

# Variable Support Control for the Wave Equation

## A Multiplier Approach

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Abstract: We study the controllability of the multidimensional wave equation in a bounded domain with Dirichlet boundary condition, in which the support of the control is allowed to change over time.

The exact controllability is reduced to the proof of the observability inequality, which is proven by a multiplier method. Besides our main results, we present some applications.

## 1 INTRODUCTION

The controllability of the wave equation or more generally of partial differential equations has been studied intensively in the last 30 years. Exact controllability for evolutive systems is a challenging mathematical problem, also relevant in engineering applications. The exact controllability of the wave equation with Dirichlet boundary condition using the multiplier method is studied in (Lions, 1988) see also (Komornik, 1994) for a systematic study of this method and, for an approach using theory of semi-groups see (Tucsnak and Weiss, 2009). All these results do not allow the support of the control to change over time, while this variability is required in some applications (see (Dessi et al., 2004)-(Carcattera et al., 2014)).

The numerical implementation of controllability problems is widely studied in literature (see for instance (Glowinski and Lions, 1995), (Castro et al., 2008), (Münch, 2005), (Zuazua, 2005) and the references therein). In this works, the abstract procedure used to show the controllability plays an important role to prove the convergence of the scheme; see e.g. (Castro et al., 2008). The present results provide the abstract framework to the numerical implementation in the variable support control case.

In this paper we provide a controllability result which extends the classical controllability results and admits variability of the control support over time. We have to point out that even in the fixed support case, the subset of the boundary, on which the control acts, cannot be chosen arbitrarily; for an extensive discussion on these topics see (Bardos et al., 1992).

The common strategy to prove the exact controllability is to study an equivalent property i.e. the exact observability for the adjoint system (for more on this see (Komornik and Loreti, 2005), (Tucsnak and Weiss, 2009)). Our approach is based on the multiplier method (see (Komornik, 1994), (Lions, 1988)), which seems the most powerful in the multidimensional case. Although the strategy of the proof is quite classical and follows essentially (Lions, 1988), this approach leads to some unexpected results and it opens some questions on the optimality of these results; see (Agresti et al., 2017) for more on this.

Here we dwell more on the multidimensional case which seems to be closer to applications, providing some explicit examples in special geometries.

Let us begin with some notations:

- Let  $\Omega$  be a bounded domain of  $\mathbf{R}^d$  with  $d \geq 1$ , of class  $C^2$  or convex. From the hypothesis on the boundary, we know that the exterior normal vector  $\mathbf{v}$  is well defined  $\mathcal{H}^{d-1}$ -a.e. on  $\partial\Omega$ ; where  $\mathcal{H}^{d-1}$  is the  $d - 1$  dimensional Hausdorff measure (see (Evans and Garipey, 2015)). Moreover, we denote  $d\Gamma$  the measure  $\mathcal{H}^{d-1}$  restricted to  $\partial\Omega$ .
- For each  $t \in (0, T)$  and  $T > 0$ ,  $\Gamma(t)$  denotes an open subset of  $\partial\Omega$ .
- Finally, we define

$$\Sigma := \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\}; \quad (1.1)$$

and we suppose it to be  $\mathcal{H}^{d-1} \otimes \mathcal{L}^1$ -measurable (as defined in (Evans and Garipey, 2015) Chapter 1).

In this paper we want to study the following property.

**Definition 1.1** (Exact Controllability). *We say that the system*

$$\begin{cases} w_{tt} - \Delta w = 0, & (x, t) \in \Omega \times (0, T), \\ w = 0, & (x, t) \in \partial\Omega \times (0, T) \setminus \Sigma, \\ w = v, & (x, t) \in \Sigma, \\ w = w_0, & (x, t) \in \Omega \times \{0\}, \\ w_t = w_1, & (x, t) \in \Omega \times \{0\}; \end{cases} \quad (1.2)$$

is exactly controllable in time  $T > 0$ , if for all  $w_0, z_0 \in L^2(\Omega)$  and  $w_1, z_1 \in H^{-1}(\Omega)$  there exists a control  $v \in L^2(\Sigma)$  such that the unique solution  $w \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$  of (1.2) satisfies

$$w(x, T) = z_0, \quad w_t(x, T) = z_1. \quad (1.3)$$

Of course, the problem (1.2) has to be intended in a weak sense, which we give below (see Definition 1.3).

In the following, it will be useful to know some properties of the solution of the wave equation with null Dirichlet boundary condition, i.e.

$$\begin{cases} u_{tt} - \Delta u = 0, & (x, t) \in \Omega \times \mathbf{R}^+, \\ u = 0, & (x, t) \in \partial\Omega \times \mathbf{R}^+, \\ u = u_0, & (x, t) \in \Omega \times \{0\}, \\ u_t = u_1, & (x, t) \in \Omega \times \{0\}; \end{cases} \quad (1.4)$$

where  $\mathbf{R}^+ := [0, \infty)$ . In particular, we will need the following proposition; see (Pazy, 2012) for the notion of classical and mild solution.

**Proposition 1.2.** *The following holds true:*

- i) *For each  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ , then the problem (1.4) has an unique **mild** solution  $u$  in the class*

$$C^1(\mathbf{R}^+; H_0^1(\Omega)) \cap C(\mathbf{R}^+; L^2(\Omega)).$$

- ii) *For each  $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ , then the problem (1.4) has an unique **classical** solution  $u$  in the class*

$$C(\mathbf{R}^+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^2(\mathbf{R}^+; L^2(\Omega)).$$

*Proof.* Let  $\{f_k : k \in \mathbf{N}\}$  and  $\{\lambda_k : k \in \mathbf{N}\}$  be respectively the eigenfunctions and the eigenvalues of the Laplace operator with null Dirichlet boundary condition.

Since  $\Omega$  is of class  $C^2$  or convex, then  $f_k \in H^2(\Omega) \cap H_0^1(\Omega)$  for all  $k \in \mathbf{N}$ ; for a proof see (Taylor, 2011) or (Grisvard, 2011). Then, the solution  $u$  of the problem (1.4) is

$$u(t) = \sum_{k=0}^{\infty} \left( \cos(\lambda_k t) \hat{u}_{0,k} + \frac{\sin(\lambda_k t)}{\lambda_k} \hat{u}_{1,k} \right) f_k; \quad (1.5)$$

where

$$\hat{u}_{i,k} := (u_i, f_k)_{L^2(\Omega)} := \int_{\Omega} u_i f_k dx, \quad i = 0, 1.$$

Then the representation of the solution in (1.5) readily implies the claim in i) – ii).  $\square$

With this in hand, we can give a definition of solution for the problem (1.2).

**Definition 1.3** (Weak Solution of (1.2)). *A map  $w \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$  is a weak solution of (1.2) with  $v \in L^2(\Sigma)$  and  $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$  if and only if*

$$\begin{aligned} & - \langle w_t(s), u(s) \rangle_{H^{-1} \times H_0^1} + \langle w(s), u_t(s) \rangle_{L^2 \times L^2} = \\ & - \langle w_1, u_0 \rangle_{H^{-1} \times H_0^1} + \langle w_0, u_1 \rangle_{L^2 \times L^2} + \\ & \int_0^s \int_{\Gamma(t)} v \partial_\nu u d\Gamma dt, \end{aligned} \quad (1.6)$$

for all  $0 < s \leq T$  and for all mild solutions  $u$  of (1.4) and initial data  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ .

For the motivation see (Komornik and Loreti, 2005); for other equivalent formulations one may consult (Lions, 1988).

*Remark 1.4.* Note that, on the RHS of (1.6) it appears  $\partial_\nu u$  i.e. the normal derivative of  $u$ . Although the regularity of a mild solution  $u$  is not sufficient to have a well defined trace for  $\partial_\nu u$ , in the sense of Sobolev spaces (see (Taylor, 2011) Chapter 4), the normal derivative  $\partial_\nu u$  is a well defined element of  $L^2(\partial\Omega \times (0, t))$  for each  $t > 0$ : this result is generally called *hidden regularity*; the proof relies on a clever integration by parts argument, for details see (Komornik, 1994), (Lions, 1988) or (Lasićka and Triggiani, 1983). By this, the last term in the RHS of (1.6) is well defined since  $v \in L^2(\Sigma)$ .

Before proceeding in the analysis of the weak solutions, we recall the following well known result about the solutions of the wave equation (1.4).

**Corollary 1.5** (Energy conservation). *For each  $t \in \mathbf{R}^+$  and each mild solution  $u$  of (1.4) with initial data  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ , the energy*

$$E(t) = \frac{1}{2} \left( \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 + \|u_t(\cdot, t)\|_{L^2(\Omega)}^2 \right), \quad (1.7)$$

is constant, and it is equal to the initial energy

$$E_0 := \frac{1}{2} \left( \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right). \quad (1.8)$$

We conclude the treatment of the weak solution with the following well posedness result, for a proof see (Komornik and Loreti, 2005).

**Theorem 1.6** (Well Posedness). *For each  $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$  and  $v \in L^2(\Sigma)$ , there exist a unique weak solution of (1.2) in the sense of Definition 1.3.*

For brevity, as we said above, here we confine ourselves to the study of the following equivalent property.

**Definition 1.7** (Exact Observability). *Let  $u \in C(\mathbf{R}^+; H_0^1(\Omega)) \cap C^1(\mathbf{R}^+; L^2(\Omega))$  be the unique mild solution of (1.4) for the initial data  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ . Then the system is called exactly observable in time  $T > 0$  if there exists a constant  $C > 0$  such that*

$$\int_{\Sigma} |\partial_{\nu} u|^2 d\Gamma \otimes dt = \int_0^T \int_{\Gamma(t)} |\partial_{\nu} u|^2 d\Gamma dt \geq C \left( \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right), \quad (1.9)$$

for all  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ .

We remind that  $\partial_{\nu} u$  is a well defined  $L^2(\partial\Omega \times (0, T))$  element (see Remark 1.4).

We are now in a position to prove an identity, which is the basic tool in proving the exact observability for some  $\Sigma$ 's.

**Lemma 1.8** (A multiplier identity). *Let  $u \in C(\mathbf{R}^+; H_0^1(\Omega)) \cap C^1(\mathbf{R}^+; L^2(\Omega))$  be the mild solution corresponding to initial data  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ . Then for each  $0 \leq s < \tau$  and  $\xi \in \mathbf{R}^d$ , we have*

$$\begin{aligned} & \frac{1}{2} \int_s^{\tau} \int_{\partial\Omega} (x - \xi) \cdot \nu |\partial_{\nu} u|^2 d\Gamma(x) dt = \\ & \left[ \int_{\Omega} u_t \left( \nabla u \cdot (x - \xi) + \frac{d-1}{2} u \right) dx \right]_s^{\tau} + (\tau - s) E_0; \end{aligned} \quad (1.10)$$

where  $E_0$  is the initial energy of the system as defined in (1.8) and  $[f(t)]_s^{\tau} := f(s'') - f(s')$ .

*Proof.* For reader's convenience, we divide the proof into three steps.

**Step 1.** We first suppose that  $u_0, u_1$  are smoother, i.e.  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $u_1 \in H_0^1(\Omega)$ . By Proposition 1.2, the solution to (1.4) is a classical solution, in particular the quantities  $u_{x_k, x_j}$ , for  $1 \leq k, j \leq d$ , belong to  $C(\mathbf{R}^+; L^2(\Omega))$ ; this will be used in the proof.

Under this assumption, the wave equation is solved by  $u$  almost everywhere on  $\Omega \times \mathbf{R}^+$ , then we can multiply the wave equation by  $(x - \xi) \cdot \nabla u$  and on integrating over  $\Omega \times (s, t)$ , we obtain

$$\begin{aligned} & \int_{\Omega} \int_s^{\tau} ((x - \xi) \cdot \nabla u) u_{tt} dx dt \\ & - \int_{\Omega} \int_s^{\tau} ((x - \xi) \cdot \nabla u) \Delta u dx dt = 0. \end{aligned} \quad (1.11)$$

For the first term on the LHS in (1.11), with a simple

integration by parts argument, we obtain

$$\begin{aligned} & \int_{\Omega} \int_s^{\tau} (x - \xi) \cdot \nabla u u_{tt} dx \\ & = \left[ \int_{\Omega} (x - \xi) \cdot \nabla u u_t dx \right]_s^{\tau} \\ & \quad - \int_{\Omega} \int_s^{\tau} (x - \xi) \cdot \frac{1}{2} \nabla((u_t)^2) dx dt \\ & = \left[ \int_{\Omega} (x - \xi) \cdot \nabla u u_t dx \right]_s^{\tau} + \frac{d}{2} \int_{\Omega} \int_s^{\tau} (u_t)^2 dx dt; \end{aligned}$$

where, in the last inequality, we have used the Green's formulas,  $u_t = 0$  on  $\partial\Omega \times (s, \tau)$  and  $\nabla \cdot (x - \xi) = d$ .

For the second term in (1.11) we use a similar argument. Indeed, by Green's formulas

$$\begin{aligned} & \int_{\Omega} \int_s^{\tau} ((x - \xi) \cdot \nabla u) \cdot \Delta u dx dt \\ & = \int_{\partial\Omega} \int_s^{\tau} \partial_{\nu} u (x - \xi) \cdot \nabla u d\Gamma dt \\ & \quad - \int_{\Omega} \int_s^{\tau} \nabla((x - \xi) \cdot \nabla u) \cdot \nabla u dx dt. \end{aligned}$$

Now, using that  $((u_{x_j})^2)_{x_k} = 2u_{x_j, x_k}$  and  $\nabla(x_k - \xi_k)_{x_j} = \delta_{k,j}$  for all  $1 \leq k, j \leq d$  (here  $\delta_{k,j}$  is the Kronecker's delta), one obtains

$$\nabla((x - \xi) \cdot \nabla u) \cdot \nabla u = |\nabla u|^2 + \frac{1}{2} (x - \xi) \cdot \nabla(|\nabla u|^2);$$

pointwise. With simple computations, we have

$$\begin{aligned} & \int_{\Omega} \int_s^{\tau} ((x - \xi) \cdot \nabla u) \cdot \Delta u dx dt = \\ & \frac{1}{2} \int_{\partial\Omega} \int_s^{\tau} (x - \xi) \cdot \nu |\partial_{\nu} u|^2 + \frac{d-2}{2} \int_{\Omega} \int_s^{\tau} |\nabla u|^2 dx dt. \end{aligned}$$

Putting all together and using that  $E(t) \equiv E_0$  (cfr. Corollary 1.5), we obtain

$$\begin{aligned} & \frac{1}{2} \int_{\partial\Omega} \int_s^{\tau} (x - \xi) \cdot \nu |\partial_{\nu} u|^2 = \left[ \int_{\Omega} (x - \xi) \cdot \nabla u u_t dx \right]_s^{\tau} \\ & \quad + (t - \tau) E_0 + \frac{d-1}{2} \int_{\Omega} \int_s^{\tau} |u_t|^2 - |\nabla u|^2 dx dt. \end{aligned} \quad (1.12)$$

**Step 2.** In this step we rewrite the last term on LHS in (1.12). Indeed, under the assumption of **Step 1**, the wave equation is solved by  $u$  a.e. on  $\Omega \times (s, \tau)$ , so that on multiplying it by  $u$ , integration over  $\Omega \times (s, \tau)$  and using Green's formulas (recall that  $u = 0$  on  $\partial\Omega \times (s, \tau)$ )

$$\begin{aligned} & 0 = \int_{\Omega} \int_s^{\tau} u (u_{tt} - \Delta u) dx dt \\ & = \left[ \int_{\Omega} u u_t dx \right]_s^{\tau} - \int_{\Omega} \int_s^{\tau} |u_t|^2 - |\nabla u|^2 dx dt. \end{aligned} \quad (1.13)$$

**Step 3.** Combining the equalities (1.12)-(1.13) we obtain (1.10) for classical solutions.

In the general case, choose sequences such that  $\{u_{0,k} : k \in \mathbf{N}\} \subset H^2(\Omega) \cap H_0^1(\Omega)$  and  $\{u_{1,k} : k \in \mathbf{N}\} \subset H_0^1(\Omega)$ , such that

$$u_{0,k} \rightarrow u_0 \quad \text{in } H_0^1(\Omega), \quad u_{1,k} \rightarrow u_1 \quad \text{in } L^2(\Omega).$$

Then passing to the limit in the identity (1.10) valid for the solution  $u_k$  for initial data  $(u_{0,k}, u_{1,k})$ , one obtains the claim.  $\square$

*Remark 1.9.* The proof of Corollary 3.1 is based on the multiplication of the wave equation against the function  $m(x) \cdot \nabla u = (x - \xi) \cdot \nabla u$ , which is called *multipplier*, which justifies the name of the identity.

## 2 ALTERNATING OBSERVATION

For alternating observation we mean that there exists a partition  $0 =: t_{-1} < t_0 < t_1 < \dots < t_{N-1} < t_N =: T$  of the interval  $[0, T]$  such that

$$\Gamma(t) \equiv \Gamma_j, \quad \forall t \in (t_{j-1}, t_j) \quad j = 0, \dots, N; \quad (2.1)$$

where  $\Gamma_j \subset \partial\Omega$  is a fixed subset for each  $j = 0, \dots, N$ . Under the previous hypothesis, by (1.1) we have

$$\Sigma = \bigcup_{j=0}^N \Gamma_j \times (t_{j-1}, t_j), \quad (2.2)$$

up to a set of measure 0.

As pointed out in Section 1 the family  $\{\Gamma_j\}_{j=0, \dots, N}$  cannot be chosen arbitrarily; we construct this family in a special form (see (2.5) below).

To do this, let  $\{x_i\}_{i=0, \dots, N}$  be an arbitrary family of points in  $\mathbf{R}^d$ ;

- For each  $i = 0, \dots, N$ , define

$$R_i = \max\{|x - x_i| \mid x \in \overline{\Omega}\}, \quad (2.3)$$

and for each  $i = 0, \dots, N-1$ , define

$$R_{i+1,i} = |x_{i+1} - x_i|. \quad (2.4)$$

- For each  $i = 0, \dots, N$ , define

$$\Gamma_i = \{x \in \partial\Omega \mid (x - x_i) \cdot \nu > 0\}. \quad (2.5)$$

**Theorem 2.1** (Alternating Observability). *In the previous notations, for each real number  $T$  such that*

$$T > R_N + \sum_{i=0}^{N-1} R_{i+1,i} + R_0, \quad (2.6)$$

*the system is exactly observable in time  $T$ , in the sense of Definition 1.7 for  $\Sigma$  as in (2.2).*

*Proof.* For each  $i = 0, \dots, N$ , use the identity (1.10) for  $\xi = x_i$ ,  $s = t_{i-1}$  and  $\tau = t_i$ . On summing over  $i = 0, \dots, N$  such identities, we have

$$\begin{aligned} & \frac{1}{2} \sum_{i=0}^N \int_{t_{i-1}}^{t_i} \int_{\Gamma_i} (x - x_i) \cdot \nu |\partial_\nu u|^2 d\Gamma dt = \\ & \sum_{i=0}^N \left[ \int_{\Omega} u_t \left( \nabla u \cdot (x - x_i) + \frac{d-1}{2} u dx \right) \right]_{t_{i-1}}^{t_i} \\ & \quad + T E_0, \quad (2.7) \end{aligned}$$

since  $\sum_{i=0}^N (t_i - t_{i-1}) = T$ , where we take as above  $T = t_N$ .

It will be useful to adopt the following notation

$$u^s(x) := u(x, s), \quad u_t^s(x) := u_t(x, s),$$

where  $s \in [0, T]$  and  $x \in \Omega$ . In the sum on the RHS of (2.7) some cancellations are possible:

$$\begin{aligned} & \sum_{i=0}^N \left[ \int_{\Omega} u_t \left( \nabla u \cdot (x - x_i) + \frac{d-1}{2} u \right) dx \right]_{t_{i-1}}^{t_i} = \\ & \int_{\Omega} u_t^T \left( \nabla u^T \cdot (x - x_N) + \frac{d-1}{2} u^T \right) dx \\ & - \sum_{i=0}^{N-1} \int_{\Omega} u_t^{t_i} \left( \nabla u^{t_i} \cdot [(x - x_{i+1}) - (x - x_i)] \right) dx \\ & - \int_{\Omega} u_1 \left( \nabla u_0 \cdot (x - x_0) + \frac{d-1}{2} u_0 \right) dx. \quad (2.8) \end{aligned}$$

For the first and last term on the RHS of (2.8) we have

$$\left| \int_{\Omega} u_t^T \left( \nabla u^T \cdot (x - x_N) + \frac{d-1}{2} u^T \right) dx \right| \leq R_N E_0, \quad (2.9)$$

$$\left| \int_{\Omega} u_1 \left( \nabla u_0 \cdot (x - x_0) + \frac{d-1}{2} u_0 \right) dx \right| \leq R_1 E_0. \quad (2.10)$$

Moreover, for the terms in the sum on the RHS of (2.8), we have

$$\left| \int_{\Omega} u_t^{t_i} \left( \nabla u^{t_i} \cdot (x_i - x_{i+1}) \right) dx \right| \leq R_{i+1,i} E_0, \quad (2.11)$$

for each  $i = 0, \dots, N$ .

For convenience, we postpone the proof of the inequalities (2.9)-(2.11), see Lemma 2.2 below.

By definition of  $\Gamma_i$  in (2.5) and  $R_i$  in (2.3), clearly we have

$$\begin{aligned} & R_i \int_{t_{i-1}}^{t_i} \int_{\Gamma_i} |\partial_\nu u|^2 d\Gamma dt \geq \\ & \int_{t_{i-1}}^{t_i} \int_{\partial\Omega} (x - x_i) \cdot \nu |\partial_\nu u|^2 d\Gamma dt, \quad (2.12) \end{aligned}$$

for each  $i = 0, \dots, N$ .

Using the equation (2.7), the identity (2.8) and the estimates (2.9)-(2.12), we have

$$\left( \max_{i=0, \dots, N} R_i \right) \sum_{i=0}^N \int_{t_{i-1}}^{t_i} \int_{\Gamma_i} |\partial_\nu u|^2 d\Gamma dt \geq 2(T - R_N - R_{N,N-1} - \dots - R_{1,0} - R_0)E_0. \quad (2.13)$$

Now, by assumption  $T > R_N + \sum_{i=1}^{N-1} R_{i+1,i} + R_1$ , so there exists a constant  $C = C_T > 0$ , which depends on  $T$ , such that

$$\sum_{i=0}^N \int_{t_{i-1}}^{t_i} \int_{\Gamma_i} |\partial_\nu u|^2 d\Gamma dt \geq C_T E_0, \quad (2.14)$$

which is exactly the observability inequality, as defined in Definition 1.7.  $\square$

We now prove the estimates (2.9)-(2.11); in particular (2.9)-(2.10) and (2.11) follow respectively by *i*) and *ii*) of the following Lemma.

**Lemma 2.2.** *For each mild solution  $u \in C(\mathbf{R}^+; H_0^1(\Omega)) \cap C^1(\mathbf{R}^+; L^2(\Omega))$  of (1.4), the following holds.*

*i) For each  $\xi \in \mathbf{R}^d$  and  $s \in \mathbf{R}$ , then*

$$\left| \int_{\Omega} u_t^s \left( \nabla u^s \cdot (x - \xi) + \frac{d-1}{2} u^s \right) dx \right| \leq R_\xi E_0,$$

where  $R_\xi = \max\{|x - \xi| : x \in \bar{\Omega}\}$ .

*ii) For each  $\xi, \eta \in \mathbf{R}^d$  and  $s \in \mathbf{R}^+$ , then*

$$\left| \int_{\Omega} u_t^s (\nabla u^s \cdot (\xi - \eta)) dx \right| \leq R_{\xi, \eta} E_0,$$

where  $R_{\xi, \eta} = |\xi - \eta|$ .

*Proof.* *i)* By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left| \int_{\Omega} u_t^s \left( \nabla u^s \cdot (x - \xi) + \frac{d-1}{2} u^s \right) dx \right| \\ & \leq \|u_t^s\|_{L^2} \left\| \nabla u^s \cdot (\cdot - \xi) + \frac{d-1}{2} u^s \right\|_{L^2} \\ & \leq \frac{R_\xi}{2} \|u_t^s\|_{L^2}^2 + \frac{1}{2R_\xi} \left\| \nabla u^s \cdot (\cdot - \xi) + \frac{d-1}{2} u^s \right\|_{L^2}^2; \end{aligned} \quad (2.15)$$

here  $L^2 := L^2(\Omega)$ . Note that,

$$\begin{aligned} & \left\| \nabla u^s \cdot (\cdot - \xi) + \frac{d-1}{2} u^s \right\|_{L^2}^2 \\ & = \left\| \nabla u^s \cdot (\cdot - \xi) \right\|_{L^2}^2 + (d-1) (\nabla u^s \cdot (\cdot - \xi), u^s)_{L^2} \\ & \quad + \frac{(d-1)^2}{4} \|u^s\|_{L^2}^2. \end{aligned}$$

Since  $u = 0$  on  $\partial\Omega$ , then using Green's identity the middle term in the RHS of the previous equation is equal to

$$\begin{aligned} & \int_{\Omega} (\nabla u^s \cdot u^s) \cdot (x - \xi) dx \\ & = \frac{1}{2} \int_{\Omega} (\nabla(u^s)^2) \cdot (x - \xi) dx \\ & = -\frac{1}{2} \int_{\Omega} (u^s)^2 (\nabla \cdot (x - \xi)) dx = -\frac{d}{2} \|u^s\|_{L^2}^2. \end{aligned}$$

This implies

$$\begin{aligned} & \left\| \nabla u^s \cdot (\cdot - \xi) + \frac{d-1}{2} u^s \right\|_{L^2}^2 = \\ & \left\| \nabla u^s \cdot (\cdot - \xi) \right\|_{L^2}^2 \\ & + \left[ -(d-1) \frac{d}{2} + \frac{(d-1)^2}{4} \right] \|u^s\|_{L^2}^2 \\ & \leq \left\| \nabla u^s \cdot (\cdot - \xi) \right\|_{L^2}^2 \leq R_\xi^2 \left\| \nabla u^s \right\|_{L^2}^2, \end{aligned}$$

since

$$-\frac{d(d-1)}{2} + \frac{(d-1)^2}{4} = -\frac{d^2}{2} + \frac{1}{4} < 0,$$

for all  $d \geq 1$ .

Now returning to (2.15), we have

$$\begin{aligned} & \left| \int_{\Omega} u_t^s \left( \nabla u^s \cdot (x - \xi) + \frac{d-1}{2} u^s \right) dx \right| \\ & \leq \frac{R_\xi}{2} \|u_t^s\|_{L^2}^2 + \frac{1}{2R_\xi} R_\xi^2 \left\| \nabla u^s \right\|_{L^2}^2 \\ & = \frac{R_\xi}{2} (\|u_t^s\|_{L^2}^2 + \left\| \nabla u^s \right\|_{L^2}^2) = R_{\xi, \eta} E_0; \end{aligned}$$

where the last inequality follows by the energy conservation (cfr. Corollary 1.5).

*ii)* The proof the second part of the Lemma is easier. Indeed, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left| \int_{\Omega} u_t^s \nabla u^s \cdot (\xi - \eta) dx \right| \\ & \leq \|u_t^s\|_{L^2} \left\| \nabla u^s \cdot (\xi - \eta) \right\|_{L^2} \\ & \leq R_{\xi, \eta} \|u_t^s\|_{L^2} \left\| \nabla u^s \right\|_{L^2} \leq R_{\xi, \eta} E_0; \end{aligned}$$

where, the last inequality follows by the energy conservation.  $\square$

## 2.1 The Role of $\{t_i\}_{i=-1, \dots, N}$

One may wonder what is the role of the family  $\{t_i\}_{i=-1, \dots, N}$ , since in Theorem 2.1 only the sum  $T = \sum_{i=0}^N (t_i - t_{i-1})$  appears.

To explain the role of these values, we have to recall that if  $N = 0$  (i.e. fixed support control), Theorem 2.1 implies that the exact controllability holds for any

$T > 2 \max\{|x - x_0| \mid x \in \overline{\Omega}\}$ , where  $x_0 \in \mathbf{R}^d$  is a fixed point. So, if for an index  $j \in \{0, \dots, N\}$  we have

$$|t_j - t_{j-1}| > 2 \max\{|x - x_j| \mid x \in \overline{\Omega}\}, \quad (2.16)$$

then we can construct a control  $v$  such that  $\text{supp } v \subset \overline{\Gamma}_j \times [t_{j-1}, t_j]$  by using the fixed support case of Theorem 2.1. Indeed, fix  $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$  and let be  $w$  the unique weak solution of the problem (1.2) for  $T = t_j$  and  $v = 0$ ; so by Theorem 1.6 we have that  $(w(t_{j-1}), w_t(t_{j-1}))$  is well defined as an element of  $L^2(\Omega) \times H^{-1}(\Omega)$ . Since the inequality in (2.16) holds, the fixed case of Theorem 2.1 provides a control  $\tilde{v}$  such that the solution at time  $t = t_j$  satisfies the null condition; so after defining  $v = 0$  for  $t < t_{j-1}$  or  $t_j < t < T$  and  $v = \tilde{v}$  for  $t_{j-1} < t < t_j$  we have constructed a control for the initial condition  $(w_0, w_1)$ . So if the inequality (2.16) holds for an index  $j$  the controllability results follows by classical results, (see for instance Theorem 6.1, Chapter 1 of (Lions, 1988)). In the following Section we produce explicit examples in which the inequality (2.16) is not satisfied by any  $i = 0, \dots, N$ ; so our investigation produces new results on controllability for the wave equation.

By the way, we point out that Theorem 2.1 allows us to apply the Hilbert uniqueness method (or HUM, see (Komornik and Loreti, 2005)-(Lions, 1988)); with some effort one can prove that the control provided by this method minimizes the *energy* of the control over the possible controls (see Chapter 7 of (Lions, 1988)). For this reason even in the case when (2.16) holds our Theorem yields the existence of a minimizer control; this remarkable property of the HUM control can be useful in applications. For brevity we do not reproduce the needed calculations.

### 3 VARIABLE OBSERVATION

In this Section we state a fairly general observability Theorem in which the subset  $\Gamma(t)$  of observation at time  $t$  can vary at each time  $t \in (0, T)$ .

As explained in Section 1, the family  $\{\Gamma(t)\}_{t \in (0, T)}$  cannot be arbitrary and will be constructed in a similar fashion to  $\{\Gamma_i\}_{i=0, \dots, N}$  (see (2.5) in Section 2).

To do this, let  $\varphi : [0, T] \rightarrow \mathbf{R}^d$  be a continuous and piecewise differentiable curve in  $\mathbf{R}^d$  of finite length, i.e.

$$L(\varphi) := \int_0^T |\varphi'(t)| dt < +\infty. \quad (3.1)$$

For such a  $\varphi$ , define

$$\Gamma_\varphi(t) = \{x \in \partial\Omega \mid (x - \varphi(t)) \cdot \nu > 0\}, \quad (3.2)$$

$$\Sigma_\varphi = \bigcup_{t \in (0, T)} \Gamma_\varphi(t) \times \{t\}, \quad (3.3)$$

$$c_i = \max_{\overline{\Omega}} |x - \varphi(i)|, \quad i = 0, T. \quad (3.4)$$

Hereafter we assume that  $\Sigma_\varphi$  is  $\mathcal{H}^{d-1} \otimes \mathcal{L}^1$ -measurable; for further discussion on this topic see (Agresti et al., 2017).

Also, let  $P = \{0 = t_{-1} < t_0 < \dots < t_{N-1} < t_N = T\}$  be a partition of the interval  $[0, T]$ ; define

$$x_{\varphi, j} = \varphi(t_{j-1}), \quad (3.5)$$

$$\Gamma_{\varphi, j} = \{x \in \partial\Omega \mid (x - x_{\varphi, j}) \cdot \nu > 0\}, \quad (3.6)$$

for all  $j = 0, \dots, N$ . For future convenience, we set

$$\Sigma_\varphi^P := \bigcup_{j=0}^N \Gamma_{\varphi, j} \times (t_{j-1}, t_j). \quad (3.7)$$

In next theorem, we have to consider a sequence of partitions  $\{P_k\}_{k \in \mathbf{N}}$  of the interval  $[0, T]$ , so we will add the upper index  $k$  in (3.5)-(3.6) in order to keep trace of the dependence on  $P_k$ ; and we will set  $\Sigma_\varphi^k := \Sigma_\varphi^{P_k}$ . Now we are ready to state the main result of this Section.

**Theorem 3.1** (Variable Support Observability). *Under the above hypothesis, suppose there exists a sequence of partitions  $\{P_k\}_{k \in \mathbf{N}}$  of the interval  $[0, T]$ , such that  $\sup_{\{t_{j-1}^k, t_j^k\} \in P_k} |t_j^k - t_{j-1}^k| \searrow 0$  as  $k \nearrow \infty$  and*

$$\lim_{k \rightarrow \infty} \left( \mathcal{H}^{d-1} \otimes \mathcal{L}^1 \right) \left( \Sigma_\varphi \Delta \Sigma_\varphi^k \right) = 0, \quad (3.8)$$

where  $P_k = \{0 = t_{-1}^k < t_0^k < \dots < t_N^k = T\}$ ,  $\mathcal{L}^1$  is the Lebesgue measure on  $\mathbf{R}^1$  and  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ . Furthermore, suppose that  $T$  verifies

$$T > c_0 + \int_0^T |\varphi'(t)| dt + c_N. \quad (3.9)$$

Then the system is exactly observable in time  $T$  (see Definition 1.7) for  $\Sigma = \Sigma_\varphi$ .

For brevity we do not report the proof of Theorem 3.1; for the proof, other applications and further extension of this result we refer to (Agresti et al., 2017).

### 4 APPLICATIONS

In this Section we give some applications of Theorems 2.1 and 3.1 in order to show the potentiality of these results.

To begin, we focus our attention to  $\Omega = \mathcal{B}_1 := \{x \in \mathbf{R}^2 \mid |x| < 1\}$ , i.e. the ball with center 0 and radius 1. As usual, we denote with  $\mathcal{S}^1 := \partial\mathcal{B}_1$  the unit circle with center 0.

**Corollary 4.1** (1-time alternating - Circle case). *Let  $T > 2(1 + \sqrt{2})$  be a real number, define*

$$d_0 = \left\{ z \in \mathcal{S}^1 \mid \frac{\pi}{2} < \arg z < 2\pi \right\}, \quad (4.1)$$

$$d_1 = \left\{ z \in \mathcal{S}^1 \mid 0 < \arg z < \frac{3\pi}{2} \right\}. \quad (4.2)$$

*Then the system (1.2) is exactly controllable for  $\Omega = \mathcal{B}_1$ ,  $t_0$  is an arbitrary element of  $(0, T)$ , and*

$$\Sigma = d_0 \times (0, t_0) \cup d_1 \times (t_0, T).$$

*Proof.* It is an easy consequence of Theorem 2.1 with the choice  $x_0 = (1, 1)$  and  $x_1 = (1, -1)$ . Indeed, by (2.3)-(2.4) it is clear that

$$R_0 = R_1 = \sqrt{2} + 1, \quad R_{0,1} = 2\sqrt{2}.$$

Moreover, the condition (2.6) in Theorem 2.1, implies  $T > 2(1 + \sqrt{2})$ . Lastly, with simple geometrical consideration, one can see  $\Gamma_0 = d_0$  and  $\Gamma_1 = d_1$  by (2.5); see also Figure 1. This concludes the proof.  $\square$



Figure 1: On the top (resp. bottom) we have represented in red  $d_0$  (resp.  $d_1$ ) and  $x_0$  (resp.  $x_1$ ), to point out the connection with the formula (2.5).

*Remark 4.2.* Note that in Corollary 4.1 there is no assumption on the value  $t_0 \in (0, T)$ ; due to the discussion in Subsection 2.1 it is clear that Corollary 4.1 does not follow trivially by known results if

$$|t_0| < 2(1 + \sqrt{2}). \quad (4.3)$$

We now extend Corollary 4.1 to the  $N$ -times alternating case.

**Corollary 4.3** ( $N$ -times alternating - Circle case). *Let  $T > 2(N + 1) + 2\sqrt{2}$  be a real number and let  $d_0, d_1$  be as in (4.1)-(4.2). Then the system (1.2) is exactly*

*controllable for  $\Omega = \mathcal{B}_1$ ,  $\{t_i\}_{i=0, \dots, N-1}$  any increasing finite subfamily of  $(0, T)$ , and*

$$\Sigma = \left( \bigcup_{i=0, i \in 2\mathbb{N}}^N d_0 \times (t_{j-1}, t_j) \right) \cup \left( \bigcup_{i=0, i \in 2\mathbb{N}+1}^N d_1 \times (t_{j-1}, t_j) \right). \quad (4.4)$$

*Proof.* It is similar to the proof of Corollary 4.1. In this case, we have to choose  $x_i \equiv x_0 = (1, 1)$  if  $i$  is even or  $x_i \equiv x_1 = (1, -1)$ . Similar to Corollary 4.1, we have

$$R_0 = R_1 = \sqrt{2} + 1, \quad R_{i,i+1} = 2\sqrt{2},$$

for all  $i = 0, \dots, N-1$ . As in the proof of Corollary 4.1, by (2.5) we have that  $\Gamma_i \equiv d_0$  if  $i$  is even, otherwise  $\Gamma_i \equiv d_1$  and  $\Sigma$  is as in (4.4).  $\square$

*Remark 4.4.* As did in Remark 4.2 for Corollary 4.1, we observe that in Corollary 4.3 there is no assumption on the values  $t_i$  for  $i = 0, \dots, N-1$ . In this case we may require that

$$|t_j - t_{j-1}| < 2(1 + \sqrt{2}), \quad \forall j = 0, \dots, N; \quad (4.5)$$

to avoid triviality of the result in the previous Corollary.

Below we give an interesting application of Theorem 3.1.

**Corollary 4.5.** *Let  $\alpha \in \mathbf{R}^+$  be a positive real number, such that*

$$\alpha < \frac{\pi}{4(1 + \sqrt{2}) + \pi\sqrt{2}}. \quad (4.6)$$

*Then the system (1.2) is exactly controllable in time  $T = \pi/(2\alpha)$  with  $\Omega = \mathcal{B}_1$  and  $\Sigma = \cup_{t \in (0, T)} \Gamma(t) \times \{t\}$ ; where*

$$\Gamma(t) = \left\{ z \in \mathcal{S}^1 \mid \frac{\pi}{4} + \alpha t < \arg z < \frac{7\pi}{4} + \alpha t \right\}. \quad (4.7)$$

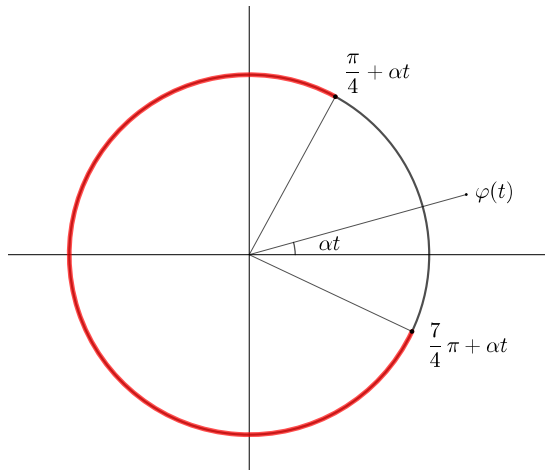
*Proof.* Set  $\varphi(t) = \sqrt{2}(\cos(\alpha t), \sin(\alpha t))$  for  $t \in (0, \pi/(2\alpha))$ . It is easy to check that  $\Gamma(t) = \Gamma_\varphi(t)$  (see Figure 2), to prove the Corollary we have only to check the hypothesis of Theorem 3.1.

Indeed, it is clear that the condition (3.8) holds by taking a sequence of partition  $P_k = \{t_j^k\}_{j=0, \dots, k}$  with  $t_j^k = (j/k)T$  and  $T := \pi/(2\alpha)$ .

By construction, the length of the curve is  $L(\varphi) = \sqrt{2}(\pi/2)$ , and  $c_i = 1 + \sqrt{2}$  for  $i = 0, T$ . So condition (3.9) is satisfied if

$$\frac{\pi}{2\alpha} > 2(1 + \sqrt{2}) + \sqrt{2}\frac{\pi}{2}, \quad (4.8)$$

which is equivalent to (4.6).  $\square$


 Figure 2: Construction of  $\Gamma_\varphi(t)$ , in accordance with (3.2).

We now analyse a case of a convex domain; i.e. the interior of the hexagon with side 1 and center 0, it will be denoted by  $\mathcal{H} \subset \mathbf{R}^2$  and we set  $\mathcal{E} := \partial\mathcal{H}$ . In this situation, we have analogous result to Corollaries 4.1, 4.3 and 4.5:

**Corollary 4.6** (1-time alternating - Hexagon case I). *Let  $T > 6$  be a real number, define*

$$e_0 = \left\{ (x, y) \in \mathcal{E} \mid x < \frac{1}{2} \right\}, \quad (4.9)$$

$$e_1 = \left\{ (x, y) \in \mathcal{E} \mid x > -\frac{1}{2} \right\}. \quad (4.10)$$

*Then the system (1.2) is exactly controllable for  $\Omega = \mathcal{H}$ ,  $t_0$  is an arbitrary element of  $(0, T)$ , and*

$$\Sigma = e_0 \times (0, t_0) \cup e_1 \times (t_0, T).$$

*Proof.* As in the Proof of Corollary 4.1, we use Theorem 2.1 with the choice  $x_0 = (1, 0)$  and  $x_1 = (-1, 0)$ ; it is also clear that  $\Gamma_0 = e_0$  and  $\Gamma_1 = e_1$  by definition (2.5) (see Figure 3).

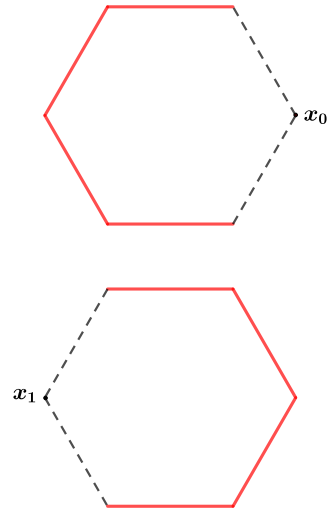
Furthermore, by (2.3)-(2.4), we have

$$R_0 = R_1 = R_{0,1} = 2;$$

then by (2.6) the claim follows.  $\square$

**Corollary 4.7** ( $N$ -times alternating - Hexagon case). *Let  $T > 2(2+N)$  be a real number and let  $e_0, e_1$  be as in (4.9)-(4.10). Then the system (1.2) is exactly controllable for  $\Omega = \mathcal{H}$ ,  $\{t_i\}_{i=0, \dots, N-1}$  any increasing finite subfamily of  $(0, T)$ , and*

$$\Sigma = \left( \bigcup_{i=0, i \in 2N}^N e_0 \times (t_{j-1}, t_j) \right) \cup \left( \bigcup_{i=0, i \in 2N+1}^N e_1 \times (t_{j-1}, t_j) \right).$$


 Figure 3: On the top (resp. bottom) we have represented in red  $e_0$  (resp.  $e_1$ ) and  $x_0$  (resp.  $x_1$ ), to point out the connection with the formula (2.5).

*Proof.* Is similar to the proof of Corollaries 4.3 and 4.6.

In this case, we have to choose  $x_i \equiv x_0 = (1, 0)$  if  $i$  is even or  $x_i \equiv x_1 = (-1, 0)$  otherwise; see Figure 3. Moreover, by (2.3)-(2.4) we obtain

$$R_0 = R_N = R_{i+1, i} = 2.$$

Then condition (2.6) is equivalent to  $T > 4 + 2N = 2(2+N)$  and the claim follows.  $\square$

The last application consists in the following:

**Corollary 4.8** (1-time alternating - Hexagon case II). *Let  $T > 5\sqrt{3}$  be a real number, define*

$$e'_0 = \left\{ (x, y) \in \mathcal{E} \mid y < \frac{\sqrt{3}}{2}x \right\},$$

$$e'_1 = \left\{ (x, y) \in \mathcal{E} \mid y > \frac{\sqrt{3}}{2}x \right\}.$$

*Then the system (1.2) is exactly controllable for  $\Omega = \mathcal{H}$ ,  $t_0$  is an arbitrary element of  $(0, T)$ , and*

$$\Sigma = e'_0 \times (0, t_0) \cup e'_1 \times (t_0, T).$$

*Proof.* As in the Proof of Corollary 4.6, we use Theorem 2.1 with the choice

$$x_0 = \left( \frac{3}{2}, \frac{\sqrt{3}}{2} \right), \quad x_1 = \left( -\frac{3}{2}, -\frac{\sqrt{3}}{2} \right).$$

By (2.3)-(2.4), we have

$$R_0 = R_1 = \frac{3\sqrt{3}}{2}, \quad R_{0,1} = 2\sqrt{3}.$$

In this case, the condition (2.6) is  $T > 2(3\sqrt{3})/2 + 2\sqrt{3} = 5\sqrt{3}$  as stated in the Corollary.



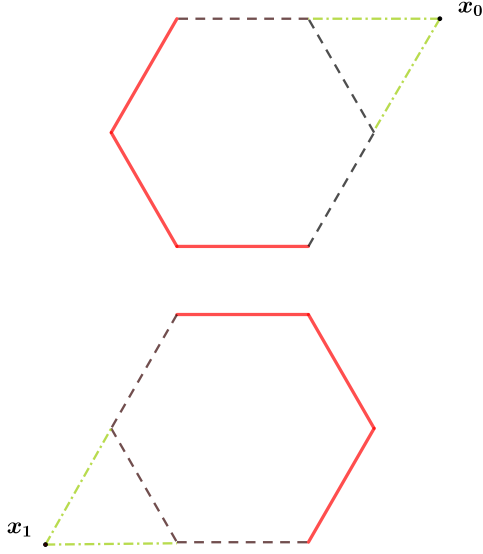


Figure 4: On the top (resp. bottom) we have represented in red  $e'_0$  (resp.  $e'_1$ ) and  $x_0$  (resp.  $x_1$ ), to point out the connection with the formula (2.5); the green lines are the prolongation of the hexagon sides.

Furthermore, with this choice of  $x_0, x_1$ , by (2.5) one can prove that  $\Gamma_0 = e'_0$  and  $\Gamma_1 = e'_1$  (see also Figure 4 for a geometrical construction); then the claim follows.  $\square$

The analogous of Corollary 4.7 for the choice of  $x_0, x_1$  made in Corollary 4.8 can be easily proven by the same argumentations (in this case  $T > \sqrt{3}(3 + 2N)$ ). In order to avoid repetition we do not record the proof here; the same considerations are valid for the content of Remarks 4.2 and 4.4.

We point out that Theorems 2.1 and 3.1 can be applied to a very wide range of domains in  $\mathbf{R}^d$  for any  $d > 1$ ; we have chosen only two cases in this Section in order to give a flavour of the possible applications. Note that, Theorem 3.1 needs the additional condition (3.8), which is not in general immediate to prove, though it can be proven to be valid for smooth  $\Omega$  and  $\varphi$ ; this observation can be useful in applications.

## 5 CONCLUSIONS

To conclude we have shown that the exact controllability holds if  $T$  is sufficiently large and the subset of observation  $\Sigma$  is suitable.

More specifically:

- Let  $\{x_i\}_{i=0, \dots, N}$  be an arbitrary family of points in  $\mathbf{R}^d$ ,  $\{t_i\}_{i=-1, \dots, N}$  an increasing family of positive numbers such that  $t_{-1} = 0$  and  $T := t_N$ ; then the

system (1.2) is exactly controllable if

$$T > R_0 + \sum_{i=0}^{N-1} R_{i+1,i} + R_N;$$

where  $R_i, R_{i+1,i}$  are defined in (2.3)-(2.4) and

$$\Sigma = \bigcup_{i=0}^N \Gamma_i \times (t_i, t_{i-1}),$$

$$\Gamma_i := \{x \in \partial\Omega \mid (x - x_i) \cdot \nu > 0\}.$$

See Theorem 2.1.

- Let  $\varphi : [0, T] \rightarrow \mathbf{R}^d$  be a continuous and piecewise differentiable curve of finite length (see (3.1)). If

$$T > c_0 + \int_0^T |\varphi'(t)| dt + c_N;$$

$$c_i := \max_{x \in \Omega} |x - \varphi(i)|, \quad i = 0, T,$$

the system (1.2) is exactly controllable with

$$\Sigma = \bigcup_{t \in (0, T)} \Gamma_\varphi(t) \times \{t\},$$

$$\Gamma_\varphi(t) := \{x \in \partial\Omega \mid (x - \varphi(t)) \cdot \nu > 0\}.$$

See Theorem 3.1.

These results can be helpful in engineering applications in which one has to control the evolution of a structure (whose dynamics is governed by the wave equation) by means of an action on a portion of the boundary (see (Dessi et al., 2004)-(Carcattera et al., 2014)), but for structural reasons, one cannot act for a long time on the same portion of the boundary, so that the *switch* of the control is necessary.

Finally we mention that, the linear theory for fixed control support has been extended to nonlinear wave equations, see for instance (Zuazua, 1993) or (Coron and Trélat, 2006). With this in mind, we expect that the variable support control results, proven before, can be applied to show the exact controllability of systems driven by nonlinear wave equations with variable support control (see (Debnath, 1994) for explicit physical examples).

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