Lowest Unique Bid Auctions with Resubmission Opportunities

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Abstract: The recent online platforms propose multiple items for bidding. The state of the art, however, is limited to the analysis of one item auction. In this paper we study multi-item lowest unique bid auctions (LUBA) in discrete bid spaces under budget constraints. We show the existence of mixed Bayes-Nash equilibria for an arbitrary number of bidders and items. The equilibrium is explicitly computed in two bidder setup with resubmission possibilities. In the general setting we propose a distributed strategic learning algorithm to approximate equilibria. Computer simulations indicate that the error quickly decays in few number of steps by means of speedup techniques. When the number of bidders per item follows a Poisson distribution, it is shown that the seller can get a non-negligible revenue on several items, and hence making a partial revelation of the true value of the items.

1 INTRODUCTION

With the increasing impact of information technology, the traditional structure of economic and financial markets has been revolutionized. Today, the market includes millions of economic agents worldwide and reaches an annual transaction of billions U.S. dollars. In this paper, we study a little-studied type auction which is called lowest unique bid auction (LUBA). Because of a limited number of items provided by the auctioneers and budget restrictions, bidders need to manage their behaviors in a strategic way.

Literature Review

Auctions: In 1961 in the work of Vickrey (Vickrey, 1961), the theory of auctions as games of incomplete information is first proposed. Auctions with homogeneous valuation distributions (symmetric auctions) and self-interested non-spiteful bidders are well-investigated in the literature. However, auctions with asymmetric bidders remain a challenging open problem (see (Maskin and Riley, 2000; Lebrun, 1999; Lebrun, 2006) and the references therein). In asymmetric auction scenarios, the expected “revenue equivalence theorem” (Myerson 1981) does not hold, i.e., the revenue of the auctioneer depends on the auction mechanism employed. In addition, there is no ranking revenue between the auction mechanisms (first, second, English or Deutch). Most of research articles (Bang-Qing, 2003; C. Yi, 2015 ; N. Wang, 2014; Rituraj and Jagannatham, 2013) on multi-item auctions works provide computer simulation or numerical experiments results. However, no analysis of the outcome of the multi-item auction is available. There is no analysis of the equilibrium seeking algorithm therein.

LUBA: LUBA is different from the lowest cost auction called procurement auction which is widely used in e-commerce(J. Zhao, 2015), hybrid cloud computing or in demand-supply matching in power grids (R. Zhou, 2015). As a special case of unmatched bid auctions, the single-item LUBAs have been studied by other researchers (H. Houba, 2011; Stefan and Norman, 2006; Rapoport, 2007; Erik, 2015; J.Eichberger, 2008).

The authors (Stefan and Norman, 2006) run laboratory experiments with one bid min bid auctions. They consider the results of a Monte Carlo simulation under the restriction of one bid per player. The work in (Erik, 2015) considers a lowest unique positive integer experiment in a single bid per player setting. The observed behaviors are compared with the solution of a Poisson game. The authors in (Rapoport, 2007) consider both high and low unique bid auctions, and they also assume that bidders are restricted to a single bid. A numerical approximation of the solution for a game-theoretic model is provided. The so-

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lutions are compared with the results of a laboratory experiment. (J.Eichberger, 2008; J.Eichberger, 2015; J. Eichberger, 2016) conduct field experiments on Lowest-Unmatched Price Auctions with high prizes involving large numbers of participants (tens and hundreds of thousands). The work in (Dong-Her, 2011) discusses security and privacy issues of multi-item reverse Vickrey auction. They provide more secure protocol design. Most of the above works restrict the number of submissions per bidder to one. The recent focus within the auction field has been multi-item auctions. The bidder can also place multiple bids for each item. It has been of practical importance in Internet auction sites and has been widely executed by them.

2 PROBLEM STATEMENT

A LUBA operates under the following three main rules: Whoever bids the lowest unique positive bid wins; If no unique bid exits then no one wins; No participant can see bids placed by other participants. "Lowest unique bid" means the lowest amount that nobody else has bid.

In the multi-item LUBA, there are $n \geq 2$ bidders exerting effort for m items (objects) proposed by the auctioneers. Each bidder has the information as follow. Valuation vector: Each bidder j has a vector of values $v_j = (v_{ji})_{i \in I}$, for assessing the worth of each object i offered for the auction. The random variable v_{ji} has support $[\underline{v}, \bar{v}]$ where $0 < \underline{v} < \bar{v}$. A bidder may have his/her own valuation vector but not the valuation vector of the others. Budget: Each bidder j has a initial total budget of $\bar{b}_j$ to be used for all items. Registration fee: To participate in the LUBA, each bidder needs to pay a one time registration fee c_r. Submission fee: To participate in the auction on item i, he/she needs to pay a submission fee c_i for each submission.

Note that each bidder can resubmit bids a certain number of times subject to his or her available buject. If bidder j has (re)submitted n_{ji} times on item i, his/her total submmission/bidding cost would be $n_{ji}c_i$ in addition to the registration fee. The set that contains all the bids of bidder j on item i is denoted by $B_{ji} \subset \mathbb{N}$. Thus, we can obtain the set of bidders who are submitting b on item i by $N_{i,b} = \{j \in \mathcal{J} \mid b \in B_{ji}\}$. We use $|N_{i,b}|$ to represent the cardinality of $N_{i,b}$. Obviously, $|N_{i,b}| = 1$ yields that b was chosen by only one bidder. Then, the set of all positive natural numbers that were unique on item i can be calculated by $B_i^* = \{b > 0 \mid |N_{i,b}| = 1\}$. $B_i^* = \emptyset$ yields no winner on item i at that round. If $B_i^* \neq \emptyset$, there is a winner on item i, and the winning bid is $\inf B_i^*$. At the same time, the winner can be calulated by $j^* \in N_{i,\inf B_i^*}$.

2.1 The Payoff of Participants

The cost of bidder j on item i consists of the registration fee c_r, the submission fee $|B_{ji}| * c_i$, and the bid fee which is conditional on moves by the other bidders. If he/she is the winner, the bid fee is $\inf B_i^*$ calculated by the proceeding subsection. Losing the LUBA yields no bid fee. Thus, the payoff of bidder j on item i at a round would be $r_{ji} = v_{ji} - |B_{ji}|c_i - \inf B_i^* - c_r$, if j is a winner on item i, and $r_{ji} = -|B_{ji}|c_i - c_r$, if j is not a winner on item i. The payoff of bidder j on item i is zero if B_{ji} is reduced to $\{0\}$ (or equivalently the empty set). Thus, it is easy to conclude that

$$\begin{aligned} & r_{ji}(B) \\ = \quad & [-c_r - c_i|B_{ji}| - (v_{ji} - b_{ji}) \mathbb{1}_{\{b_{ji} = \inf B_i^*\}}] \mathbb{1}_{\{B_{ji} \neq \{0\}\}}, \end{aligned} \tag{1}$$

where the infininum of the empty set is zero. Overall, the total payoff of bidder j is $r_j(B) = \sum_{i \in I} r_{ji}(B)$. The instant payoff of the auctioneer of item i can be calculated by

$$r_{a,i} = \left(\sum_j c_r \mathbb{1}_{\{B_{ji} \neq \emptyset\}} + \inf B_i^* + \sum_{j=1} |B_{ji}| c_i \right) - v_{a,i},$$

where $v_{a,i}$ is the realized valuation of the auctioneer for item i. Obviously, the instant payoff of the auctioneer of a set of item I is $r_{a,I} = \sum_{i \in I} r_{a,i}$. In terms of reward seeking, bidders are interested in optimizing their payoffs, and auctioneers are interested in their revenue.

3 SOLUTION APPROACH

Since the game is of incomplete information, the strategies of participating in the game must be specified as a function of the information structure. We introduce the definition of pure strategy and mixed strategy as follows.

Definition 1. *A pure strategy of a bidder is a choice of a subset of natural numbers contingent on the own-value and own-budget. Thus, given a bidder's own valuation vector $v_j = (v_{ji})_i$, the bidder j will choose an action $(B_{ji})_i$ that satisfies the budget constraints, which is*

$$\sum_i c_r \mathbb{1}_{\{B_{ji} \neq \{0\}\}} + \sum_{i=1}^{m} [\inf B_i^*] \mathbb{1}_{B_{ji} \cap [\inf B_i^*]} + \sum_{i=1}^{m} |B_{ji}| c_i \leq \bar{b}_j.$$

The set of multi-item bid space for bidder j is

$$\begin{aligned} \mathcal{B}_j(v_j, \bar{b}_j) = \{ (B_{ji})_i \mid \ & B_{ji} \subset \{0, 1, \ldots, \bar{b}_j - c_r\}, \\ & \textstyle\sum_i c_r \mathbb{1}_{\{B_{ji} \neq \{0\}\}} + \sum_{i=1}^{m} [\inf B_i^*] \mathbb{1}_{B_{ji} \cap [\inf B_i^*]} \\ & + \textstyle\sum_{i=1}^{m} |B_{ji}| c_i \leq \bar{b}_j \}. \end{aligned}$$

- *A pure strategy is a mapping $v_j \mapsto B_j \subset \mathbb{N}$. A constrained pure strategy is a mapping $v_j \mapsto B_j \in \mathcal{B}_j$.*
- *A mixed strategy is a probability measure over the set of pure strategies.*

The action set $\mathcal{B}_j(v_j, \bar{b}_j)$ is finite because of budget limitations; hence, a bid $b_{ji} \in B_{ji}$ is less than $\min(\bar{b}_j, v_{ji} - c_r)$. We define a solution concept of the above game with incomplete information: Bayes-Nash equilibrium.

Definition 2. *A mixed Bayes-Nash strategy equilibrium is a profile $(s_j(v_j))_j$ such that for all bidders j $\mathbb{E}_{s_j,s_{-j}} r_j(B_j(v_j), B_{-j}|\, v_j) \geq \mathbb{E}_{s'_j,s_{-j}} r_j(B'_j, B_{-j}|\, v_j)$, for any strategy s'_j.*

The information structure of the auction is as follows. Each bidder knows its own-value and own-bid but not the valuation of the other bidders. Each bidder has the valuation cumulative distribution of the others. The structure of the game is common knowledge. We are interested in the equilibria, the equilibrium payoffs of the bidders, and the revenue of the seller.

Existence of Bayes-Nash Equilibrium

Proposition 1. *The multi-item Bayesian LUBA game (without resubmission) but with arbitrary number of bidders has at least one Bayes-Nash equilibrium in mixed strategies under budget restrictions.*

Proposition 1 provides the existence of at least one Bayes-Nash equilibrium. However, it does not tell us what are those equilibria or how can we target them. The next subsection computes some of the equilibria in specific setups.

Computation of Bayes-Nash Equilibrium

Proposition 2. *Any strategy b_j such that $b_{ji} > \min\left(v_{ji} - \tilde{c} - 1, \bar{b}_j\right)$ is strictly dominated by strategy"0". If $v_{ji} < c$, bidder j is better off of not participating on item i.*

Thus, the bid space of j can be reduced to $\prod_{i=1}^{m} \{0, 1, 2, \ldots, \min(v_{ji} - \tilde{c} - 1, \bar{b}_j)\}$. Now, we analyze some of the equilibria in specific setups.

Two Participants. Let $n = 2$, and the action space is restricted to $\{0, 1, 2, \ldots, v - \tilde{c}\}$, where $\tilde{c} = c + c_r$.

Proposition 3. *Suppose $v > \tilde{c} + 1$ and $n = 2$. With the non-participation option, i.e., when the bid can be zero, $(0, \ldots, 0, 1, 0, \ldots, 0)$ and its permutations are equilibria.*

We can easily derive proposition 3 from the payoff matrix, which is shown in Table 1.

Table 1: Payoff matrix of 2 bidders bidding 1 item.

	0	1	2	3
0	$0, 0$	$(0, v - \tilde{c} - 1)^*$	$0, v - \tilde{c} - 2$	$0, v - \tilde{c} - 3$
1	$(v - \tilde{c} - 1, 0)^*$	$-\tilde{c}, -\tilde{c}$	$v - \tilde{c} - 1, -\tilde{c}$	$v - \tilde{c} - 1, -\tilde{c}$
2	$v - \tilde{c} - 2, 0$	$-\tilde{c}, v - \tilde{c} - 1$	$-\tilde{c}, -\tilde{c}$	$v - \tilde{c} - 2, -\tilde{c}$

Proposition 4. *When the budgets are identical and equal to k, and $n = 2$, There is a partially mixed equilibrium which is symmetric and is explicitly given by*

$$(X_0, X_1, X_2, \ldots, X_k) = (\frac{\tilde{c}}{v-1}, 1 - \frac{\tilde{c}}{v-1}, 0, \ldots, 0).$$

Three Participants. With three participants, the action profile $(1, 1, 1)$ is not an equilibrium anymore: if players 1 and 2 bid 1 cent then player's 3 best response is to bid 2 and player 3 will be the winner of the auction. The game is not a dominant solvable game.

Table 2: Payoff matrix of 3 bidders targetting 1 item.

	0	1
0	$0, 0, 0$	$(0, v - \tilde{c} - 1, 0)^*$
1	$(v - \tilde{c} - 1, 0, 0)^*$	$-\tilde{c}, -\tilde{c}, 0$

	0	1
0	$(0, 0, v - \tilde{c} - 1)^*$	$0, -\tilde{c}, -\tilde{c}$
1	$-\tilde{c}, 0, -\tilde{c}$	$-\tilde{c}, -\tilde{c}, -\tilde{c}$

Proposition 5. *When the budgets are identical and $n \geq 3$, There is a partially mixed equilibrium which is symmetric and is explicitly given by*

$$(X_0, X_1, X_2, \ldots, X_k) = (\sqrt[n-1]{\frac{\tilde{c}}{v-1}}, 1 - \sqrt[n-1]{\frac{\tilde{c}}{v-1}}, 0, \ldots, 0).$$

3.1 Multi-item LUBA Game

Now we investigate the property of multi-item LUBA game. First, we analyze a simple setup, then we extend it to a general form. We assume $n = 2$, $\overline{b_1} = 4 + \tilde{c}$, and $\overline{b_2} = 3 + \tilde{c}$. Thus, the action profiles of each bidder can be expressed as $A_1 = \{b_1 : b_{11} + b_{12} \leq 4\}$ and $A_2 = \{b_2 : b_{21} + b_{22} \leq 3\}$. More specifically,

$$\begin{cases} A_1 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), \\ (0,3), (2,1), (1,2), (4,0), (3,1), (0,4), (1,3), (2,2)\} \end{cases}$$

$$\begin{cases} A_2 = \{(0,0), (1,0), (0,1), (2,0), (0,2), (1,1), (3,0), \\ (0,3), (1,2), (2,1)\} \end{cases}$$

From the action files, we can derive that $\{(0,0), (1,1)\}$, $\{(1,1), (0,0)\}$, $\{(0,1), (1,0)\}$, and $\{(1,0), (1,0)\}$ are pure Nash equilibria. For the extended version, the game can be defined as $G(J, (\bar{b}_j)_{j \in J}, (c, c_r), m, (F_j)_{j \in J})$. When $\bar{b}_j$ is given, we can derive that the action profile $A_j = D_0 \cup D_1 \cup D_2 \cup \ldots \cup D_{\overline{b}_j}$, where $D_k = \{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid a_i \geq 0, \sum_{i=1}^{m} a_i = k\}$ is the set of all possible decomposition of the integer k.

Proposition 6. *If $\bar{b}_j > 1 + \tilde{c}$, the set of pure Nash equilibria of multi-item auction game $G(J, (\bar{b}_j)_{j\in J}, (c, c_r), m, (F_j)_{j\in J})$, $n \geq 1, m \geq 1$ without resubmission is the set of matrices*

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2m} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nm} \end{bmatrix}$$

such that $\forall i, j, b_{ji} \in \{0,1\}$, $\forall i \in \{1,\dots,m\}, \sum_{j=1}^{n} b_{ji} = 1$ *and* $\forall j \in \{1,\dots,n\}$, $\sum_{i=1}^{m} b_{ji} \leq \bar{b}_j - \tilde{c}$. *The pure Nash equilibria are also global optima of the game.*

3.2 Learning Algorithm

Imitative learning plays a role in many applications (Tembine, 2012) such as security & reliability, cloud networking, power grid and evolution of protocols and technologies. It has been successfully utilized in order to capture human learning and animal behavior. We calculate the instant payoff of bidder j targeting item i at tth round through $R^t_{ji} = -c_r - c + \mathbb{1}_{\text{j wins}}(v_{ji} - b^t_{ji})$. Bidder j wins item i at round t if the placing bid b^t_{ji} is the lowest unique bid on item i. $R^t_{ji} = 0$ if bidder j does not participate to item i. This algorithm below describes how to update the reward and the strategy in the bid space. $\hat{R}^t_{ji}(k)$ is the estimation of the reward a bidder j can obtain by holding the bid k. Denote the estimate of $R^t_{ji}(k)$ by $\hat{R}^t_{ji}(k)$.

Algorithm 1: The Imitative learning algorithm.

Initialization: Generate $\hat{R}^0_{ji}(0), \hat{R}^0_{ji}(1), \dots, \hat{R}^0_{ji}(\bar{b}(0))$ from the uniform distribution.
For Round $t+1$:
1.**Update the reward estimation:**$\hat{R}^t_{ji}(k) : \hat{R}^{t+1}_{ji}(k) = \hat{R}^t_{ji}(k) + \mathbb{1}_{\{b^t_{ji}=k\}} \alpha^t_{ji}(R^t_{ji} - \hat{R}^t_{ji}(k))$
2.**Update the mixed strategy:** $X^{t+1}_{ji}(k) = X^t_{ji}(k)(1+\lambda^t_{ji})^{\hat{R}^t_{ji}(k)}$
3.**Normalize the mixed strategy.**
Iterate until convergence.

The next result provides a convergence of the algorithm to pure equilibria.

Proposition 7. *If the algorithm starts from interior point, it converges to an ending point in a stable steady state of the replicator equation. This counts for an arbitrary number of participants. Therefore it is a Nash equilibrium. In the asymmetric case, the mixed equilibrium is unstable and the algorithm converges to one of pure Nash equilibria.*

We give a simple example to show that the mixed equilibrium is unstable, and the algorithm converges to one of pure Nash equilibria. We analyze the situation where $n = 2$, $v = 4$, $c = 1$. In this specific parameter setting, the ordinary differential equations (ODEs) are given by

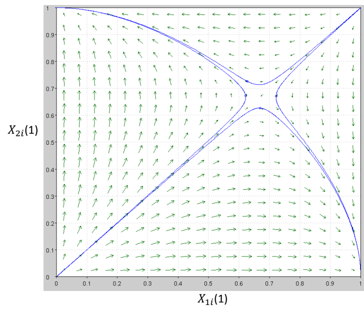

Figure 1: The vector field of ODEs where $n = 2$, $v = 4$, $c = 1$.

$$\begin{cases} \dot{X}_{1i}(1) = X_{1i}(1)(1 - X_{1i}(1))(2 - 3X_{2i}(1)) \\ \dot{X}_{2i}(1) = X_{2i}(1)(1 - X_{2i}(1))(2 - 3X_{1i}(1)) \end{cases}$$

Figure 1 shows the vector field of the ODEs. By analyzing the vector field, we can observe that the algorithm always converges to one of pure Nash equilibria except the symetric situation; however, the probability of symmetric situation is zero.

Proposition 8. *In the symmetric case, the mixed equilibrium is stable and the dynamical system converges to it. However, the probability of obtaining equal numbers by stochastic processes is nearly zero.*

4 EXPERIMENT

In this part, we do experiments to show the effectiveness of the imitative learning algorithm. First, we present a numerical investigation focusing on learning symmetric equilibria for two bidders setting; then, we extend to the asymmetric situation. Table 3 shows the experiment setting. We compare and contrast the results with the theory value presented in proposition 4.

Table 3: Summary of two bidders setting.

Symbol	Setting
n	2
I	1
c	1
b_{min}	1
Initial of $\hat{R}^0_{ji}$	[1 1 1]
Budget	Static budget constraint

As shown in Figure 2, the proposed algorithm can effectively learn the Nash equilibrium at a fast convergence rate. More specifically, for big v, the learning algorithm converges in a few steps. For the asymmetric situation, the experiments in Figure 3 show that the learning process can also converge to one of pure equilibria in a few steps.

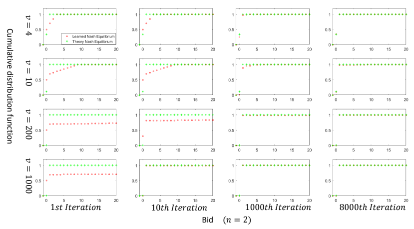

Figure 2: Nash equilibrium. The red squares present the strategy learned by the proposed algorithm and the green stars represent the theoretical equilibrium.

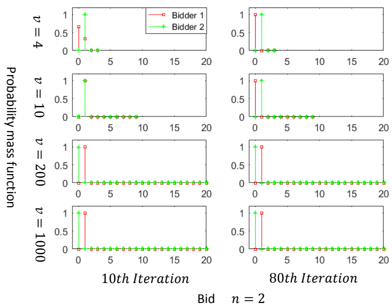

Figure 3: Learned strategies. The red squares present bidder 1's learned strategy under the proposed algorithm and the green stars represent bidder 2's learned strategy.

For multi-item under budget constrains, we also do a simulation experiment to show the performance of the proposed learning algorithm. The experiment settings are shown in Table 4. We utilize a dynamic setting with changing budget in this experiment by setting different v_{ji} and different initial budget. Considering the resources of item is limited, we also assign resources of each item. Figure 4 shows the results.

Table 4: Summary of Multi-Items Experiment Setting.

Symbol	Setting
$\mathcal{N}$	4
I	2
c	1
b_{min}	1
Resource for item 1 and 2	[2000 1500]
Initial of Budget	[100 120 80 90]
Initial of $\hat{R}^0_{ji}$	0.0001
v for $j \in \{1,2,3,4\}$	[40,102],[42,109],[38,100],[36,110]
$\bar{b}_{ji}$	$\min(v_{ji}, budget_{ji})$
Assumption	Description
Budget	Budget evolves with the iteration.

4.1 Statistic Properties

In order to show the robustness of the proposed algorithm, we do an experiment analysis under the static budget constraint and analyze its statistic properties. Table 5 presents the experiment setting.

We evaluate the convergence rate by introducing the root-mean-square error (RMSE). The probability distributions of sequential round strategies

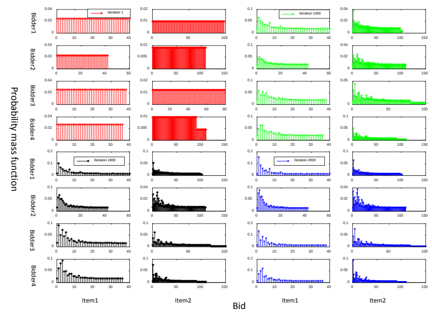

Figure 4: Probability mass function of four players and two items in the auction system with budget update.

Table 5: Summary of Experiment Setting.

Symbol	Setting
$\mathcal{N}$	4
I	1
α	0.1
λ	0.1
c	1
b_{min}	1
Initial of X^0_{ji}	Uniform distribution
Initial of $\hat{R}^0_{ji}$	Unidrnd(20,1,Budget)
v_{ji} for $j \in \{1,2,3,4\}$	[147,165,170,177]
$\bar{b}_{ji}$ for $j \in \{1,2,3,4\}$	[100,100,100,100]
Budget	Static Budget Constraint

are utilized to calculate the RMSE. $RMSE_{t,t-1} = \sqrt{\sum_{j=1}^{\mathcal{N}} \sum_{k=1}^{\bar{b}_j(t)} (X^t_{ji}(k) - X^{t-1}_{ji}(k))^2}$. In order to analyze the statistic property of the experiment results, we reconduct the experiment 200 times using the experiment setting in the Table 5. The analysis of the statistic properties are shown in Figure 5.

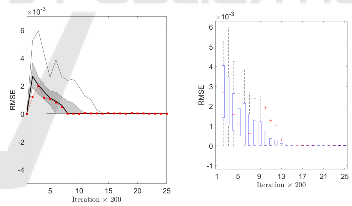

Figure 5: Statistic information on the RSME of the strategies.

Figure 5 presents the distribution of RMSE (percentiles P5 P25 P50 P75 P90) obtained from the proposed learning algorithm. The results in this figure show all the repeat experiment converge to zero and a narrow distribution of RMSE over the whole period.

4.2 Impact of Parameters

We investigated the impact of the parameters in the proposed learning algorithm described in Sect.3.2. Table 6 shows experiment setting details. Figure 6 and 7 present the statistic properties of RMSE evolution obtained by the proposed learning algorithm. Compared to $\alpha = 0.1$ and $\lambda = 0.1$, the results show a fast convergence rate. The Boxplot shows that the large parameter setting results less outliers in the experiment results. According to the results in Figure 6 and 7, the parameter α and λ influence the convergence of the proposed algorithm equally and a large α can reduce disturbance and outliers more effectively.

Table 6: Summary of Experiment Setting in Parameters Impact Investigation.

Symbol	Original Setting	Compared Setting
α	0.1	1
λ	0.1	1
Others	Same as Table 5	

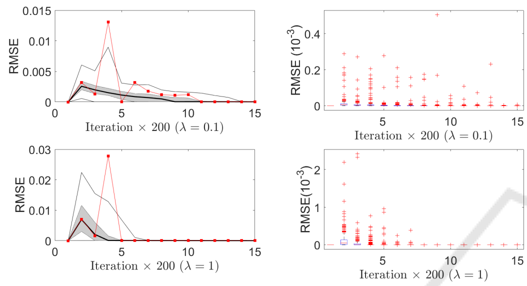

Figure 6: Statistic information on RSME on strategies.

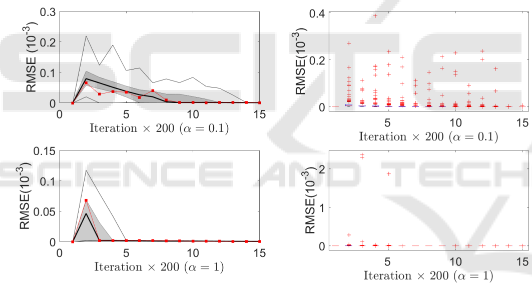

Figure 7: Statistic information on RSME.

5 LUBA WITH RESUBMISSION

In this section, we assume that each bidder can resubmit bids a certain number of times subject to her available budget, each resubmission for item i will cost c_i. If bidder j has (re)submitted n_{ji} times on item i, his or her total submission cost would be $n_{ji}c_i$ in addition to the registration fee. Denote by the set that contains all the bids of bidder j on item i by B_{ji}. Thus, $n_{ji} = |B_{ji}|$ is the cardinality of the strictly positive bids by j on item i. The set of bidders who are submitting b on item i is denoted by $N_{i,b} = \{j \in \mathcal{N} \mid b \in B_{ji}\}$. In order to get the set of all unique bids, we introduce the following: The set of all positive natural numbers that were chosen by only one bidder on item i is $B_i^* = \{b > 0 \mid |N_{i,b}| = 1\}$. If $B_i^* = \emptyset$ then there is no winner on item i at that round (after all the resubmission possibilities). If $B_i^* \neq \emptyset$ then there is a winner on item i and the winning bid is $\inf B_i^*$ and winner is $j^* \in N_{i,\inf B_i^*}$. The payoff of bidder j on item i at that round would be $r_{ji} = v_{ji} - |B_{ji}|c_i - \inf B_i^*$, if j is a winner on item i, and $r_{ji} = -|B_{ji}|c_i$, if j is not a winner on item i. A pure strategy of a bidder is a choice of a subset of natural numbers. That set is finite because of budget limitation. Thus, given its own valuation vector $(v_{ji})_i$ bidder j will choose an action $(B_{ji})_i$ that respect the budget constraints

$$\forall j,\ \ c_r + \sum_{i=1}^{m} [\inf B_i^*]\, \mathbb{1}_{B_{ji}\cap[\inf B_i^*]} + \sum_{i=1}^{m} |B_{ji}|c_i \leq \bar{b}_j.$$

A bid is hence less than $\min(\bar{b}_j, v_{ji} - c)$. The instant payoff of the auctionner of item i is

$$r_{a,i} = \left(\sum_j c_r \mathbb{1}_{\{B_{ji}\neq\emptyset\}} + \inf B_i^* + \sum_{j=1} |B_{ji}|c_i\right) - v_{a,i}.$$

The instant payoff of the auctionner of a set of item I is $r_{a,I} = \sum_{i\in I} r_{a,i}$. The following result provides existence of equilibria in behavioral mixed strategies.

Proposition 9. *The multi-item LUBA game with resubmission has at least one equilibrium in behavioral mixed strategies.*

Proposition 10. *Let $n = 2$ and each bidder can resubmit a certain number of bids, each resubmission will cost $c < v - 1$. The game has a partially mixed equilibrium which is explicitly given by*

$$y^* = \left(\frac{c}{v-1}, \frac{c}{v-2}, \ldots, \frac{c}{v-k}, 1 - \sum_{l=0}^{k-1} \frac{c}{v-(l+1)}, 0, \ldots, 0\right)$$

where k is the maximum number such that $\sum_{l=0}^{\min(\bar{b},\frac{v}{c})} \frac{\tilde{c}}{v-l} < 1$.

We now investigate how much money the online platform can make by running multi-item LUBA. Since the platform will be running for a certain time before the auction ends, each bidder is facing a a random number of other bidders, who may bid in a stochastic strategic way. Their valuation is not known. We need to estimate the set of bids B_i and the bid values on item i. We denote by n_{ib} the random number of bidders who bid on b. If $n_{i,b}$ follows a Poisson distribution with parameter $\lambda_{i,b}$, and that all variables $n_{i,b}$ are independent. The value $\lambda_{i,b}$ is assumed to be non-decreasing with b. The expected payoff of the seller on item i is $\sum_j c_r + \sum_b \mathbb{E} n_{i,b} c_i + \mathbb{E}\inf B_i^* - v_i$.

Proposition 11. *Let $\lambda_{i,b} = \frac{v_i}{c_i}\frac{1}{(1+b)^z}$, with $z > 0$. The expected revenue of the seller on item i is $\sum_j c_r + \mathbb{E}\inf B_i^* + v_i[\sum_b \frac{1}{(1+b)^z}]$, exceeds the value v_i of the item i for small value of z whenever $\sum_{b\leq\min(\bar{b},\frac{v_i}{c_i})} \frac{1}{(1+b)^z} > 1 - \frac{\sum_j c_r + \mathbb{E}\inf B_i^*}{v_i}$*

6 CONCLUSION

We propose and analyze a multi-item LUBA game with budget constraint, registration fee and resubmission cost. We show that the analysis can be reduced into a finite game (with incomplete information) by eliminating the bids that are higher than the value of the item or by the bid that are higher the total available budget. Using classical fixed-point theorem, there is at least one Bayes-Nash equilibrium in mixed strategies. Next, we address the question of computation and stability of such an equilibrium. We provide explicitly the equilibrium structure in simple cases. In the general setting, we provide a learning algorithm that is able to locate equilibria. We propose an imitative combined fully distributed payoff and strategy learning (imitative CODI- PAS learning) that is adapted to LUBA. We examine how the bidders of the game are able to learn about the online system output using their own-independent learning strategies and own-independent valuation. The revenue of the auctionner is explicitly derived in a situation where a random number of bids are placed.

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PROOFS

Proof of Proposition 2. By budget constraint, j's bids must fulfill $\sum_i b_{ji} \leq \bar{b}_j$. If $b_{ij} > v_{ji} - \tilde{c}$, j gets $v_{ji} - \tilde{c} - b_{ji}$ which is negative (loss), and j could guarantee zero by not participating. Therefore the strategy 0 dominates any b_{ji} higher than $v_{ji} - \tilde{c}$.

Proof of Proposition 1. By Proposition 2 the constrained game has a finite number of actions. By standard fixed-point theorem, the multi-item Bayesian LUBA game has at least one Bayes-Nash equilibrium in mixed strategies.

Proof of Proposition 3. Suppose $v > \tilde{c}+1$. Then the payoff when only one agent places a bid of 1 on item i is $v-\tilde{c}-1 > 0$ for that agent, and others get 0. When a single devient change any other bidders' decision to bid 1 instead of non-participation, there is a collision and the bid 1 is not unique anymore. Both agents gets $-\tilde{c} < 0$ and the rest of the agents gets nothing.

Proof of Proposition 4. In order to find a strictly mixed equilibrium for each bidder, we can introduce the indifference condition. Now we can easily calculate X_m by its general form and the initial condition. $X_m = \max(0, y_{m-1}-y_{m-2}) = \max(0, \frac{\tilde{c}}{v-(m-1)} - \frac{\tilde{c}}{v-m}) = 0$, and $X_1 = 1-\frac{\tilde{c}}{v-1}$, where $\frac{\tilde{c}}{v-1} < 1$. Therefore $X^* = (\frac{\tilde{c}}{v-1}, 1-\frac{\tilde{c}}{v-1}, 0, \ldots, 0)$ is a partially mixed equilibrium, and the expected equilibrium payoff at (X^*, X^*) is zero.

Proof of Proposition 5. As the payoff for bidder $i=1$ is equal to 0 when he or she bids 0, we can deduce that the first equation in the indifference condition is equal to 0. When bidder $i=1$ bids 1, his or her payoff can be presented by $(v-\tilde{c}-1)*P_{11}(x)-\tilde{c}*(1-P_{11}(x))$, where $P_{11}(x) = {X_0}^{n-1}$ is the probability that bidder 1 wins under bid 1.

Proof of Proposition 7. In the proposed algorithm, we update the strategy according to $X_{ji}^{t+1}(k) = \frac{X_{ji}^t(k)(1+\lambda_{ji}^t)^{\hat{R}^t_{ji}(k)}}{\sum_k X_{ji}^t(k)(1+\lambda_{ji}^t)^{\hat{R}^t_{ji}(k)}}$. Rewriting it by subtracting $X_{ji}^t(k)$, then dividing the result by λ_{ji}^t, we can conclude $\frac{X_{ji}^{t+1}(k)-X_{ji}^t(k)}{\lambda_{ji}^t} = X_{ji}^t(k)[\frac{(1+\lambda_{ji}^t)^{\hat{R}^t_{ji}(k)}}{\lambda_{ji}^t\Sigma_k} - \frac{\Sigma_k}{\lambda_{ji}^t\Sigma_k}]$, where Σ_k denotes $\sum_k X_{ji}^t(k)(1+\lambda_{ji}^t)^{\hat{R}^t_{ji}(k)}$. As $\lim_{\lambda\to 0}\frac{(1+\lambda)^n-1}{\lambda} = \frac{1+\binom{n}{1}\lambda+\binom{n}{2}\lambda^2+\cdots\binom{n}{n}\lambda^n-1}{\lambda} = n$, and $\lim_{\lambda\to 0}(1+\lambda)^n = 1$ we can get the $\dot{X_{ji}} = X_{ji}(\hat{R}^t_{ji}(k) - \sum_k X_{ji}\hat{R}^t_{ji}(k))$. According to Proposition 5, we only consider $k=0$ and $k=1$ and derive that $\dot{X_{ji}} = X_{ji}(1-X_{ji})[(v-1)\prod_{p\neq j}(1-X_{pi})-c]$. Obviously, an point $X_{ji} = 1-\sqrt[n-1]{\frac{c}{v-1}}$ for $j\in[1,\ldots,n]$ is a steady point of ordinary differential equations (ODEs), and its corresponding matrix is A. Assuming $\det(A-\lambda I) = (-1)^n(\lambda-a)^{n-1}(\lambda+a(n-1)) = 0$, where $a = c(1-\sqrt[n-1]{\frac{c}{v-1}})$, we can deduce the eigenvalues vector of A is [-(n-1)a, a ,a,…,a]. There is at least one positive eigenvalue, which means the steady point is not stable. So, the mixed Nash equilibria is not stable.

Proof of Proposition 8. In the symmetric situation, using the methodology used in the proof of Proposition 7, we can derive that the eigenvalues of the matrix corresponding steady point have negative real parts. Thus, the steady point is stable.

Proof of Proposition 9. A proof can be obtained following similar lines as in Proposition 1 with a notable difference that here the action is a choice of subset of the budget-constrained bid space.

Proof of Proposition 10. Let k be the largest integer such that $y_k = \mathbb{P}(\{0,1,...,k\}) > 0$, $k \leq \bar{b}$. The expected payoff of bidder 1 when playing $\{0,...,l\}$ is therefore given by

$$\begin{aligned}
&\text{Action}\{0\} : r_{1i}(\{0\}, y) = 0.\text{Action}\{01\} : \\
&r_{1i}(\{01\}, y) = (v-c-1)y_0 - cy_1 - c(y_2+\ldots+y_k), \ldots \\
&\text{Action}\{012\ldots l\} : \\
&r_{1i}(\{012\ldots l\}, y) = (v-lc-1)y_0 + (v-lc-2)y_1 \\
&+\ldots+(v-lc-l)y_{l-1} - lcy_l - lc(y_{l+1}+\ldots+y_k) \\
&\ldots
\end{aligned}$$

It turns out that

$$\begin{cases}
(v-1)y_0 = c. \quad (v-1)y_0+(v-2)y_1 = 2c \\
\ldots \\
(v-1)y_0+(v-2)y_1+\ldots+(v-l)y_{l-1} = lc \\
\ldots \\
(v-kc-1)y_0+(v-kc-2)y_1+\ldots+(v-kc-k)y_{k-1} = kc.
\end{cases}$$

For l between 1 and $k-1$ we make the difference between line $l+1$ and line l to get:

$$\begin{cases}
y_0 = \frac{c}{v-1} \quad y_1 = \frac{c}{v-2} \ldots y_l = \frac{c}{v-(l+1)} \ldots y_{k-1} = \frac{c}{v-k} \\
y_k = 1-(y_0+y_1+\ldots+y_{k-1}) > 0 \quad y_{k+1+s} = 0.
\end{cases}$$

Thus, the partially mixed strategy

$$y^* = (\frac{c}{v-1}, \frac{c}{v-2}, \ldots, \frac{c}{v-k}, 1-\sum_{l=0}^{k-1}\frac{c}{v-(l+1)}, 0, \ldots, 0)$$

is an equilibrium strategy. The equilibrium payoff is zero.

Proof of Proposition 11.

The expected payoff of the seller on item i is equal to $\sum_j c_r + \sum_b \mathbb{E}n_{i,b}c_i + \mathbb{E}\inf B_i^* - v_i$. We can calculate that $\mathbb{E}n_{i,b} = \lambda_{i,b} = \frac{v_i}{c_i}\frac{1}{(1+b)^z}$. Rewrite the expected payoff of the seller, we can get it is equal to $\sum_j c_r + \sum_b \mathbb{E}n_{i,b}c_i + \mathbb{E}\inf B_i^* - v_i$. Then we can easily induce the condition for the expected revenue of the seller exceeds the value of v_i is

$$\sum_{b\leq\min(\bar{b}, \frac{v_i}{c_i})}\frac{1}{(1+b)^z} > 1 - \frac{\sum_j c_r + \mathbb{E}\inf B_i^*}{v_i}.$$