}_1^{OR} = (15, 16)$	0.0239	Reject the null hypothesis
2	$\tilde{H}_0^T = (14.8, 15.0, 15.2)$	$\tilde{H}_1^{OR} = (16, 17)$	0.0239	Reject the null hypothesis
3	$\tilde{H}_0^T = (14.8, 15.0, 15.2)$	$\tilde{H}_1^{OR} = (16, 16)$	0.0239	Reject the null hypothesis
4	$\tilde{H}_0^T = (15.0, 15.0, 15.0)$	$\tilde{H}_1^{OR} = (15, 16)$	0.0098	Reject the null hypothesis
5	$\tilde{H}_0^T = (14.6, 15.0, 15.4)$	$\tilde{H}_1^{OR} = (15, 16)$	0.0494	Slightly Reject the null hypothesis
6	$\tilde{H}_0^T = (14.0, 15.0, 16.0)$	$\tilde{H}_1^{OR} = (15, 16)$	0.1699	Not Reject the null hypothesis
7	$\tilde{H}_0^T = (14.9, 15.0, 15.1)$	$\tilde{H}_1^{OR} = (15, 16)$	0.0156	Reject the null hypothesis

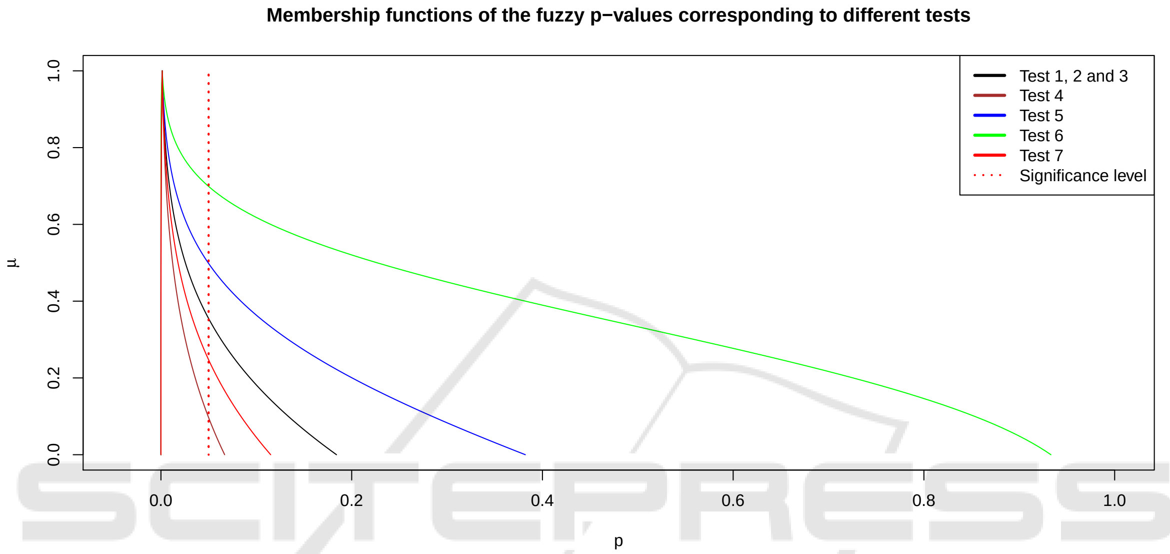

Figure 1: Membership functions of the fuzzy p-values corresponding to the tests in Table 5.1 at the significance level $\delta = 0.05$.

The aim is to test on the significance level $\delta = 0.05$ *the following hypotheses*

$$\tilde{H}_0: \quad \mu \text{ is near } 100,$$
$$\tilde{H}_1: \quad \mu \text{ is away from } 100.$$

Let us suppose that the null and alternative hypotheses are fuzzy and given by the fuzzy triangular numbers $\tilde{H}_0^T = (99.7, 100, 100.3)$ *and the alternative one by* $\tilde{H}_1^T = 1 - \tilde{H}_0^T$ *as seen in Figure 4. The* α*-cuts of* $\tilde{H}_0^T$ *are:*

$$(\tilde{H}_0^T)_\alpha = \begin{cases} (\tilde{H}_0^T)_\alpha^L = 99.7 + 0.3\alpha; \\ (\tilde{H}_0^T)_\alpha^R = 100.3 - 0.3\alpha. \end{cases} \quad (41)$$

Furthermore let us assume that the α*-cuts of the observed fuzzy sample mean* $\overline{\tilde{X}}$ *are given by the following:*

$$\overline{\tilde{X}}_\alpha = \begin{cases} \overline{\tilde{X}}_\alpha^L = 100.4 + 0.6\alpha; \\ \overline{\tilde{X}}_\alpha^R = 101.6 - 0.6\alpha. \end{cases} \quad (42)$$

The rejection region in this two-sided test is given by the equation (13)*. We note that in a standardized normal distribution, the median is equal to the mean, thus we consider that the fuzzy median is equal to the one expressed by the null hypothesis. The* α*-cuts* $\tilde{p}_\alpha$ *of the fuzzy p-value can be computed by equation* (26)*. And since we are in the case of* $A_l \leq A_r$*, we get that* $\tilde{p}_\alpha$ *is as follows:*

$$\tilde{p}_\alpha = [2\int_{\theta_1(\alpha)}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) du,$$
$$2\int_{\theta_2(\alpha)}^{\infty} (2\pi)^{-\frac{1}{2}} \exp(\frac{-u^2}{2}) du]. \quad (43)$$

where $\theta_1(\alpha)$ *et* $\theta_2(\alpha)$ *are the following functions of* α*:*

$$\theta_1(\alpha) = \frac{\overline{\tilde{X}}_\alpha^R - (\tilde{H}_0^T)_\alpha^L}{\sigma / \sqrt{n}} = 0.9 + 8.1 \times \alpha$$

and

$$\theta_2(\alpha) = \frac{\overline{\tilde{X}}_\alpha^L - (\tilde{H}_0^T)_\alpha^R}{\sigma / \sqrt{n}} = 17.1 - 8.1 \times \alpha.$$

Figure 5 shows the fuzzy p-value at the significance level $\delta = 0.05$.

We defuzzify this fuzzy p-value by the signed distance method using the equation (36) *and we obtain the following distance:*

$$
\begin{aligned}
d(\tilde{p},\tilde{0}) &= \frac{1}{2}\int_0^1 2\times(P_{\theta_L}(T \geq \tilde{t}^R_\alpha) + 2\times P_{\theta_R}(T \geq \tilde{t}^L_\alpha))\mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 (2\times\int_{0.9+8.1\times\alpha}^{\infty}(2\pi)^{-\frac{1}{2}}\exp(\frac{-u^2}{2})\mathrm{d}u + \\
&\qquad 2\times\int_{17.1-8.1\times\alpha}^{\infty}(2\pi)^{-\frac{1}{2}}\exp(\frac{-u^2}{2})\mathrm{d}u)\mathrm{d}\alpha \\
&= 0.0123989.
\end{aligned}
$$

The defuzzified p-value (0.0123989) is smaller than the significance level (0.05), then the decision will be to reject the null hypothesis at the level $\delta = 0.05$.

6 CONCLUSION

In this work, we presented a hypothesis testing procedure when both data and hypotheses are fuzzy. We introduced as well a fuzzy p-value with its α-cuts. We discussed after the defuzzification of this fuzzy p-value by the so-called "signed distance method". We finally proposed numerical examples of one-sided and two-sided tests, in addition to a small comparison between different null and alternative hypotheses with the same hypothetical sample at the same significance level. To conclude, despite the fact that the defuzzification step reduces the amount of information contained in a fuzzy p-value, we thought that in many cases defuzziying these p-values with the signed distance can be of a high relevance. In addition, since testing hypotheses on linguistic variables is in most cases complicated and not feasible in classical statistics, proposing such an approach to deal with fuzziness and obtaining a p-value deserves merit in decision making. Indeed, we can see the defuzzied p-value as an "informal indicator" of rejecting or not a given null hypothesis. For further researches, we will be interested in testing the method with many other statistical distributions.

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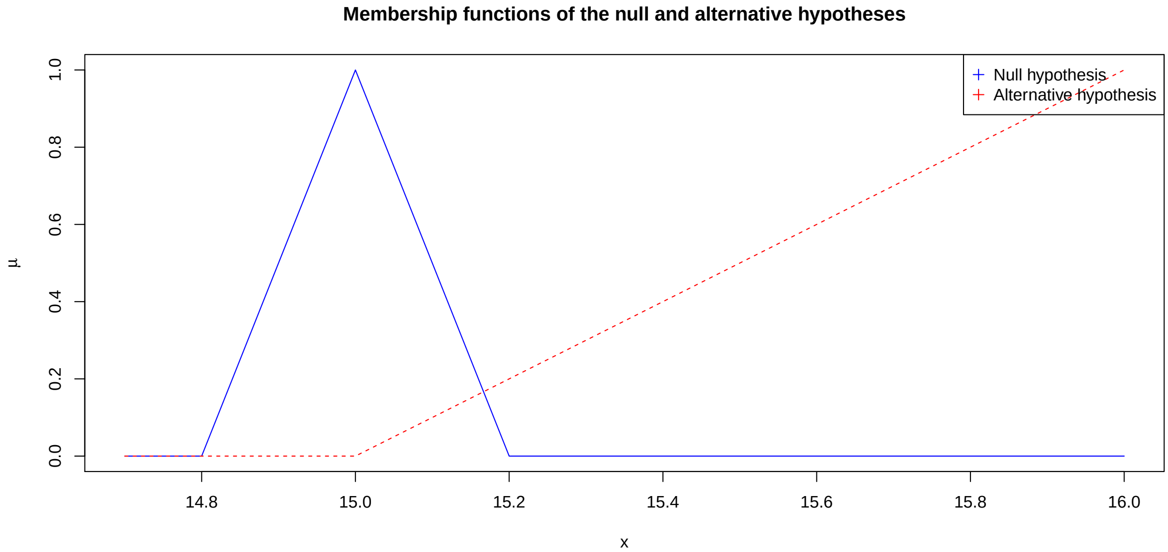

Figure 2: The membership functions of the null and the alternative hypotheses - Example 5.1.

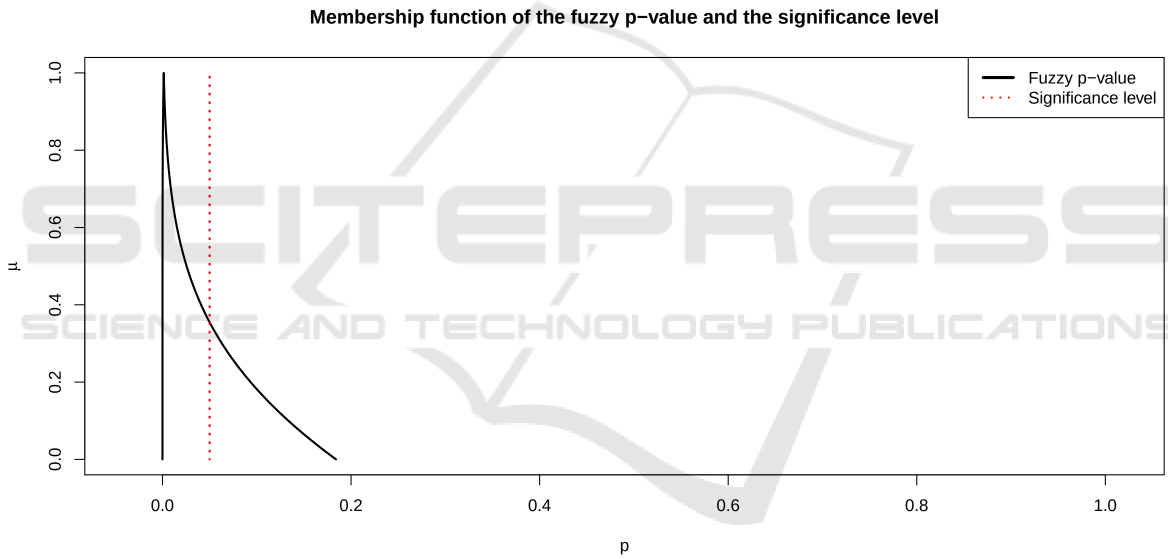

Figure 3: The membership function of the fuzzy p-value $\tilde{p}_{\alpha}$ - Example 5.1.

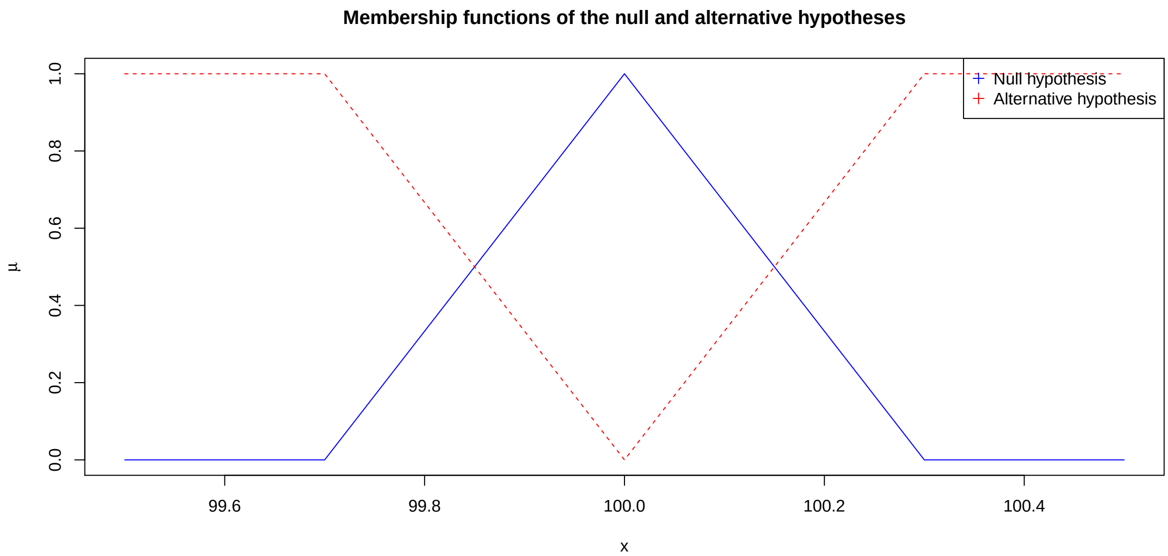

Figure 4: The membership functions of the null and the alternative hypotheses - Example 5.2.

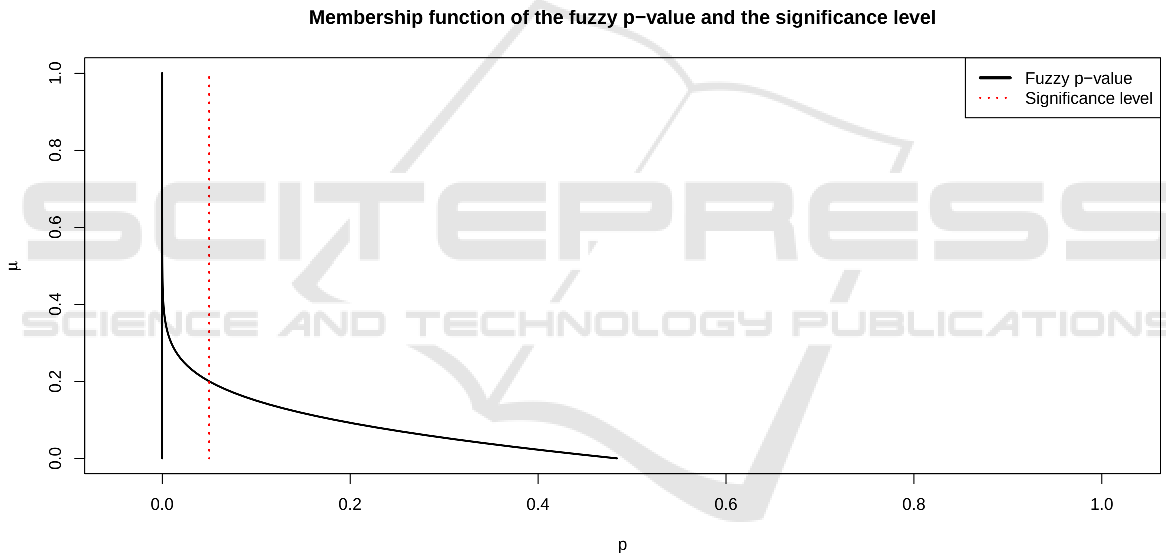

Figure 5: The membership function of the fuzzy p-value $\tilde{p}_{\alpha}$ - Example 5.2.