Fredholm Integral Equation for Finite Fresnel Transform

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Abstract: The fundamental formula in an optical system is Rayleigh diffraction integral. In practice, we deal with Fresnel diffraction integral as approximate diffraction formula. We seek the function that its total power is maximized in finite Fresnel transform plane, on condition that an input signal is zero outside the bounded region. This problem is a variational one with an accessory condition. This leads to the eigenvalue problems of Fredholm integral equation of the first kind. The kernel of the integral equation is Hermitian conjugate and positive definite. Therefore, eigenvalues are nonnegative and real number. By discretizing the kernel, the problem depends on the eigenvalue problem of Hermitian conjugate matrix in finite dimensional vector space. By using the Jacobi method, we compute the eigenvalues and eigenvectors of the matrix. We applied it to the problem of approximating a function and evaluated the error.

1 INTRODUCTION

The integral theorem of Helmholts and Kirchhoff plays an important role in the development of the scalar theory of diffraction (Goodman, 2005). Although scalar wave propagation is fully described by a single scalar wave equation, fundamental formula in an optical system is Rayleigh diffraction integral. In practice, we deal with Fresnel diffraction integral as approximate diffraction formula. The Fresnel transform has been studied mathematically and revealed the topological properties in Hilbert space (Aoyagi, 1973). In recently, it is also used in image processing, optical information processing, optical waveguides and computer-generated holograms. The extension of optical fields through on optical instrument is practically limited to some finite area. This leads to the spatially band-limited problem. The effect of band limitation has been studied for an optical Fourier transform, namely in the region of the Fraunhofer diffraction. Up to now, sampling theorem have been derived from band-limited effect in Fourier transform plane and applied to application areas. Moreover, sampling function system are orthonormal system in Hilbert space. An orthonormal function system may be considered as coordinate system in some functional space.

In sampling theorem, it is important to develop the orthogonal functional systems (Ogawa, 2009). It has been also revealed the function to minimize the norm of error on condition that L^2-norm of a function in finite Fourier plane is not exceeding a constant (Kida, 1994). In the literature, there are many examples of band-limited function in the Fourier transform, its applications and reference therein (Jerri, 1977). However, the band-limited effect in Fresnel transform plane is not revealed sufficiently.

In this paper, we seek the function that its total power is maximized in finite Fresnel transform plane, on condition that an input signal is zero outside the bounded region. This problem is a variational one with an accessory condition. This leads to the eigenvalue problems of Fredholm integral equation of the first kind. The kernel of the integral equation is Hermitian conjugate and positive definite. Therefore, eigenvalues are real non-negative numbers. By discretizing the kernel and using the value of the representative points, the problem depends on the eigenvalue problem of Hermitian conjugate matrix in finite dimensional vector space. By using the Jacobi method, we compute the eigenvalues and eigenvectors of the matrix. In general finite dimensional vector spaces ($\mathbb{C}^n$), the eigenvalues of Hermitian matrix are real numbers and then eigenvectors from different eigenspaces are orthogonal. We consider the application of the eigenvectors to the problem of approximating a function and evaluate the error between original test functions and approximating functions.

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2 FRESNEL TRANSFORM

Assume that we place a diffracting screen on the $z = 0$ plane. The parameter z represents the normal distance from the input plane. Let ξ, η be the coordinates of any point in that plane. Parallel to the screen at z is a plane of observation. Let x, y be the coordinates of any point in this latter plane. If $f(\xi,\eta)$ represents the amplitude transmittance, then the Fresnel transform is defined by

$$g(x,y;z) = \frac{k\exp(ikz)}{i2\pi z}\iint_{-\infty}^{\infty} f(\xi,\eta) \times \exp\left[\frac{ik}{2z}\{(x-\xi)^2+(y-\eta)^2\}\right] d\xi d\eta, \quad (1)$$

where k is the wave number and $i = \sqrt{-1}$. The inverse Fresnel transform is defined by

$$f(\xi,\eta) = -\frac{k\exp(-ikz)}{i2\pi z}\iint_{-\infty}^{\infty} g(x,y;z) \times \exp\left[-\frac{ik}{2z}\{(x-\xi)^2+(y-\eta)^2\}\right] dxdy. \quad (2)$$

Figure 1 shows a general optical system and its coordinate system. Fresnel transform is a bounded, linear, additive and unitary operator in Hilbert space.

3 EIGENVALUE PROBLEM

To simplify the discussion, we consider only one-dimensional Fresnel transform. The one-dimensional Fresnel transform is defined by

$$F(x,z) = \frac{1}{\sqrt{i2\pi z}}\int_{-\infty}^{\infty} f(\xi) \times\exp\left\{\frac{i}{2z}(x-\xi)^2\right\} d\xi, \quad (3)$$

where we set the wave number unit. The inverse Fresnel transform is defined by

$$f(\xi) = \sqrt{\frac{i}{2\pi z}}\int_{-\infty}^{\infty} F(x,z)$$

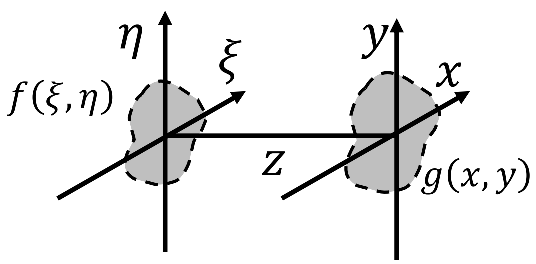

Figure 1: Sketch of a general optical system.

$$\times\exp\left\{-\frac{i}{2z}(x-\xi)^2\right\} dx. \quad (4)$$

Assume that $f(\xi)$ is limited within the finite region R on the ξ-plane and its total power P_R , namely the inner product of the function, is constant.

$$P_R = \int_{-\infty}^{\infty} |f(\xi)|^2 \, d\xi = \int_R |f(\xi)|^2 \, d\xi = \int_R f(\xi)f^*(\xi)\, d\xi = const. \quad (5)$$

where $f^*(\xi)$ denotes the complex conjugate function of $f(\xi)$. Assume that $g(x)$ is the Fresnel transform of the function $f(\xi)$ which is bounded by a finite region R, that is,

$$g(x) = \frac{1}{\sqrt{i2\pi z}}\int_{-\infty}^{\infty} f(\xi)\exp\left\{\frac{i}{2z}(x-\xi)^2\right\} d\xi = \frac{1}{\sqrt{i2\pi z}}\int_R f(\xi)\exp\left\{\frac{i}{2z}(x-\xi)^2\right\} d\xi. \quad (6)$$

Then, the total power P_S of $g(x)$ in the bounded region S is

$$P_S = \int_S |g(x)|^2 \, dx = \int_S g^*(x)g(x)\, dx$$
$$= \int_S \frac{1}{\sqrt{-i2\pi z}}\int_R f^*(\xi)\exp\left\{-\frac{i}{2z}(x-\xi)^2\right\} d\xi \times \frac{1}{\sqrt{i2\pi z}}\int_R f(\xi')\exp\left\{\frac{i}{2z}(x-\xi')^2\right\} d\xi' dx$$
$$= \int_R \int_R K_S(\xi,\xi')f^*(\xi)f(\xi')\, d\xi' d\xi, \quad (7)$$

where the kernel function $K_S(\cdot,\cdot)$ is defined by

$$K_S(\xi,\xi') = \frac{1}{2\pi z}\exp\left\{-\frac{i}{2z}(\xi^2-\xi'^2)\right\} \times \int_S \exp\left\{\frac{i}{z}(\xi-\xi')x\right\} dx. \quad (8)$$

We seek the function $f(\xi)$ that maximizes P_S provided that the total power P_R is fixed. This problem is a variational one with an accessory condition. We use the method of Lagrange multiplier to solve this problem.

Let us define two functions, $H(\xi)$ and $G(\xi)$ as followings.

$$G(\xi) \equiv \int_R f(\xi)f^*(\xi)\, d\xi - const. \quad (9)$$

$$H(\xi) \equiv \int_S |g(x)|^2 \, dx = \int_R \int_R K_S(\xi,\xi')f^*(\xi)f(\xi')\, d\xi' d\xi. \quad (10)$$

We want to maximize the function $H(\xi)$, subject to the constraint $G(\xi)$. Setting λ the Lagrange multiplier, Lagrangian functional is defined by

$$L(\xi) \equiv H(\xi) - \lambda G(\xi). \qquad (11)$$

We set the gradient of Lagrangian to zero, that is,

$$\nabla L(\xi) = \nabla H(\xi) - \lambda \nabla G(\xi) = 0, \qquad (12)$$

where ∇ indicate the gradient.
By Eq. (7) and Eq. (9), we obtain

$$\int_R K_S(\xi,\xi') f^*(\xi) f(\xi')\, d\xi' - \lambda f(\xi) f^*(\xi) = 0. \qquad (13)$$

We conclude that

$$\int_R K_S(\xi,\xi') f(\xi')\, d\xi' = \lambda f(\xi). \qquad (14)$$

This is the Fredholm integral equations of the first kind. This equation corresponds to some modification of the integral equation for prolate spheroidal wave functions (Slepian and Pollak, 1961).
According to Eq. (7), we can write

$$\int_R \int_R K_S(\xi,\xi') f^*(\xi) f(\xi')\, d\xi' d\xi = \int_S \left| \frac{1}{\sqrt{i2\pi z}} \int_R f(\xi) \exp\left\{ \frac{i}{2z}(x-\xi)^2 \right\} d\xi \right|^2 dx \geq 0. \qquad (15)$$

Therefore, the kernel $K_S(\xi,\xi')$ of the integral equation is positive definite.
To prove the eigenvalues of above integral equation are nonnegative, it is necessary to show $\lambda\|\varphi\| \geq 0$. In this case λ is an eigenvalue, φ is an eigenvector and ||·|| indicates the norm in Hilbert space (Yosida, 1980). Therefore, by replacing $f(\xi)$ with $\varphi(\xi)$ and taking Eq. (14) into consideration, we can write

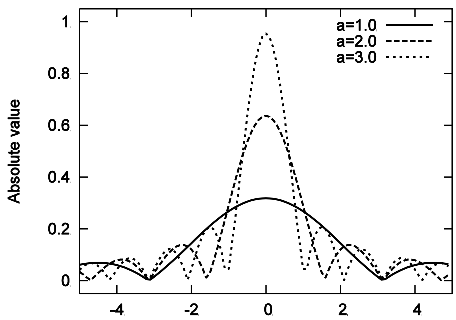

Figure 2: Absolute values of the kernel of the integral equation. Variable ‘a’ is the non-zero region on Fresnel transform plane, i.e. |x|≤a.

$$\lambda \int_R |\varphi(\xi)|^2 d\xi = \int_R \int_R K_S(\xi,\xi') \varphi^*(\xi) \varphi(\xi') d\xi d\xi' = \int_S \left| \frac{1}{\sqrt{i2\pi z}} \int_R \varphi(\xi) \exp\left\{ \frac{i}{2z}(x-\xi)^2 \right\} d\xi \right|^2 dx \geq 0. \qquad (16)$$

Therefore, the eigenvalues of the integral equation are nonnegative and real number.

Let us consider the kernel of the integral equation. If object plane and Fresnel transform plane are bounded by finite regions, the kernels of the integral equation are calculated analytically as the kernel function. We set the finite region S in Fresnel transform plane $-a \leq x \leq a$.

$$K_{[-a,a]}(\xi,\xi') = \frac{1}{2\pi z} \exp\left\{ -\frac{i}{2z}(\xi^2 - \xi'^2) \right\} \times \int_{-a}^{a} \exp\left\{ \frac{i}{z}(\xi - \xi')x \right\} dx = \frac{1}{\pi(\xi - \xi')} \exp\left\{ -\frac{i}{2z}(\xi^2 - \xi'^2) \right\} \times \sin\left(\frac{\xi - \xi'}{z} a \right). \qquad (17)$$

Figure 2 shows the absolute value of the kernel in Eq. (17). In this case we set $z = 1$.
Let us consider the complex conjugate of the kernel of the integral equation.

$$\overline{K_{[-a,a]}(\xi,\xi')} = \frac{1}{\pi(\xi - \xi')} \exp\left\{ \frac{i}{2z}(\xi^2 - \xi'^2) \right\} \times \sin\left(\frac{\xi - \xi'}{z} a \right) = \frac{-1}{\pi(\xi' - \xi)} \exp\left\{ -\frac{i}{2z}(\xi'^2 - \xi^2) \right\} \times (-1)\sin\left(\frac{\xi' - \xi}{z} a \right) = \frac{1}{\pi(\xi' - \xi)} \exp\left\{ -\frac{i}{2z}(\xi'^2 - \xi^2) \right\} \sin\left(\frac{\xi' - \xi}{z} a \right) = K_{[-a,a]}(\xi',\xi) \qquad (18)$$

Therefore, the kernel is of Hermitian symmetry.

4 COMPUTER CALCULATION

4.1 Eigenvalues and Eigenvectors

It is difficult in general to seek the strict solution of the integral equation. So we desire to seek the approximate solution in practical exact accuracy (Kondo, 1954). By discretizing the kernel function and using the value of the representative points, we can write

$$\sum_{j=1}^{N} K_{ij} x_j = \lambda x_i, \tag{19}$$

where i, j are the natural number, $1 \leq i \leq M$. The matrix K_{ij} is the Hermitian matrix if the kernel is discretized evenly-spaced and $M = N$. Therefore, the eigenvalue problems of the integral equation depend on a one of the Hermitian matrix in finite dimensional vector space. However, although the diagonal elements of the matrix are indeterminate form, we seek the limit value. If $\xi = \xi'$, we can write

$$\lim_{\xi \to \xi'} K_{[-a,a]}(\xi, \xi')$$
$$= \lim_{\xi \to \xi'} \frac{1}{\pi} \exp(\xi - \xi') \frac{\sin\left\{\frac{a}{z}(\xi - \xi')\right\}}{\xi - \xi'}. \tag{20}$$

We can replace $\xi - \xi'$ with X.

$$\lim_{\xi - \xi'} K_{[-a,a]}(\xi, \xi') = \lim_{X \to 0} \frac{1}{\pi} \exp(0) \frac{\sin\left\{\frac{a}{z} X\right\}}{X}$$
$$= \frac{1}{\pi} \exp(0) \lim_{X \to 0} \frac{\sin\left\{\frac{a}{z} X\right\}}{\frac{a}{z} X} \frac{a}{z}$$
$$= \frac{1}{\pi} \frac{a}{z} \lim_{X \to 0} \frac{\sin\left\{\frac{a}{z} X\right\}}{\frac{a}{z} X} = \frac{a}{\pi z}. \tag{21}$$

In general finite dimensional vector spaces ($\mathbb{C}^n$), the eigenvalues of Hermitian matrix are real numbers and then eigenvectors from different eigenspaces are orthogonal (Anton and Busby, 2003).

We use the Jacobi method to compute eigenvalues and eigenvectors of the matrix (Press et al., 1992). Jacobi method is a procedure for the diagonalization of complex symmetric matrices, using a sequence of plane rotations through complex angles (Seaton, 1969). It works by performing a sequence of orthogonal similarity updates $A \leftarrow Q^t A Q$ with the property that each new A is more diagonal than its predecessor. In this update Q is an orthogonal matrix. Eventually, the off-diagonal elements are small enough to be declared zero (Golub and Van Loan, 1996). Finally, it can calculate all eigenvalues and eigenvectors.

Figure 3 shows the eigenvalues in descending order, if z is 1.0, 2.0 and 3.0. They are nonnegative and real number. Figure 4 shows the real part of the eigenvectors for the largest eigenvalue. Because of 10 dimensional vector space, except for this, there are 9 eigenvectors. Figure 5 shows the imaginary part of the eigenvectors for the largest eigenvalue. In this case, the finite region S in Fresnel transform is $|x| \leq 3.0$. Its vector space is spanned by these eigenvectors.

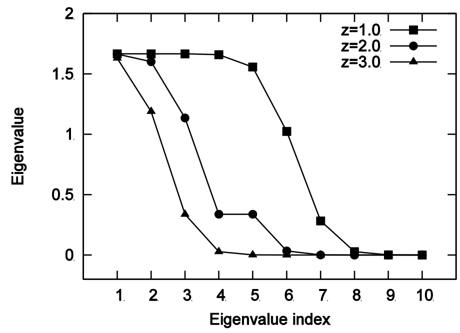

Figure 3: Plots of the eigenvalues in descending order. a=3.0.

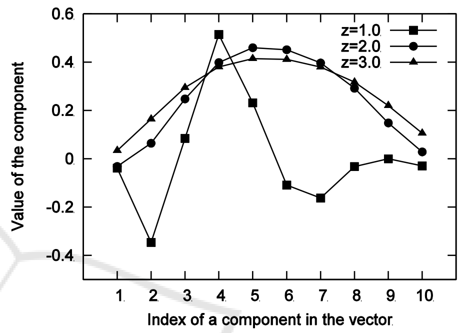

Figure 4: Plots of the eigenvectors for the largest eigenvalue. Its real part. a=3.0.

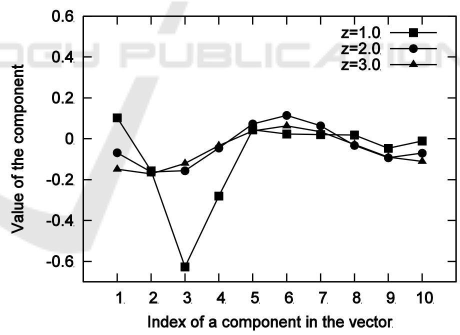

Figure 5: Plots of the eigenvectors for the largest eigenvalue. Its imaginary part. a=3.0.

4.2 Evaluation of Functions

We consider the application of the above eigenvectors to the problem of approximating a function.

Theoretically, we deal with a problem of expressing an arbitrary element on a finite N-dimensional Hilbert space H_N with an orthonormal basis.

In H_N, every element can be expressed as a linear combination of orthonormal basis (Reed and Simon, 1972). For any element $\boldsymbol{v}$ in H_N, by using orthonormal basis $\{\psi_n\}_{n=1}^{N}$, we can write

$$\mathbf{I} = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|, \qquad (22)$$

and

$$|\boldsymbol{v}\rangle = \mathbf{I}|\boldsymbol{v}\rangle = \sum_{n=1}^{N} \langle\psi_n|\boldsymbol{v}\rangle|\psi_n\rangle, \qquad (23)$$

where **I** indicates an identity operator and $\langle\cdot\mid\cdot\rangle$ is a bracket.

Now, we set $N = 10$. Let us consider the set $\mathbb{C}^{10}$ of all 10-tuples

$$\boldsymbol{v} = (v_1, v_2, \cdots, v_{10}), \qquad (24)$$

where $v_1, v_2, \cdots, v_{10}$ are complex numbers. Figure 6 shows the original test vector in $\mathbb{C}^{10}$, which real part is (0, 0, 1, 1, 1, 1, 1, 1, 0, 0) and imaginary part is all zero. Figure 7 shows another original test vector in $\mathbb{C}^{10}$, which real part is (0, 0, 0.5, 1, 0.5, -0.5, -1, -0.5, 0, 0) and imaginary part is all zero. Figure 8 shows third original test vector in $\mathbb{C}^{10}$, which real part is (0, 0, 0.5, 1, 0.5, -0.5, -1, -0.5, 0, 0) and imaginary part is (0, -0.5, -0.5, -0.5, 0, 0, 0.5, 0.5, 0.5, 0). The eigenvectors which are calculated by the Jacobi method are automatically orthonormal. So, by using Eq. (23), we have evaluated the error between the original test vector and the approximating vectors which are consisted of the eigenvectors. Figure 9 illustrates the mean square error versus the number of eigenvectors. The mean square error is defined by

$$\mathrm{Error}(n) = \frac{\|\boldsymbol{v}_n - \boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2}, \qquad (25)$$

where $\boldsymbol{v}_n$ is the sum in Eq. (23) up to n, $\boldsymbol{v}$ is the original vector, and $\|\cdot\|_2$ is the ℓ^2-norm. From Fig. 9, we can see that the error decreases with increasing number of eigenvectors used in the expansion. The problem of time consuming arises with increasing vector dimension.

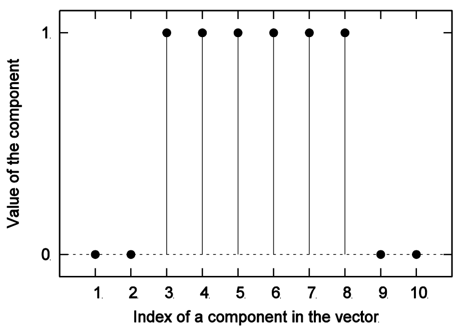

Figure 6: Example 1: Plots of the real parts of the original test vector.

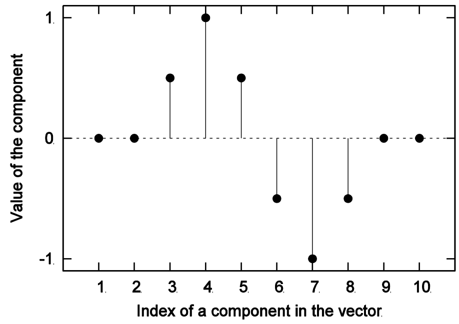

Figure 7: Example 2: Plots of the real parts of the original test vector.

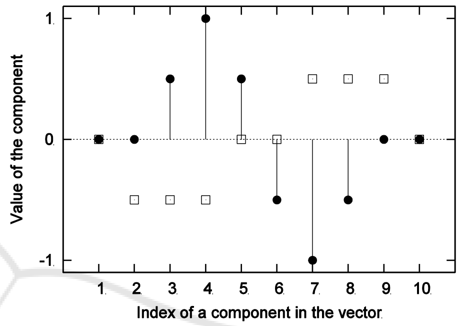

Figure 8: Example 3: Plots of the original test vector. Filled squares indicate the imaginary part.

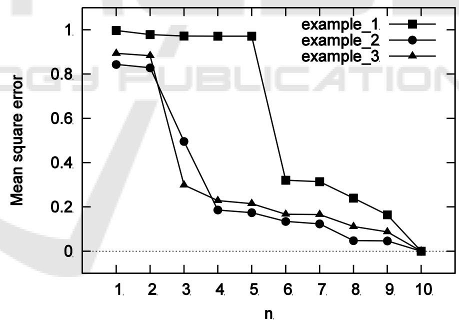

Figure 9: Plots of the normalized mean square error versus the number of eigenvectors.

5 CONCLUSIONS

We have sought the function that its total power is maximized in finite Fresnel transform plane, on condition that an input signal is zero outside the bounded region. We have showed that this leads to the eigenvalue problems of Fredholm integral equation of the first kind. By discretizing the kernel and using the value of the representative points, the problem depends on the eigenvalue problem of Hermitian conjugate matrix in finite dimensional vector space. By using the Jacobi method, we compute the eigenvalues and eigenvectors of the matrix. Furthermore, we applied it to the problem of approximating a function and evaluated the error. We confirmed the validity of the eigenvectors for the Fresnel transform by computer simulations. In this study, there are many parameters, especially, band-limited area S, R, and z. It is necessary to consist of orthogonal functional systems with the optimal parameters for finite Fresnel transform in application of an optical system. In our Hermitian matrix, its elements depend on the parameter a/z. It is necessary to reveal the property of the matrix with such parameter and its effect of the eigenvalues and the eigenvectors. Moreover, in general, the matrix given by discretizing the kernel of the integral equation is not the Hermitian matrix. If so, it is difficult to compute accurately all eigenvalues and eigenvectors. It is also necessary to consider other computational methods for this. These become the future problems. In our further problem, theoretically, it is important to search for a spectral representation of finite Fresnel transform in Hilbert space.

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