Upper Bounds for the Total Chromatic Number of Join Graphs and Cobipartite Graphs

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Abstract: We concern ourselves with the combinatorial optimisation problem of determining a minimum total colouring of a graph G for the case wherein G is a join graph $G = G_1 * G_2$ or a cobipartite graph $G = (V_1 \cup V_2, E(G))$. We present algorithms for computing a feasible, not necessarily optimal, solution for this problem, providing the following upper bounds for the total chromatic numbers of these graphs (let $n_i := |V_i|$ and $\square_i := \square(G_i)$ for $i \in \{1,2\}$ and $\square \in \{\Delta, \chi, \chi', \chi''\}$): $\chi''(G) \leqslant \max\{n_1, n_2\} + 1 + P(G_1, G_2)$ if G is a join graph, wherein $P(G_1, G_2) := \min\{\Delta_1 + \Delta_2 + 1, \max\{\chi'_1, \chi''_2\}\}$; $\chi''(G) \leqslant \max\{n_1, n_2\} + 2(\max\{\Delta^B_1, \Delta^B_2\} + 1)$ if G is cobipartite, wherein $\Delta^B_i := \max_{u \in V_i} d_{G[\partial_G(V_i)]}(u)$ for $i \in \{1,2\}$. Our algorithm for the cobipartite graphs runs in polynomial time. Our algorithm for the join graphs runs in polynomial time if $P(G_1, G_2)$ is replaced by $\Delta_1 + \Delta_2 + 1$ or if it can be computed in polynomial time. We also prove the Total Colouring Conjecture for some subclasses of join graphs, such as some joins of indifference (unitary interval) graphs.

1 INTRODUCTION

Many variants of graph colouring problems have been developed and studied in the last century, each with its importance, applications, and open questions[3]. Although most of these combinatorial optimisation problems are NP-hard, there are some polynomial-time algorithms which compute feasible colourings using upper bounds for the optimal number of colours. In the particular case of total colourings, which have applications e.g. in scheduling and in task management in networks (Leidner, 2012), some upper bounds for the total chromatic number of a general n-order graph G of maximum degree Δ are:

- $\chi''(G) \leqslant n + 1$ (Behzad et al., 1967);
- $\chi''(G) \leqslant \chi'(G) + 2\sqrt{\chi(G)}$ (Hind, 1990);
- $\chi''(G) \leqslant \Delta + 10^{26}$ (Molloy and Reed, 1998);
- $\chi''(G) \leqslant \Delta + 8(\ln \Delta)^8$ (Hind et al., 2000).

This paper presents potentially better upper bounds for the case wherein G is a join or a cobipartite graph.

The *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 * G_2$, is the graph defined by $V(G_1 * G_2) := V_1 \cup V_2$ and $E(G_1 * G_2) := E_1 \cup E_2 \cup \{v_1 v_2 : v_1 \in V_1 \text{ and } v_2 \in V_2\}$. A *join graph* is the result of the join of two graphs. Remark that, if G_1 and G_2 are not the same K_1 graph, we can assume without loss of generality that they are disjoint (Zorzi and Zatesko, 2016). A *cobipartite graph* is the complement of a bipartite graph. Since a graph is a join graph if and only if it is the K_1 or its complement is disconnected, join graphs and cobipartite graphs can be recognised in linear time using, for instance, the algorithms presented in (Ito and Yokoyama, 1998).

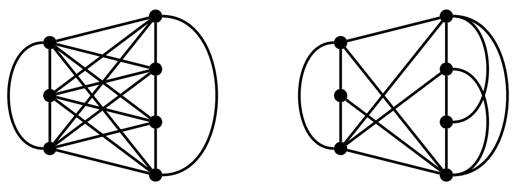

Figure 1: In the left, the join graph $K_3 * C_4$. In the right, a cobipartite graph with $n_1 = 3$ and $n_2 = 4$.

Join graphs and cobipartite graphs have already been studied by several works in the context of edge-colourings, with some partial results been found (Simone and de Mello, 2006; Simone and Galluccio, 2007; Simone and Galluccio, 2009; Machado and

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[3] For an introduction on Graph Colouring we refer the reader to (Jensen and Toft, 1994).

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de Figueiredo, 2010; Simone and Galluccio, 2013; Lima et al., 2015; Zorzi and Zatesko, 2016; Zatesko et al., 2017). One of the reasons why hard graph problems are studied when restricted to join graphs, for example, is the straightforward observation that the class of the join graphs includes some other very important graph classes, such as graphs with a spanning star, complete multipartite graphs, and connected cographs. It is worthy mentioning that almost every graph can be turned into a cograph with no more than $\frac{1}{2}\binom{n}{2} - \frac{1}{5}\eta n^{\frac{1}{2}}$ edge addition or deletion operations (Corneil et al., 1985; Alon and Stav, 2008).

This paper is structured as follows. The remaining of this section introduces some of the definitions used and some known facts relevant for the results we present. In Sections 2 and 3, we present the upper bounds obtained for the total chromatic number of cobipartite graphs and join graphs, respectively. In Section 4, we discuss how to improve the bound for some join graphs, and this improvement enlarges the class of join graphs for which the Total Colouring Conjecture is known to be true. Finally, Section 5 concludes with remarks for future works.

Other Definitions and Related Results

In this paper, we use the term *graph* to refer always to a simple graph, i.e. an undirected loopless graph without multiple edges. Definitions concerning to graph-theoretical concepts follow their usual meanings and notation. Particularly, the *degree* of a vertex u in a graph G is denoted by $d_G(u) := |N_G(u)| = |\partial_G(u)|$, wherein $N_G(u)$ and $\partial_G(u)$ denote, respectively, the *set of the neighbours* of u in G and the *set of the edges* incident to u in G. Also, for any $X \subseteq V(G)$, $\partial_G(X)$ denotes the *cut* defined by X in G, i.e. the set of the edges of G with exactly one endpoint in X.

Let G be a graph and $\mathscr{C}$ be a set of t colours. A *t-vertex-colouring* is a function $\varphi\colon V(G) \to \mathscr{C}$ injective in $\{u,v\}$ for all $uv \in E(G)$. The least t for which G is t-vertex-colourable is the *chromatic number* of G, denoted by $\chi(G)$. A *t-edge-colouring* is a function $\varphi\colon E(G) \to \mathscr{C}$ injective in $\partial_G(u)$ for all $u \in V(G)$. The least t for which G is t-edge-colourable is the *chromatic index* of G, denoted by $\chi'(G)$. A *t-total colouring* is a function $\varphi\colon V(G) \cup E(G) \to \mathscr{C}$ injective in $\{u,v\}$ and injective in $\partial_G(u) \cup \{u\}$ for all $u \in V(G)$ and all $v \in N_G(u)$. The least t for which G is t-total colourable is the *total chromatic number* of G, denoted by $\chi''(G)$. Obviously, $\chi''(G) \leqslant \chi(G) + \chi'(G)$.

If $uv \in E(G)$, $G - uv$ is t-edge-colourable, and $d_{G-uv}(w) < t$ for all $w \in N_G(u) \cup \{u\}$, then G is also t-edge-colourable (Vizing, 1964). Vizing's proof for this statement is constructive and often referred as *Vizing's Recolouring Procedure*. Also, it implies that $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$, in which case G is said to be *Class 1* or *Class 2*, respectively. Although Vizing's Recolouring Procedure yields a polynomial-time algorithm for computing a $(\Delta(G)+1)$-edge-colouring of any graph G, deciding if G is *Class 1* is NP-complete (Holyer, 1981), even restricted to perfect graphs (Cai and Ellis, 1991), a class of graphs for which optimal vertex-colourings can be computed in polynomial time using linear programming — see, for example, (Grötschel et al., 1988, Chapter 9).

If G is a graph on n vertices with maximum degree $\Delta > n/3$, the *Overfull Graph Conjecture* (Chetwynd and Hilton, 1984; Chetwynd and Hilton, 1986; Hilton and Johnson, 1987) states that G is *Class 2* if and only if it satisfies a property known as *subgraph-overfullness*, which can be tested in polynomial time (Padberg and Rao, 1982; Niessen, 1994; Niessen, 2001). Join graphs and connected cobipartite graphs satisfy $\Delta \geqslant n/2$ by definition, but no polynomial-time algorithm is known for computing the chromatic index of all join or cobipartite graphs. Recall that all bipartite graphs are *Class 1* (Kőnig, 1916).

Except for complete graphs and odd cycles, which have $\chi(G) = \Delta + 1$, $\chi(G) \leqslant \Delta$ by Brooks's Theorem (Brooks, 1941). Therefore, $\chi''(G) \leqslant 2\Delta + 2$. The *Total Colouring Conjecture*, proposed independently by (Behzad, 1965) and (Vizing, 1968), states that $\chi''(G) \leqslant \Delta + 2$ for every graph G. In view of that $\chi''(G) \geqslant \Delta + 1$ by definition, graphs with $\chi''(G) = \Delta + 1$ and $\chi''(G) = \Delta + 2$ have been called *Type 1* and *Type 2*, respectively. This conjecture was proved for some graph classes, such as the complete graphs and the complete bipartite graphs (Behzad et al., 1967), and graphs with $\Delta \geqslant \frac{3}{4}n$ (Hilton and Hind, 1993). In particular, the complete graph K_n is *Type 1* if n is odd, or *Type 2* otherwise, and the complete bipartite graph K_{n_1,n_2} is *Type 1* if $n_1 \neq n_2$, or *Type 2* otherwise (Behzad et al., 1967). Recall that computing the total chromatic number of a graph is NP-hard (Sánchez-Arroyo, 1989), even if restricted to bipartite graphs (McDiarmid and Sánchez-Arroyo, 1994).

A *pullback* from a graph G_1 to a graph G_2 is a homeomorphism $f\colon V(G_1) \to V(G_2)$ (i.e. a function such that $f(u)f(v) \in E(G_2)$ for all $uv \in E(G_1)$) injective in $N_{G_1}(u) \cup \{u\}$ for all $u \in V(G_1)$. If there is a pullback from G_1 to G_2, then $\chi'(G_1) \leqslant \chi'(G_2)$ and $\chi''(G_1) \leqslant \chi''(G_2)$ (de Figueiredo et al., 1999).

Now, let $\mathscr{C}$ be a set of colours, no matter how many. Under an assignment of a *list* $L(u) \subseteq \mathscr{C}$ for each $u \in V(G)$, a *vertex-list-colouring* is a vertex-colouring $\varphi\colon V(G) \to \mathscr{C}$ such that $\varphi(u) \in L(u)$ for all $u \in V(G)$. The graph G is said to be *t-vertex-choosable* if it is vertex-list-colourable under *any* assignment of lists to the vertices with at least t colours in each list. The least t for which G is t-vertex-choosable is the *vertex-choosability* of G, denoted by $\mathrm{ch}(G)$. Analogously, under assignments of lists to the edges, we define *edge-list-colourings*, and the least t for which G is *t-edge-choosable* is the *edge-choosability* of G, denoted by $\mathrm{ch}'(G)$. Clearly, $\mathrm{ch}(G) \geqslant \chi(G)$, $\mathrm{ch}'(G) \geqslant \chi'(G)$, and $\chi''(G) \leqslant \mathrm{ch}'(G)+2$.

The *Edge-List-Colouring Conjecture*[4] states that $\mathrm{ch}'(G) = \chi'(G)$ for every graph G. It is worthy remarking that the similar statement concerning to vertex-list-colourings is known to be false, since one can construct a graph with $\chi(G) = 2$ and $\mathrm{ch}(G)$ arbitrarily large (Gravier, 1996), although it is true that $\mathrm{ch}(G) \leqslant \Delta+1$ (Vizing, 1976; Erdős et al., 1979). The Edge-List-Colouring Conjecture has been shown only for a few graphs, such as the bipartite graphs (Janssen, 1993; Galvin, 1995) and the K_n with n odd (Häggkvist and Janssen, 1997) or $n-1$ prime (Schauz, 2014). For the K_n with n even and $n-1$ composite, it is only known that $\mathrm{ch}'(K_n) \leqslant \Delta(K_n)+1 = n$ (Häggkvist and Janssen, 1997). Observe that the K_n is *Class 1* if and only if n is even, a standard result which can be found e.g. in (Fiorini and Wilson, 1977).

2 COBIPARTITE GRAPHS

Throughout this section, G is a connected cobipartite graph with $V(G) = V_1 \cup V_2$, wherein V_1 and V_2 are two disjoint cliques with $|V_1| =: n_1$ and $|V_2| =: n_2$. Connectivity is assumed without loss of generality because the total chromatic number of a graph is the maximum amongst the total chromatic numbers of its connected components. Ergo, $\partial_G(V_1) = \partial_G(V_2) \neq \emptyset$, and we define $B_G := G[\partial_G(V_1)] = G[\partial_G(V_2)]$. Note that B_G is bipartite.

Theorem 1. *Let $\Delta_i^B := \max_{u \in V_i} d_{B_G}(u)$ for $i \in \{1,2\}$. Then, $\chi''(G) \leqslant \max\{n_1,n_2\} + 2(\max\{\Delta_1^B, \Delta_2^B\}+1)$.*

Proof. Let $\mathscr{C}$ be a set with $\max\{n_1,n_2\}+2(\Delta_1^B+1)$ colours, assuming without loss of generality that $\Delta_1^B \geqslant \Delta_2^B$. We shall construct a total colouring φ for G using the colours of $\mathscr{C}$.

Step 1. Choose n_1 colours from $\mathscr{C}$ and assign each one of them to a vertex of V_1.

Step 2. For each $uv \in E(B_G)$ with $u \in V_1$ and $v \in V_2$, create the list $L(uv)$ with any $\Delta(B_G)$ colours of $\mathscr{C}$ distinct from $\varphi(u)$. As $\Delta(B_G) = \mathrm{ch}'(B_G)$, by (Galvin, 1995), we can assign to each uv a colour of $L(uv)$.

Step 3. Now, for each $v \in V_2$, the set $X(v)$ of the colours assigned to the neighbours of v in B_G and to the edges incident to v in B_G has at most $2d_{B_G}(v)$ colours. Hence, if we take the list $L(v) := \mathscr{C} \setminus X(v)$, we have
$$|L(v)| \geqslant \max\{n_1,n_2\} + 2(\Delta_1^B+1) - 2\Delta_2^B \geqslant n_2.$$
Since $\mathrm{ch}(K_{n_2}) = n_2$ is a straightforward result, we can assign to each $v \in V_2$ a colour of $L(v)$.

Step 4. Finally, in order to complete φ, it remains to colour the edges of $E(G[V_1]) \cup E(G[V_2])$. For each uv amongst them, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of B_G adjacent to uv in G. Define then the list $L(uv) := \mathscr{C} \setminus X(uv)$. Since $|X(uv)| \leqslant 2\Delta_1^B + 2$, $|L(uv)| \geqslant \max\{n_1,n_2\}$. Thus, by the result of (Häggkvist and Janssen, 1997) according to which $\mathrm{ch}'(K_n) \leqslant n$, we can assign to each $uv \in E(G[V_1]) \cup E(G[V_2])$ a colour of $L(uv)$. □

Because all the colourings taken in the proof of Theorem 1 can be obtained in polynomial time, our proof is a polynomial-time algorithm to construct a $(\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1))$-total colouring. Recall that $\Delta(G) = \max\{n_1-1+\Delta_1^B, n_2-1+\Delta_2^B\}$, which means that the upper bound provided in Theorem 1 is better than the bounds for general graphs listed in Section 1, as long as Δ_1^B and Δ_2^B are not *too large*, in the sense that the propositions below clarify.

Proposition 2. *If*
$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant \frac{\min\{n_1,n_2\}}{2} - 1,$$
then $\max\{n_1,n_2\} + 2(\max\{\Delta_1^B,\Delta_2^B\}+1)$ is strictly less than $|V(G)|+1$, the upper bound for $\chi''(G)$ by (Behzad et al., 1967).

Proof.
$$\begin{aligned} &\max\{n_1,n_2\} + 2(\max\{\Delta_1^B,\Delta_2^B\}+1) \\ &\leqslant \max\{n_1,n_2\} + \min\{n_1,n_2\} \\ &= n_1+n_2 < |V(G)|+1. \end{aligned}$$
□

Proposition 3. *If*
$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant 5 \times 10^{25} - 2,$$
then $\max\{n_1,n_2\} + 2(\max\{\Delta_1^B,\Delta_2^B\}+1)$ is strictly less than $\Delta(G)+10^{26}$, the upper bound for $\chi''(G)$ by (Molloy and Reed, 1998).

Proof.
$$\begin{aligned} &\max\{n_1,n_2\} + 2(\max\{\Delta_1^B,\Delta_2^B\}+1) \\ &\leqslant \max\{n_1-1+\Delta_1^B, n_2-1+\Delta_2^B\} + 10^{26} - 2 \\ &< \Delta(G) + 10^{26} \end{aligned}$$
□

Proposition 4. *If*
$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant \sqrt{\max\{n_1,n_2\}} - \frac{3}{2},$$
then $\max\{n_1,n_2\} + 2(\max\{\Delta_1^B,\Delta_2^B\}+1)$ is strictly less than $\chi'(G) + 2\sqrt{\chi(G)}$, the upper bound for $\chi''(G)$ by (Hind, 1990).

[4]For more on the origin and the history of this conjecture, see (Jensen and Toft, 1994, Chapter 12).

Proof. $\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1)$
$\leqslant \Delta(G)+2\sqrt{\max\{n_1,n_2\}}-1$
$< \Delta(G)+2\sqrt{\chi(G)} \leqslant \chi'(G)+2\sqrt{\chi(G)}$. □

Proposition 5. *If*

$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant 4\big(\ln(\max\{n_1+\Delta_1^B,n_2+\Delta_2^B\})\big)^8-\frac{3}{2}$$

then $\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1)$ *is strictly less than* $\Delta(G)+8(\ln\Delta(G))^8$*, the upper bound for* $\chi''(G)$ *by (Hind et al., 2000).*

Proof. $\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1)$
$\leqslant \Delta(G)+8(\ln(\Delta(G)))^8-1$. □

3 JOIN GRAPHS

Throughout this section and the next, G is the join of two disjoint graphs G_1 and G_2 with, respectively, n_1 and n_2 vertices and maximum degrees Δ_1 and Δ_2. Now, B_G denotes the complete bipartite graph $G-(E_1\cup E_2)$. For simplicity, we write $\chi_1 := \chi(G_1)$, $\chi_2' := \chi'(G_2)$ etc. Note that $\Delta(G)=\max\{\Delta_1+n_2,\Delta_2+n_1\}$.

Theorem 6. *Let*

$$P(G_1,G_2) := \min\{\Delta_1+\Delta_2+1,\max\{\chi_1',\chi_2''\}\}. \quad (1)$$

Then, $\chi''(G) \leqslant \max\{n_1,n_2\}+1+P(G_1,G_2)$.

Proof. Let $t := \max\{n_1,n_2\}+1+P(G_1,G_2)$ and take two disjoint sets $\mathscr{C}_A$ and $\mathscr{C}_B$ with, respectively, χ_1 and $\max\{\chi_1',\chi_2''\}$ colours. As it can be straightforwardly verified that $|\mathscr{C}_A|+|\mathscr{C}_B| \leqslant t$, take a set $\mathscr{C}$ with t colours having $\mathscr{C}_A$ and $\mathscr{C}_B$ as subsets. We shall construct a total colouring $\varphi\colon V(G)\cup E(G)\to\mathscr{C}$.

Step 1. Take a χ_1-vertex-colouring of G_1 using only the colours of $\mathscr{C}_A$.

Step 2. Take a χ_1'-edge-colouring of G_1 and a χ_2''-total colouring of G_2, both using only the colours of $\mathscr{C}_B$. Since $\mathscr{C}_A$ and $\mathscr{C}_B$ are disjoint, no colour conflict has been created.

Step 3. Now, for each edge $uv \in B_G$, with $u \in V_1$, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of $G_1\cup G_2$ adjacent to uv in G. It is clear that $|X(uv)| \leqslant 1+P(G_1,G_2)$. Define then the list $L(uv) := \mathscr{C}\setminus X(uv)$. Since $|L(uv)| \geqslant t-1-P(G_1,G_2) = \max\{n_1,n_2\}$ and $\mathrm{ch}'(B_G)=\max\{n_1,n_2\}$ (Galvin, 1995), we can assign to each $uv\in E(B_G)$ a colour of $L(uv)$. □

Remark in Theorem 6 that, from the definition of $P(G_1,G_2)$ in (1), the choice of the graphs for the roles of G_1 or G_2 may lead to a better or a worse upper bound. Moreover, if $P(G_1,G_2)$ is known, or if it can be computed in polynomial time, then our proof is a polynomial-time algorithm, provided that the underlying colourings are also known or can be computed. Replacing $P(G_1,G_2)$ by some upper bound on it, such as $\Delta_1+\Delta_2+1$, also makes our algorithm polynomial.

Similar to the bound for the cobipartite graphs, the upper bound presented in Theorem 6 is better than the upper bounds for general graphs listed in Section 1 if $P(G_1,G_2)$ is not *too large*, in the sense that the propositions below clarify.

Proposition 7. *If* $P(G_1,G_2) \leqslant \min\{n_1,n_2\}-1$*, then* $\max\{n_1,n_2\}+1+P(G_1,G_2) < |V(G)|+1$.

Proof. $\max\{n_1,n_2\}+1+P(G_1,G_2)$
$\leqslant \max\{n_1,n_2\}+\min\{n_1,n_2\} = |V(G)|$. □

Proposition 8. *If* $P(G_1,G_2) \leqslant 10^{26}-1$*, then* $\max\{n_1,n_2\}+1+P(G_1,G_2) < \Delta(G)+10^{26}$.

Proof. $\max\{n_1,n_2\}+1+P(G_1,G_2)$
$< \max\{n_1+\Delta_2,n_2+\Delta_1\}+10^{26}$. □

Proposition 9. *If* $P(G_1,G_2) \leqslant 2\sqrt{\chi_1+\chi_2}-1$*, then* $\max\{n_1,n_2\}+1+P(G_1,G_2) < \chi'(G)+2\sqrt{\chi(G)}$.

Proof. $\max\{n_1,n_2\}+1+P(G_1,G_2)$
$< \max\{n_1+\Delta_2,n_2+\Delta_1\}+2\sqrt{\chi_1+\chi_2}$
$\leqslant \chi'(G)+2\sqrt{\chi(G)}$. □

Proposition 10. *If*

$$P(G_1,G_2) \leqslant 8\big(\ln(\max\{n_1+\Delta_1^B,n_2+\Delta_2^B\})\big)^8-1,$$

then $\max\{n_1,n_2\}+1+P(G_1,G_2)$ *is strictly less than* $\Delta(G)+8(\ln\Delta(G))^8$.

Proof. $\max\{n_1,n_2\}+1+P(G_1,G_2)$
$< \max\{n_1+\Delta_2,n_2+\Delta_1\}+8(\ln\Delta(G))^8$. □

4 IMPROVING THE BOUND FOR JOIN GRAPHS

Following (Simone and de Mello, 2006), we denote by G_M the graph $(G_1\cup G_2)+M$ for any perfect matching M on B_G. Inspired by an observation in the same work, we show how the upper bound of Theorem 6 may be lowered in some cases. In the statements, as it is usual for functions $f\colon A\to B$, we denote by $f(X)$ the set $\bigcup_{x\in X} f(x)$ for all $X\subseteq A$.

Theorem 11. *Let* φ *be a total colouring of* G_M *for some perfect matching* M *on* B_G*. If the sets* $\varphi(V_1)$ *and* $\varphi(E_1\cup M\cup V_2\cup E_2)$ *are disjoint and*

$$|\varphi(E_1\cup M\cup V_2\cup E_2)| \leqslant \max\{\chi_1',\chi_2''\} \leqslant \Delta_1+\Delta_2+3,$$

then $\chi''(G) \leqslant \max\{n_1,n_2\}+\max\{\chi_1',\chi_2''\}$.

Proof. Let $\mathscr{C}$ be a set with $t := \max\{n_1, n_2\} + \max\{\chi_1', \chi_2''\}$ colours having $\mathscr{C}_A := \varphi(V_1)$ and $\mathscr{C}_B := \varphi(E_1 \cup M \cup V_2 \cup E_2)$ as subsets. In order to obtain a t-total colouring of G using the colours of $\mathscr{C}$, we start with the total colouring φ of G_M, remaining to colour only the edges of $B_G - M$.

We proceed now as in Step 3 of the proof of Theorem 6. For each edge $uv \in B_G - M$, with $u \in V_1$, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of G_M adjacent to uv in G. Clearly

$$\begin{aligned} |X(uv)| &\leqslant 1 + \min\{\Delta_1 + \Delta_2 + 3, \max\{\chi_1', \chi_2''\}\} \\ &= 1 + \max\{\chi_1', \chi_2''\}. \end{aligned}$$

Therefore, if we define the list $L(uv) := \mathscr{C} \setminus X(uv)$, we have $|L(uv)| = \max\{n_1, n_2\} - 1 = \Delta(B_G)$. Since $\Delta(B_G) = \mathrm{ch}'(B_G)$ (Galvin, 1995), we can assign to each $uv \in E(B_G)$ a colour of $L(uv)$. $\square$

Corollary 12. *If G has a total colouring φ of G_M, for some perfect matching M on B_G, satisfying the preconditions of Theorem 11, and if $\max\{n_1, n_2\} + \max\{\chi_1', \chi_2''\} \leqslant \max\{n_1 + \Delta_2 + 2, n_2 + \Delta_1 + 2\}$, then the Total Colouring Conjecture is true for G, i.e. $\chi''(G) \leqslant \Delta(G) + 2$.*

Theorem 13. *If there is a graph G_3 such that*

1. $\max\{\chi_3'', \Delta_3 + 2\} \leqslant \Delta_1 + \Delta_2 + 3$,
2. *there are a pullback f_{13} from G_1 to G_3 and a pullback f_{23} from G_2 to G_3, and*
3. *there is a perfect matching M on B_G such that, for all $uv \in M$ with $u \in V_1$, $f_{13}(u) = f_{23}(v)$,*

then $\chi''(G) \leqslant \max\{n_1, n_2\} + \max\{\chi_3'', \Delta_3 + 2\}$.

Proof. Let $\mathscr{C}_A$ be a set with χ_1 colours and take any optimal vertex-colouring of G_1. Let $\mathscr{C}_B$ be a set with $\max\{\chi_3'', \Delta_3 + 2\}$ colours, disjoint from $\mathscr{C}_A$, and ψ be a total colouring of G_3 using the colours of $\mathscr{C}_B$. By (de Figueiredo et al., 1999), the function $\varphi_1 : E_1 \to \mathscr{C}_B$ defined by

$$\varphi(uv) = \psi(f_{13}(u) f_{13}(v)), \qquad \forall uv \in E_1,$$

is a proper edge-colouring of G_1, as the function $\varphi_2 : V_2 \cup E_2 \to \mathscr{C}_B$ defined by

$$\begin{aligned} \varphi(u) &= \psi(f_{23}(u)), & \forall u \in V_2, \\ \varphi(uv) &= \psi(f_{23}(u) f_{23}(v)), & \forall uv \in E_2, \end{aligned}$$

is a proper total colouring of G_2. Since it is clear that $\max\{\chi_3'', \Delta_3 + 2\} > \Delta_3 + 1$, at least one colour $\alpha_x \in \mathscr{C}_B$ is *missing at* each $x \in V_3$, i.e. α_x is not the colour assigned by ψ to x nor to any edge incident to x. Ergo, for all $uv \in M$ with $u \in V_1$, the colour $\alpha_{f(u)}$ is missing at both u and v and thence can be assigned to uv. This yields a $(\max\{\chi_3'', \Delta_3 + 2\})$-total colouring φ of G_M with $\varphi(V_1)$ and $\varphi(E_1 \cup M \cup V_2 \cup E_2)$ disjoint and

$$\begin{aligned} |\varphi(E_1 \cup M \cup V_2 \cup E_2)| &\leqslant \max\{\chi_3'', \Delta_3 + 2\} \\ &\leqslant \Delta_1 + \Delta_2 + 3. \end{aligned}$$

The rest of the proof follows as the proof for Theorem 11, but with $\max\{\chi_3'', \Delta_3 + 2\}$ instead of $\max\{\chi_1', \chi_2''\}$. $\square$

Corollary 14. *If there is a graph G_3 satisfying the preconditions of Theorem 13, and if $\max\{n_1, n_2\} + \max\{\chi_3'', \Delta_3 + 2\} \leqslant \max\{n_1 + \Delta_2, n_2 + \Delta_1\} + 2$, then the Total Colouring Conjecture is true for G.*

Theorem 15 and Corollary 16 below deal with the joins of unitary interval graphs. Unitary interval graphs are also known as *indifference* graphs, whose edge-colourings have been studied by (de Figueiredo et al., 1997; de Figueiredo et al., 2000; de Figueiredo et al., 2003), with some partial results been found.

Theorem 15. *If G_1 and G_2 are indifference graphs, then $\chi''(G) \leqslant \max\{n_1, n_2\} + \max\{\Delta_1, \Delta_2\} + 2$.*

Proof. The proof follows from Theorem 13 by taking $G_3 := K_{\max\{\Delta_1, \Delta_2\}+1}$. By (de Figueiredo et al., 1997), there is a pullback from any indifference graph D on k vertices to the K_ℓ for any $\ell > k$ (de Figueiredo et al., 1999), and, moreover, if $0, \ldots, k-1$ is an indifference order of D, a pullback can be given by the function $f(i) = i \bmod \ell$, under $V(K_\ell) = \{0, \ldots, \ell - 1\}$. Therefore, back to our join graph G, it is clear that a matching M on B_G satisfying the requirements of Theorem 13 can be taken. $\square$

Corollary 16. *If G_1 and G_2 are indifference graphs, and if $n_1 = n_2$ or $\Delta_1 = \Delta_2$, then the Total Colouring Conjecture is true for G.*

5 FUTURE WORKS

In order to obtain a total colouring, the algorithms presented in the proofs of Theorems 1 and 6 decompose the input graph and, considering the parts of the decomposition in an appropriate order, work with the solutions for other colouring problems for each part. All these problems are hard combinatorial optimisation problems, and we hope further investigation on them considering the restricted cases of the graphs obtained by our decompositions can lead to better upper bounds for the total chromatic number of join and cobipartite graphs. We also encourage future works to investigate other decompositions.

Theorem 15 and Corollary 16 form an example of how Theorem 13 can be applied in order to prove the Total Colouring Conjecture for a noteworthy subclass of join graphs. Similar results could also be obtained for other subclasses, such as the joins of graphs with no more than $\max\{\Delta_1, \Delta_2\} + 1$ per connected component, since a pullback from each component to the $K_{\max\{\Delta_1,\Delta_2\}+1}$ could be easily obtained. We encourage future works to investigate more applications.

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