Stability Analysis in Three Dimensions for the Incompressible Navier-Stokes Equation

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Abstract: This research discusses about stability on three dimensional incompressible Navier-Stokes equations in steady state $(0, p_s)$ and with Navier boundary condition. The analysis is performed in a region geometrically of the form box hollow. The result shows the shape of stability (or instability) depends on energy, and strengthen the slip length and viscosity. With the presence of critical viscosity, it can also be shown the stability in three-dimensional domain hold by using of normed spaces.

1 INTRODUCTION

In mathematically, the Navier-Stokes equations in three dimensions are formed by viscosity. So, the equations from $\mathcal{F}$ is described by the following system

$$\rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) = \mu\nabla v - \nabla\pi - \mathbf{F}(x,t)$$

$$\nabla.v = 0$$

where t is the time, x is the point of Ω; ρ is the density, v is the velocity, π is the corresponding pressure and the positive constant μ is the velocity coefficient. So,

$$\boldsymbol{v}.\nabla\boldsymbol{v} = \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i}$$

The basic of stability analysis depends on which is function $\mathbf{f}(x,t)$. In the case, the autonomous system means that the systems are not depend on the time t. Therefore,

$$\dot{x} = \boldsymbol{f}(x), t \geq 0, \boldsymbol{x}(0) = \boldsymbol{x}_0$$

The stability is defined by two Lyapunov Stability and Asymptotic Stability (Jiang F and Jiang S, 2014). The concept of Lyapunov stability is $\ddot{x}(t)$ is said to be stable if given $\varepsilon > 0$, there exist a $\boldsymbol{\delta} = \delta(\varepsilon) > 0$ such that, for any other solution, $y(t)$ satisfying $|\ddot{x}(t_0) - y(t_0)| < \boldsymbol{\delta}$, then $|\ddot{x} - y(t)| < \varepsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$. The Asymptotic Stability is defined by if there exist a constant $b > 0$ such that, if $|\ddot{x}(t_0) - y(t_0)| < b$, the $\lim_{t\to\infty}|\ddot{x}(t_0) - y(t_0)| = 0$.

There are so many researches about stability analysis in Navier-Stokes equations, the nonlinear instability in inhomogeneous incompressible flow and stability and instability of gravity (Tulus, 2012). On 2012, Tulus was obtained the stability of Taken-Bogdanov equations with numerical solution. Based on the research has not found any research about stability analysis on the three dimensional for incompressible flow. Thus, this study is about the analysis of linear stability in the three dimensions of the compressed Navier-Stokes equation. In this research, three-dimensional model that will be discussed is as follows.

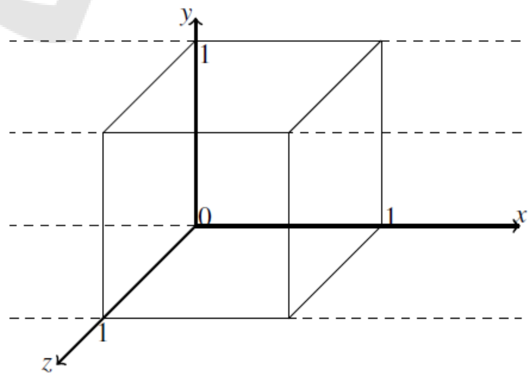

Figure 1: $\mathbb{R} \times (0,1) \times (0,1)$.

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2 METHOD

The method used in this research is

1. Define 3 dimensional incompressible Navier- Stokes equations (Tulus et al., 2017).
2. Define the theorems of stability and instability (linear or nonlinear).
3. Analysis of the Navier-Stokes equation on incompressible flow with linear stability analysis (Tulus et al., 2018).

3 DISCUSSION

3.1 3-Dimensional Incompressible Navier-Stokes

Given the Navier-Stokes equations for incompressible flow

$$\begin{cases}\partial_t \boldsymbol{v} + \boldsymbol{v}.(\partial v_1 + \partial v_2 + \partial v_3) + \\ (\partial v_1 + \partial v_2 + \partial v_3) - \mu \Delta \boldsymbol{v} = 0, \\ \nabla \boldsymbol{v} = 0\end{cases} \quad (1)$$

Where p is identified as the pressure and v is the vector field of velocity (Marpaung et al., 2018).

In this research, the Navier boundary condition is defined on domain $\Omega = \mathbb{R} \times (0,1) \times (0,1)$, then

$$\begin{cases}\partial_t \boldsymbol{v} + \boldsymbol{v}.(\partial v_1 + \partial v_2 + \partial v_3) + (\partial v_1 + \partial v_2 \\ +\partial v_3) - \mu \Delta \boldsymbol{v} = 0;\ on\ \mathbb{R} \times (0,1) \times, t \geq 0, \\ \nabla \boldsymbol{v} = 0;\ 0\ on\ \mathbb{R} \times (0,1) \times (0,1), t \geq 0\end{cases} \quad (2)$$

3.2 Linearization the Equation

On this research, for analysis the stability we use the steady state on $(v_s = 0;\ p = p_s)$(Tulus et al., 2006). Let $\boldsymbol{u}$ is the strong solution for linearization the equation with steady state $(0, p_s)$, α is the slip length, the kinetic energy system $\mathrm{E} = \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)}$. So, based on the law of energy is

$$\mathrm{E}(t) = \mathrm{E}(0) - \mu \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2\, dxdydzds + \alpha \int_0^t \int_\Omega |u|^2\, dxds$$

With $\alpha \leq 0$. There are the following steps for linearization the equation (2).

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{0}, q = p - p_s,$$

Then, $(\boldsymbol{u}, q)$ satisfies the following perturbed equations:

$$\begin{cases}\partial_t \boldsymbol{u} + \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla q - \mu \Delta \boldsymbol{u} = 0 \\ \nabla \boldsymbol{u} = 0\end{cases} \quad (3)$$

Therefore, the Navier boundary condition:

$$\begin{cases}\partial_t \boldsymbol{u} + \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla q - \mu \Delta \boldsymbol{u} = 0 \\ \nabla \boldsymbol{u} = 0\end{cases} \quad (3)$$

Let we use the perturbation formulated as below

$$\boldsymbol{u}^{(2)}(x,0,z) = \boldsymbol{u}^{(2)}(x,1,z) = \boldsymbol{u}^{(2)}(x,y,0) = \boldsymbol{u}^{(2)}(x,y,1) = 0, x \in \mathbb{R} \quad (4)$$

$$\partial_y \boldsymbol{u}^{(1)}(x,1,z) = \frac{k_1}{\mu}\boldsymbol{u}^{(1)}(x,1,z), x \in \mathbb{R} \quad (5)$$

$$\partial_y \boldsymbol{u}^{(1)}(x,0,z) = -\frac{k_0}{\mu}\boldsymbol{u}^{(1)}(x,0,z), x \in \mathbb{R} \quad (6)$$

$$\partial_z \boldsymbol{u}^{(1)}(x,y,1) = \frac{k_3}{\mu}\boldsymbol{u}^{(1)}(x,y,1), x \in \mathbb{R} \quad (7)$$

$$\partial_z \boldsymbol{u}^{(1)}(x,y,0) = -\frac{k_2}{\mu}\boldsymbol{u}^{(1)}(x,y,0), x \in \mathbb{R} \quad (8)$$

Then, linearization equation (3) with steady state $(\boldsymbol{0}, p_s)$, we obtained:

$$\begin{cases}\partial_t \boldsymbol{v} + \nabla p - \mu \Delta \boldsymbol{v} = 0 \\ \nabla \boldsymbol{u} = 0\end{cases} \quad (9)$$

3.3 Stability Analysis

For facilitate the analysis process, we will introduce some notations,

$$\Omega := \mathbb{R} \times (0,1) \times (0,1), \quad L^P := L^P(\Omega), \int := \int_\Omega,$$
$$H^K := W^{2,k}(\Omega),$$

$$H^1_\sigma := \{\boldsymbol{u} \in H^1 |\ \mathrm{div}\, \boldsymbol{u} = 0, u^2 = 0 \quad \text{at } \mathbb{R} \times (0,1) \times (0,1)\}$$

Where $H^1_0(0,1) \times (0,1)$ and $H^2(0,1) \times (0,1)$ can be written by H^1_0 and H^2.

We will define the steady state on the equations. From the equations (4)-(8), we obtained

$$\boldsymbol{v}(x,y,z;t) = \boldsymbol{w}(x,y,z)e^{\lambda}t, q(x,y,z;t) = p(x,y,z)e^{\lambda}t \quad (10)$$

For $\lambda > 0$, substituted λ and $\mathbf{w}$ on the equation,

$$\begin{cases}\lambda \boldsymbol{w} + \nabla \acute{p} - \mu \nabla \boldsymbol{w} = 0, & \text{on } \Omega, \\ \mathrm{div}\, \boldsymbol{w} = 0, & \text{on } \Omega,\end{cases} \quad (11)$$

We can write the $\boldsymbol{w}$ and $\acute{p}$ in the terminology, ψ, ϕ, π: $(0,1) \times (0,1) \to \mathbb{R}$ for all ξ, we obtained:

$$w^1(x,y,z) = -\iota\phi(z)e^1 x\xi, w^2(x,y,z) = \psi(z)e^1 x\xi,$$
$$\acute{p}(x,y,z) = \pi(z)e^1 x\xi. \quad (12)$$

Elimination π for the second equations (4) then substitution the fourth order from the ODE system for ψ, then:

$$-\lambda(\xi^2\psi - \psi^n) = \mu(\psi^4) - 2\xi^2\psi^n + \xi^4\psi, \quad z \in (0,1)\times(0,1) \tag{13}$$

$$-\lambda = \inf_{H_0^1 \cap H^2} \frac{E(\psi)}{J(\psi)}, \tag{14}$$

In this case

$$E(\psi) = \frac{\mu}{2}\int_0^1\int_0^1 [(\psi'')^2 + 2\xi^2(\psi')^2 + \xi^4\psi^2] - \frac{k_1}{2}\left(\psi'(1,z)\right)^2 - \frac{k_0}{2}\left(\psi'(0,z)\right)^2 - \frac{k_3}{2}\left(\psi'(y,1)\right)^2 - \frac{k_2}{2}\left(\psi'(y,0)\right)^2$$

and

$$J(\psi) = \frac{1}{2}\int_0^1 [\xi^2\psi^2 + (\psi')^2] \tag{15}$$

well defined on the space $H_0^1 \cap H^2$

For the positive $\lambda, \lambda(= \lambda(\xi^2))$, then

$$\frac{\mu}{2}\int_0^1\int_0^1 (\psi'')^2 - \frac{k_1}{2}\left(\psi'(1,z)\right)^2 - \frac{k_0}{2}\left(\psi'(0,z)\right)^2 - \frac{k_3}{2}\left(\psi'(y,1)\right) - \frac{k_2}{2}\left(\psi'(y,0)\right)$$

is negative for the viscosity μ. Therefore $E(\psi)$ is negative on ξ. So, the critical viscosity is defined by

$$\mu_c = \sup_{H_0^1\cap H^2, \int_0^1(\psi'')^2 \neq 0} \frac{k_1\left(\psi'(1,z)\right)^2 + k_0\left(\psi'(0,z)\right)^2}{\int_0^1\int_0^1(\psi'')^2} + \frac{k_3\left(\psi'(y,1)\right)^2 + k_2\left(\psi'(y,0)\right)^2}{\int_0^1\int_0^1(\psi'')^2} \tag{16}$$

3.4 The Critical Viscosity

From the equation (16) we obtained the shape of the critical viscosity,

$$\mu_c = \sup_{\psi \varepsilon y} Z(\psi) \tag{17}$$

where

$$(\psi) = \frac{k_1}{2}\left(\psi'(1,z)\right)^2 + \frac{k_0}{2}\left(\psi'(0,z)\right)^2 + \frac{k_3}{2}\left(\psi'^{(y,1)}\right) + \frac{k_2}{2}\left(\psi'^{(y,0)}\right) \tag{18}$$

$$y = \left\{\psi \varepsilon H_0^1 \cap H^2 \middle| \frac{1}{2}\int_0^1 (\psi'(0))^2 = 1\right\}.$$

Then, for every $f \in H_0^1 \cap H^2$,

$$\|f'\|_{L^2}^2 \geq \|f''\|_{L^2}^2 , \tag{19}$$

with applied the Poincare inequality and $f \in H_0^1$, we obtained the proof.

So that, for every $\psi \in y$,

$$|Z(\psi)| = \frac{1}{2}\left|\int_0^1\int_0^1 [((k_1+k_0)y - k_0)(\psi')^2]'_y [((k_3+k_2)z - k_2)(\psi')^2]'_z \, dydz\right.$$
$$= \left|\frac{k_1+k_0+k_3+k_2}{2}\right|\int_0^1\int_0^1 (\psi')^2 \, dydz + \int_0^1\int_0^1 [((k_1+k_0)y - k_0][((k_3+k_2)z - k_2] \psi'\psi'' \, dydz \,|$$
$$\leq \frac{1}{2}\int_0^1\int 0^1 C_3(\psi'')^2 dydz = C_3$$

for every positive constant $C_1; C_2$, dan C_3 depends on k_1 dan k_0. This statement proof $\sup_{\psi\in y} Z(\psi)$ is exist and bounded. For $\mu_c := \sup_{\psi\in y} Z(\psi)$, we can distinguish become two propositions

Propositions. For $\mu_c \varepsilon(-\infty, +\infty)$. If max $v_0, v_1, v_2, v_3 > 0$ then $\mu_c > 0$

Proof. Let v_0, v_1, v_2 and v_3 are non-positive. Then we have

$$\mu_c = \sup_{\psi\in y,z}\left[\frac{v_1}{2}(\psi(1,z))^2 + \frac{v_0}{2}(\psi(0,z))^2 + \frac{v_3}{2}(\psi'(y,1)) + \frac{v_2}{2}(\psi'(y,0))\right] \leq 0. \tag{20}$$

We have known the explicit from the critical viscosity

$$\mu_c = \begin{cases} 0, v_1 \geq 0, v_0 \geq 0, v_3 \geq 0, \text{and } v_2 \geq 0 \\ \frac{v}{6} v_1 = v_0 = v_3 = v_2 := v > 0 \\ \frac{(v_1+v_0+v_3+v_2) + \sqrt{v_1^2+v_0^2+v_3^2+v_2^2 - v_1v_0}}{6} \end{cases}$$

others

For

$$\Psi(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{4}\right]; \\ \alpha \exp\left\{\frac{1}{\left(x-\frac{1}{2}\right)^2}, \frac{1}{16}\right\}, & x \in \left(\frac{1}{4}, \frac{3}{4}\right); \\ 0, & x \in \left[\frac{3}{4}, 1\right]; \end{cases} \tag{21}$$

on y, moreover $\Psi \in C_0^\infty([0,1,z])$ and $\Psi_1'(1,z) = \Psi_0'(0,z) = 0 \leq z \leq 1$, which implies $Z(\Psi_1) = 0$ implies that $\mu_c = 0$.

4 CONCLUSIONS

On this research the linear stability analysis is obtained from the instability depends on the strength viscosity and slip length. There are three theorems, which are linear instability theorem, nonlinear instability theorem, and linear stability asymptotic. Moreover, there are the critical viscosity which distinguish the linear stability and nonlinear instability.

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REFERENCES

Fei, J., Song, J., 2014. On instability and stability of three-dimensional gravity driven viscous flows in a bounded domain. *Jornal of Advances in Mathematics*. Elsevier.

Marpaung, TJ., Tulus, Suwilo, S., 2018. Cooling optimization in tubular reactor of palm oil waste processing. *Bulletin of Mathematics*. Talenta USU.

Tulus, Ariffin, AK., Sharir, A., Mohamad, N., 2006. Finite element analysis for heat transfer in the insulator on piston pin of a linear generator engine. *2nd IMT-GT Regional Conference Of Mathematics, Statistics And Applications*. University Sains Malaysia.

Tulus, 2012. Numerical study on the stability of takens-bogdanov systems. *Bulletin of Mathematics*. Talenta USU.

Tulus, Suriati, Situmorang, M., Zain, DM., 2017. Computational Analysis of Sedimentation Process in the Water Treatment Plant. *1st International Conference on Applied & Industrial Mathematics and Statistics*. IOP Conference Series: Materials Science and Engineering.

Tulus, Suriati, Marpaung, TJ., 2018 Sedimentation optimitation on river dam flow by using comsol multiphysics. *4th International Conference on Operational Research*. IOP Conference Series: Materials Science and Engineering.