Combination of Refinement and Verification for the Construction of Lyapunov Functions using Radial Basis Functions

Peter Giesl[1] and Najla Mohammed[2]

[1]*Department of Mathematics, University of Sussex, Falmer, BN1 9QH, U.K.*

[2]*Department of Mathematical Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*

Keywords: Lyapunov Function, Domain of Attraction, Radial Basis Function, Refinement Algorithm, Verification Estimates.

Abstract: Lyapunov functions are an important tool for the determination of the domain of attraction of an equilibrium point of a given ordinary differential equation. The Radial Basis Functions collocation method is one of the numerical methods to construct Lyapunov functions. This method has been improved by combining it with a refinement algorithm to reduce the number of collocation points required in the construction process, as well as a verification that the constructed function is a Lyapunov function. In this paper, we propose a combination of both methods in one, called the combination method. This method constructs a Lyapunov function with the refinement algorithm and then verifies its properties rigorously. The method is illustrated with examples.

1 INTRODUCTION

In the field of dynamical systems, the domain of attraction of an equilibrium is of great importance in the analysis and the derivation of a dynamical system. However, the analytical determination of the domain of attraction is difficult, therefore researchers have been seeking numerical algorithms to determine subsets of the domain of attraction. Most of these are based on Lyapunov functions, i.e. scalar-valued functions that decrease along trajectories of the dynamical system.

The existence of a Lyapunov function guarantees the stability of an equilibrium. Moreover, sublevel sets of a Lyapunov function are positively invariant subsets of the domain of attraction. The construction of such functions, however, is very challenging. For a review of the numerical methods that have been developed to construct Lyapunov function, see (Giesl and Hafstein, 2015).

In this paper we consider an established numerical method to construct Lyapunov functions using the Radial Basis Functions (RBF) collocation method. This method approximates the solution of linear partial differential equations (PDEs) using scattered collocation points and one of its applications is the construction of Lyapunov functions. More precisely, it considers a particular Lyapunov function that satisfies a PDE for its orbital derivatives, and approximately solves it using meshfree collocation (Giesl, 2007; Giesl and Wendland, 2007). It turns out that the RBF approximant itself is a Lyapunov function.

Recently, the construction of Lyapunov functions using Radial Basis Functions has been improved in two directions: firstly, the first refinement algorithm for this method based on Voronoi diagrams has been proposed in (Mohammed and Giesl, 2015). Starting with a coarse grid and applying the refinement algorithm the number of collocation points needed to construct Lyapunov functions has been significantly reduced. Secondly, a method to rigorously verify whether the constructed function is a Lyapunov function, i.e., has negative orbital derivative over a given compact set, has been proposed in (Giesl and Mohammed, 2018). It uses the specific form of the approximant and Taylor-type estimates in terms of the first and second derivatives of the orbital derivative.

In this paper, we will present a combination of these two methods, i.e., the refinement algorithm and the verification process, and apply it to examples. The methodology works in any dimension, however, the number of collocation points as well as the number of point evaluations for the verification grows exponentially with the dimension. The refinement algorithm manages to reduce the number of collocation points.

The outline of this paper will be as follows: Section 2 gives the necessary background on dynamical systems and Lyapunov functions. In Section 3, we

Giesl, P. and Mohammed, N.
Combination of Refinement and Verification for the Construction of Lyapunov Functions using Radial Basis Functions.
DOI: 10.5220/0006944405690578
In *Proceedings of the 15th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2018) - Volume 1*, pages 569-578
ISBN: 978-989-758-321-6

introduce the construction method of Lyapunov functions using Radial Basis Functions. Section 4 presents the refinement algorithm and in Section 5 we introduce the verification estimates. Finally, in Section 6, we propose the new combination method and illustrate it with numerical examples before we conclude in Section 7.

2 LYAPUNOV FUNCTIONS

Consider the following autonomous system of differential equations

$$\dot{x} = f(x) \tag{1}$$

where $f \in C^{\sigma}(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geq 1$ and $d \in \mathbb{N}$. We denote by $S_t\xi := x(t)$ the solution of the initial value problem $\dot{x} = f(x)$, $x(0) = \xi \in \mathbb{R}^d$ and we assume that the solution exists for all $t \geq 0$. Note that S_t defines a dynamical system on $\mathbb{R}^d$.

A point $x_0 \in \mathbb{R}^d$ is an equilibrium of (1) if $f(x_0) = 0$. Then, $x(t) = x_0$ for all $t \geq 0$ is a constant solution of (1). The stability of an equilibrium is determined by the behaviour of solutions in a neighborhood of an equilibrium. An equilibrium is called *stable*, if all solutions starting near the equilibrium stay near the equilibrium for all future times. Moreover, it is called *asymptotically stable* if it is stable and the solutions starting near the equilibrium converge to it as time tends to infinity. It is called *exponentially stable*, if the rate of convergence to the equilibrium is exponential. For an asymptotically stable equilibrium x_0 we are interested in finding the largest set of initial states from which the trajectories of solutions converge to the equilibrium as time tends to infinity. This set is called the *domain of attraction*.

Definition 1 (Domain of attraction)**.** *The domain of attraction of an asymptotically stable equilibrium x_0 is defined by*

$$A(x_0) := \left\{ x \in \mathbb{R}^d \; | S_t x \overset{t \to \infty}{\longrightarrow} x_0 \right\}. \tag{2}$$

Remark 1. *The domain of attraction $A(x_0)$ of an asymptotically stable equilibrium x_0 is non-empty and open.*

2.1 Lyapunov Functions

The method of Lyapunov functions enables us to determine subsets of the domain of attraction of an asymptotically stable equilibrium through sublevel sets of the Lyapunov function. A function $V \in C^1(\mathbb{R}^d, \mathbb{R})$ is called a (strict) Lyapunov function for the equilibrium x_0 if it has a local minimum at x_0 and a negative orbital derivative in a neighborhood of x_0.

Definition 2 (Orbital derivative)**.** *The orbital derivative of a function $V \in C^1(\mathbb{R}^d, \mathbb{R})$ with respect to (1) at a point $x \in \mathbb{R}^d$ is defined by*

$$V'(x) = \langle \nabla V(x), f(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

Remark 2. *The orbital derivative is the derivative along solutions; using the chain rule we have*

$$\frac{d}{dt} V(S_t x)\Big|_{t=0} = \langle \nabla V(x), \dot{x} \rangle = \langle \nabla V(x), f(x) \rangle = V'(x). \tag{3}$$

The following theorem shows how Lyapunov functions are used to find subsets of the domain of attraction; note that the requirement of the local minimum at x_0 is a consequence of the assumptions. The theorem states that sublevel sets of a Lyapunov function are positively invariant subsets of the domain of attraction, see e.g. (Giesl, 2007, Theorem 2.24).

Theorem 1. *Let $x_0 \in \mathbb{R}^d$ be an equilibrium, $V \in C^1(\mathbb{R}^d, \mathbb{R})$ and $K \subset \mathbb{R}^d$ be a compact set with neighbourhood B such that $x_0 \in \mathring{K}$, where $\mathring{K}$ denotes the interior of K. Moreover, let*

1. *$K = \{x \in B \mid V(x) \leq R\}$ for a $R \in \mathbb{R}$, i.e., K is a sublevel set of V.*
2. *$V'(x) < 0$ for all $x \in K \setminus \{x_0\}$, i.e., V is decreasing along solutions in $K \setminus \{x_0\}$.*

Then $K \subset A(x_0)$, K is positively invariant and V is called a (strict) Lyapunov function.

2.2 Existence of Lyapunov Functions

There are several results on converse theorems, ensuring the existence of Lyapunov functions in different contexts, e.g. already in 1949 (Massera, 1949); for a review see (Kellett, 2015). These converse theorems, however, use the explicit solution of (1) and are thus often not useful to construct a Lyapunov function explicitly. We will approximate the Lyapunov function of the following theorem, see (Giesl, 2007, Theorem 2.46).

Theorem 2. *Let x_0 be an exponentially stable equilibrium of $\dot{x} = f(x)$ with $f \in C^{\sigma}(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geq 1$.*

Then there exists a Lyapunov function $V \in C^{\sigma}(A(x_0), \mathbb{R}^d)$ satisfying

$$V'(x) = -\|x - x_0\|^2$$

for all $x \in A(x_0)$.

If $\sup_{x \in A(x_0)} \|f(x)\| < \infty$, then

$$K_R := \{x \in A(x_0) \mid V(x) \leq R\}$$

is a compact set in $\mathbb{R}^d$ for all $R \in \mathbb{R}$.

Note that we can replace the right-hand side by other functions, e.g. also with a negative constant, such as $V'(x) = -1$, however, then the function V is only defined in $A(x_0) \setminus \{x_0\}$, see also (Giesl, 2007).

3 THE CONSTRUCTION OF LYAPUNOV FUNCTIONS USING RBF

Meshfree collocation, in particular by Radial Basis Functions, is used to approximate multivariate functions and approximately solve partial differential equations (Powell, 1992; Buhmann, 2003; Schaback and Wendland, 2006). For a general introduction to meshfree collocation and reproducing kernel Hilbert spaces see (Wendland, 2005). For the application of RBF to the construction of Lyapunov functions, see (Giesl, 2007), where details for the following overview of the method can be found, as well as (Giesl and Wendland, 2007).

A Radial Basis Function is a real-valued function whose value depends only on the distance from the origin i.e., $\Psi(x) = \psi(\|x\|_2)$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. There is a one-to-one correspondence between a Radial Basis Function and its reproducing kernel Hilbert space (RKHS). The approximate solution of the PDE will be a norm-minimal interpolant in the RKHS; for more details about this relation the interested reader is referred to (Giesl and Wendland, 2007).

Now let us consider a general linear partial differential equation of the form

$$Lu = g \text{ on } \Omega \subset \mathbb{R}^d, \tag{4}$$

where L is a linear differential operator of the form

$$Lu(x) = \sum_{|\alpha|\le m} c_\alpha(x) D^\alpha u(x). \tag{5}$$

In our case, the differential operator L will be given by the orbital derivative of a function V with respect to system (1), namely

$$LV(x) := \langle \nabla V(x), f(x)\rangle = \sum_{j=1}^{d} f_j(x)\partial_j V(x) = V'(x). \tag{6}$$

The operator L in (6) is a first order differential operator of the form (5) with $c_{e_j}(x) = f_j(x)$.

Let $X_N = \{x_1, x_2, \ldots, x_N\} \subset \Omega$ be a set of N pairwise distinct points which are no equilibria. Define *Dirac's delta-operator* δ by $\delta_{y_0} g(x) = g(y_0)$. Then we have

$$(\delta_{x_k} \circ L)^x V(x) = LV(x_k) = V'(x_k),$$

where the superscript x denotes the application of the operator with respect to the variable x. The approximant $v : \mathbb{R}^d \to \mathbb{R}$ of V will be given by

$$v(x) = \sum_{k=1}^{N} \beta_k (\delta_{x_k} \circ L)^y \Psi(x-y) \tag{7}$$

where $\Psi(x)$ is the Radial Basis Function. The coefficients β_k are determined by claiming that the interpolation condition

$$(\delta_{x_j} \circ L)^x V(x) = (\delta_{x_j} \circ L)^x v(x)$$

is satisfied for all collocation points $x_j \in X_N$, or in other words that the PDE is satisfied at all points $x_j \in X_N$. This leads to a linear system for β

$$\mathbf{A}\beta = \alpha. \tag{8}$$

If the points x_j are pairwise distinct and no equilibria, then the symmetric matrix $\mathbf{A}$ is positive definite, so in particular non-singular. Hence, the system has a unique solution β. The interpolation matrix entries of $\mathbf{A} = (a_{jk})_{j,k=1,\ldots,N}$ are given by

$$a_{jk} = (\delta_{x_j} \circ L)^x (\delta_{x_k} \circ L)^y \Psi(x-y)$$

and the right-hand side $\alpha = (\alpha_j)_{j=1,\ldots,N}$ is given by

$$\alpha_j = (\delta_{x_j} \circ L)^x V(x) = LV(x_j) = V'(x_j).$$

We choose V to be the Lyapunov function from Theorem 2 and thus $\alpha_j = V'(x_j) = -\|x_0 - x_j\|^2$.

Finally, we calculate the approximant $v(x)$ and its orbital derivative $v'(x)$, using the following formulas, by evaluating and taking the orbital derivative of (7).

$$v(x) = \sum_{k=1}^{N} \beta_k \langle x_k - x, f(x_k)\rangle \psi_1(\|x-x_k\|_2), \tag{9}$$

$$\begin{aligned} v'(x) = \sum_{k=1}^{N} \beta_k \big[&\psi_2(\|x-x_k\|_2)\langle x - x_k, f(x)\rangle \times \\ &\langle x_k - x, f(x_k)\rangle \\ &- \psi_1(\|x-x_k\|_2)\langle f(x), f(x_k)\rangle \big], \end{aligned} \tag{10}$$

where ψ_1 and ψ_2 are defined as:

$$\psi_1(r) = \frac{\frac{d}{dr}\psi(r)}{r}, \quad \text{for } r > 0 \tag{11}$$

$$\psi_2(r) = \frac{\frac{d}{dr}\psi_1(r)}{r}, \quad \text{for } r > 0 \tag{12}$$

and we assume that ψ_1 and ψ_2 can be continuously extended to 0.

In the following we use a Wendland function $\phi_{\ell,k}$ (Wendland, 1998) as Radial Basis Function. Wendland functions are compactly supported Radial Basis Functions, which are polynomials on their support. The corresponding RKHS is a Sobolev space with equivalent norm, and, if the smoothness parameter k, defined below, is sufficiently large, then ψ_1 and ψ_2 can be continuously extended to 0.

The following error estimate was given in (Giesl and Wendland, 2007, Corollay 4.11). $W_2^\tau(\Omega)$ denotes the usual Sobolev space on $\Omega \subset \mathbb{R}^d$. Note that a similar estimate holds for the Lyapunov function satisfying $V'(x) = -1$ in $A(x_0) \setminus \{x_0\}$.

Theorem 3. *Let $k \in \mathbb{N}$ if d is odd or $k \in \mathbb{N} \setminus \{1\}$ if d is even and fix $\ell := \lfloor \frac{d}{2} \rfloor + k + 1$; we use the Radial Basis Function $\psi(r) = \phi_{\ell,k}(cr)$ with $c > 0$, where $\phi_{\ell,k}$ denotes the Wendland function. Set $\tau = k + (d+1)/2$ and $\sigma = \lceil \tau \rceil$.*

Consider the dynamical system defined by (1), where $f \in C^{\sigma}(\mathbb{R}^d, \mathbb{R}^d)$. Let x_0 be an exponentially stable equilibrium of (1). Let f be bounded in $A(x_0)$ and denote by $V \in W_2^{\tau}(A(x_0), \mathbb{R})$ the Lyapunov function satisfying $V'(x) = -\|x - x_0\|_2^2$. Let $\Omega \subset A(x_0)$ be a bounded domain with Lipschitz continuous boundary.

The reconstruction v of the Lyapunov function V with respect to the operator (6) and a set $X_N \subset \Omega \setminus \{x_0\}$ satisfies

$$\|v' - V'\|_{L_\infty(\Omega)} \leq C h^{k-\frac{1}{2}} \|V\|_{W_2^{k+(d+1)/2}(\Omega)} \tag{13}$$

where $h := \sup_{x \in \Omega} \min_{x_j \in X_N} \|x - x_j\|_2$ denotes the fill distance.

Remark 3. *If the collocation points are sufficiently dense, then the right-hand of the error estimate* (13) *is smaller than a given $\varepsilon > 0$. This implies that*

$$v'(x) \leq V'(x) + \varepsilon \leq -\|x - x_0\|^2 + \varepsilon < 0$$

for all $x \in \Omega \setminus B_{\sqrt{\varepsilon}}(x_0)$. Hence, v has negative orbital derivative in Ω apart from a small neighborhood of x_0. One can use the Lyapunov function of the linearized system, the so-called local Lyapunov function, to deal with this small neighborhood of x_0, for details see (Giesl, 2007), or a modified method, see (Giesl, 2008). In this paper, we will not deal with this local problem in more detail, but we exclude a small neighborhood $E \supset B_{\sqrt{\varepsilon}}(x_0)$ of x_0 in our consideration.

The error estimate in Theorem 3 establishes that by choosing the collocation points sufficiently dense, we will construct a Lyapunov function. However, there remain two questions: firstly, how do we choose the collocation points to achieve a negative orbital derivative with as few collocation points as possible? The error estimate only gives information about the error to V' based on the fill distance. We, however, only require v' to be negative, so the error could be larger further away from the equilibrium. The advantage of meshfree collocation is to be able to use scattered points, so it is natural to start with a coarse grid and introduce a refinement algorithm, see Section 4.

Secondly, the quantities on the right-hand side of (13) are not explicitly computable, so it remains to show, after having obtained an approximation v, that $v'(x)$ is negative for all x. This will be done in Section 5.

4 THE REFINEMENT ALGORITHM

In this section, we will combine the construction method with a grid refinement algorithm, aiming for a successful construction of Lyapunov functions with fewer collocation points and less computation time than the original method. For more details of the refinement algorithm see (Mohammed and Giesl, 2015).

Our proposed algorithm is iterative and uses *Voronoi diagrams*. In each step, given a set of collocation points, we generate a Voronoi diagram for these points. Then, we run a test on each Voronoi vertex and decide whether we add it to the set of collocation points or not. A point is added if the orbital derivative is non-negative.

We have used Voronoi vertices as potential new collocation points since they are equidistant to three or more previous grid points, and thus lie “in between” the grid points. Hence, we avoid collocation points too close to each other, which would result in a (nearly) singular interpolation matrix $\mathbf{A}$ of the linear system (8).

4.1 Voronoi Diagrams

A Voronoi diagram is a geometric structure that divides a d-dimensional space into cells based on the distance between sets of points in the space (Preparata and Shamos, 1985). Many algorithms for computing Voronoi diagrams have been proposed, however, we are going to explain the structure of Voronoi diagrams via a very simple but less efficient algorithm using perpendicular hyperplanes.

Let $S = \{s_1, s_2, \ldots, s_n\} \subset \mathbb{R}^d$ be a set of n arbitrarily distributed and distinct sites (points) in $\mathbb{R}^d$. The perpendicular bisector algorithm works as follows: for each pair of sites in S we construct a hyperplane perpendicular to the line segment joining these sites, which intersects the line segment in the middle. At the end of this process, we will have intersections of finitely many hyperplanes which build up cells, with a convex polygon structure, known as *Voronoi regions*. The boundaries of each region are called *Voronoi edges* and the intersections of Voronoi edges are called *Voronoi vertices*. For more details see (Berg et al., 2008; Klein, 1989; Iyengar et al., 2014).

Mathematically, the Voronoi region of a point s_i in S is defined by

$$\mathcal{V}_i = \bigcap_{j=1, j \neq i}^{n} \left\{ x \in \mathbb{R}^d \,|\, \|x - s_i\|_2 < \|x - s_j\|_2 \right\},$$

where $\|\cdot\|_2$ denotes the Euclidean distance. This means that for every point $x \in \mathbb{R}^d$ within a Voronoi region $\mathcal{V}_i$ the Euclidean distance of x to the site s_i, which is also inside the region, is smaller than the Euclidean distance of x to any other site s_j.

4.2 The Refinement Algorithm

1. Fix a compact neighbourhood $K \subset \mathbb{R}^d$ and a small neighborhood $E \subset K$ of the equilibrium x_0 as well as a Radial Basis Function. Let $n = 1$ and start with an initial set of collocation points $X_1 = \{x_1^{(1)}, x_2^{(1)}, \ldots, x_{N_1}^{(1)}\} \subset K$, not containing any equilibrium.
2. Calculate a Lyapunov function v_n using the RBF method with the collocation points $X_n = \{x_1^{(n)}, x_2^{(n)}, \ldots, x_{N_n}^{(n)}\}$.
3. Generate Voronoi vertices $Y_n = \{y_1, y_2, \ldots, y_{M_n}\} \subset \mathbb{R}^d$ for the collocation points X_n. Exclude points in Y_n which are equilibria, lie in E or outside K.
4. Run a test on each point $y_j \in Y_n$ and check whether $v_n'(y_j) < 0$ (then $y_j \in Y_n^-$) or $v_n'(y_j) \geq 0$ (then $y_j \in Y_n^+$), where $j = 1, \ldots, M_n$ and $Y_n = Y_n^- \cup Y_n^+$.
5. Define a new set of collocation points $X_{n+1} = X_n \cup Y_n^+$.
6. $n \to n+1$, repeat the steps 2. to 5. until $Y_n^+ = \emptyset$.

Note that Y_n^+ is the set of Voronoi vertices where the orbital derivative is non-negative, so they are added to the set of collocation points, while the orbital derivative at the vertices Y^- is already negative.

The algorithm terminates if all Voronoi vertices have negative orbital derivative. However, this does not guarantee that the constructed function v_n has negative orbital derivative everywhere. In (Mohammed and Giesl, 2015) the orbital derivative was checked on a very fine grid, but again, this does not rigorously show that it is negative everywhere. Hence, in this paper, we will combine this refinement method with the verification estimates, which will be introduced in the next section.

5 THE VERIFICATION ESTIMATES

The motivation of deriving the verification estimates for the RBF construction method is that the constructed function v can not be guaranteed to have negative orbital derivative in the whole set K. According to the error estimate (13), the approximant v is a Lyapunov function, if the collocation points are sufficiently dense. However, we can not tell in advance how small the fill distance h should be since this error estimate depends on unknown quantities such as V. Therefore, verification estimates that are based on a Taylor approximation and rely on the first and second derivatives of the orbital derivative have been introduced in (Giesl and Mohammed, 2018). They make use of the special form of the RBF approximant and its orbital derivative (9) and (10), respectively. The other main ingredient of these verification estimates is a checking grid Y_{od}. Here, we will only introduce the verification estimates based on the second derivatives.

In the following we consider two point sets:

1. $X_N = \{x_1, x_2, \ldots, x_N\}$, the collocation points for the calculation of the approximant v.
2. $Y_{od} = \{y_1, y_2, \ldots, y_M\}$, used to check the sign and value of the orbital derivative of the constructed Lyapunov function on a different (usually finer but not necessarily) grid than X_N.

The checking grid Y_{od} consists of vertices of a fixed triangulation. In (Giesl and Mohammed, 2018) different triangulations and thus distructions of points in Y_{od} have been considered and it turned out that, depending on whether the dimension d is odd or even, the standard or the centered triangulation are preferable, i.e. result in fewer points in Y_{od}. In this paper we focus on even dimensions and thus use the standard triangulation, for odd dimensions see (Giesl and Mohammed, 2018).

We will introduce triangulations and define the standard triangulation.

Definition 3. *A k-simplex is a set*

$$co(x_0, \ldots, x_k) = \left\{ \sum_{i=0}^{k} \lambda_i x_i \mid 0 \leq \lambda_i, \sum_{i=0}^{k} \lambda_i = 1 \right\},$$

where $x_0, \ldots, x_k \in \mathbb{R}^d$ are pairwise distinct and are called the vertices.

A triangulation in $\mathbb{R}^d$ is a set $T := \{\mathcal{T}_\nu : \nu = 1, 2, \ldots, N\}$ (or $N = \infty$) of d-simplices $\mathcal{T}_\nu$, such that any two simplices in T intersect in a common face or are disjoint. Note that a face of a d-simplex is a k-simplex, $0 \leq k < d$, so this means that the intersection of two simplices in T is either empty or the convex combinations of the common vertices of the two simplices.

We denote the set of all vertices of all simplices by $\mathcal{V}_T$ and we say that T is a triangulation of the set

$$\mathcal{D}_T := \bigcup_\nu \mathcal{T}_\nu.$$

The standard triangulation $T_S(h_1)$ with parameter $h_1 \in \mathbb{R}^+$ consists of the simplices

$$\mathcal{T}_{z\mathcal{J}\sigma} := \mathrm{co}\left(x_0^{z\mathcal{J}\sigma}, x_1^{z\mathcal{J}\sigma}, \ldots, x_d^{z\mathcal{J}\sigma}\right)$$

for all $z \in \mathbb{N}_0^d$, all $\mathcal{J} \subset \{1,2,\ldots,d\}$, and all $\sigma \in S_d$, where

$$x_i^{z\mathcal{J}\sigma} := R^{\mathcal{J}}\left(z + \sum_{j=1}^{i} e_{\sigma(j)}\right) h_1 \quad \text{for} \quad i = 0,1,2,\ldots,d. \quad (14)$$

Here, S_d denotes the set of all permutations of the numbers $1,2,\ldots,d$ and $e_1,e_2,\ldots,e_d$ denotes the standard orthonormal basis of $\mathbb{R}^d$. Further, the functions $R^{\mathcal{J}} : \mathbb{R}^d \to \mathbb{R}^d$ are defined for every $\mathcal{J} \subset \{1,2,\ldots,d\}$ by

$$R^{\mathcal{J}}(x) := \sum_{i=1}^{d} (-1)^{\mathcal{X}_{\mathcal{J}}(i)} x_i e_i,$$

where the characteristic function $\mathcal{X}_{\mathcal{J}}(i)$ is equal to one if $i \in \mathcal{J}$ and equal to zero if $i \notin \mathcal{J}$. Hence, $R^{\mathcal{J}}(x)$ puts a minus in front of the coordinates x_i of x whenever $i \in \mathcal{J}$.

Note that the set of all vertices of all simplices of the standard triangulation is $\mathcal{V}_{T_S(h_1)} = h_1\mathbb{Z}^d$. We cite Theorem 11 of (Giesl and Mohammed, 2018) and focus on even dimensions d; note that the theorem also holds for bounded sets C.

Theorem 4. *Let $C \subset \mathbb{R}^d$ be a bounded set, $v \in C^3(\mathbb{R}^d,\mathbb{R})$ and $f \in C^2(\mathbb{R}^d,\mathbb{R}^d)$. For d even let $T = \{\mathcal{T}_\nu \in T_S(h_1) \mid \mathcal{T}_\nu \subset C\}$, i.e. the simplices of the standard triangulation with side length h_1 that are fully contained in C. Let $Y_{od} = \mathcal{V}_T \subset h_1\mathbb{Z}^d$ and $\tilde{C} = \mathcal{D}_T$.*

Then $\tilde{C} \subset C$ and

$$v'(x) \leq \max_{y \in Y_{od}} v'(y) + \frac{d^2}{4}\left(\max_{x\in\tilde{C}} \max_{i,j=1,\ldots,d} \left|\frac{\partial^2 v'(x)}{\partial x_i \partial x_j}\right|\right) h_1^2 \quad (15)$$

for all $x \in \tilde{C}$.

To estimate the second derivative of the orbital derivative in (15) we use the following Theorem 10 from (Giesl and Mohammed, 2018), which makes use of the special form of the approximant v.

Theorem 5 (The second derivative of the orbital derivative)**.** *Let $f \in C^2(\mathbb{R}^d,\mathbb{R}^d)$ and $v \in C^3(\mathbb{R}^d,\mathbb{R})$ be an RBF approximant of the form (9) with a Radial Basis Function $\psi \in C^4(\mathbb{R}_0^+,\mathbb{R})$. Let $\tilde{C} \subset \mathbb{R}^d$ be a compact set and $X_N = \{x_1,\ldots,x_N\} \subset \tilde{C}$ be a set of N pairwise distinct points which are no equilibria.*

Then the second derivative of the orbital derivative v' can be bounded by

$$\begin{aligned}\left|\frac{\partial^2 v'(x)}{\partial x_i \partial x_j}\right| \quad &\leq \quad \beta\Big([\Psi_{4,4} + 6\Psi_{3,2} + 3\Psi_{2,0}]F^2 \\ &+ \quad [2\Psi_{3,3} + 6\Psi_{2,1}]FD_1 \\ &+ \quad [\Psi_{2,2} + \Psi_{1,0}]FD_2\Big), \qquad (16)\end{aligned}$$

for all $x \in \tilde{C}$ and all $i,j \in \{1,\ldots,d\}$, where

- $F := \max_{x\in\tilde{C}} \|f(x)\|_2$,
- $D_1 := \max_{x\in\tilde{C}} \max_{j\in 1,\ldots,d} \left\|\frac{\partial f(x)}{\partial x_j}\right\|_2$,
- $D_2 := \max_{x\in\tilde{C}} \max_{i,j\in 1,\ldots,d} \left\|\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right\|_2$,
- $\Psi_{i,k} := \sup_{r\in[0,\infty)} \left[|\psi_i(r)| \cdot r^k\right]$,
- $\beta := \sum_{k=1}^{N} |\beta_k|$.

Note that we only need a computer to calculate β, all other quantities can be computed by hand. Expressions for $\Psi_{i,k}$ for certain Wendland functions are given in (Giesl and Mohammed, 2018).

6 COMBINATION METHOD

In this section we propose a new method, combining the refinement and verification methods. This method will rigorously construct a Lyapunov function with fewer collocation points than previous methods.

Fix a small neighborhood E of the equilibrium and a compact and convex set $E \subset K \subset \mathbb{R}^d$. Set $C = K \setminus E$. We first apply the refinement method until no more points are added. Let $X_N \subset C$ be the set of grid points obtained after the refinement process. Then, the steps of the combination method are:

1. Calculate the quantities $F, D_1, D_2, \Psi_{i,k}, \beta$, and substitute in (16).
2. Use estimate (15) to determine the order of h_1, i.e., the density of the checking grid Y_{od}.
3. Use this h_1 to define Y_{od} and calculate the maximum value of the orbital derivative v' in Y_{od}.
4. Use estimate (15) to show that $v'(x) < 0$ for all $x \in \tilde{C}$, and thus that the constructed function v is a Lyapunov function.
5. If 4. fails, then either the initial grid of the refinement or the checking grid is not fine enough.

We will now apply the combination method to two examples – one where both methods have previously been applied to in isolation, and one new one.

Example 1. *Consider the non-linear system (Giesl and Wendland, 2007, Example 4.3)*

$$\begin{cases} \dot{x} = -x - 2y + x^3, \\ \dot{y} = -y + \frac{1}{2}x^2 y + x^3. \end{cases}$$

The system has an asymptotically stable equilibrium at $(0,0)$ and we have used the refinement algorithm to construct an approximant v of the Lyapunov function satisfying $V'(x) = -\|x\|_2^2$. In (Mohammed and Giesl, *2015), Example 3, the Wendland function* $\phi_{6,4}(r) = (1-r)_+^6(35r^2+18r+3)$*, where* $x_+ = x$ *for* $x \geq 0$ *and* $x_+ = 0$ *for* $x < 0$ *was used. The refinement process was started with an initial set* $X_{initial}$ *of* 24 *collocation points. The algorithm terminated after four refinement steps with the set* X_{final} *of* 90 *collocation points with* $K = [-1,1]^2$ *and* $E = (-0.1, 0.1)^2$*. The sign of* v' *was checked over a checking grid* X_{check} *of size* $h_{check} = 10^{-3}$*. However, this does not rigorously show that the function has negative orbital derivative everywhere.*

In (Giesl and Mohammed, 2018), Example 1, the same example has been considered, but with a regular, larger set of 360 collocation point. It has been shown, using the verification estimates, that the constructed function has negative orbital derivative.

We will now use the combination method in this paper, again starting with an initial set $X_{initial}$ *of* 24 *collocation points. We have used the Wendland function* $\phi_{6,4}$ *and* $K = [-1,1]^2$*, but have excluded the set* $E = [-0.1, 0.1]^2$ *and set* $C = K \setminus E$*. The refinement algorithm terminated after four refinement steps with the set* X_{final} *of* $N_4 = 88$ *collocation points, see Figure 1. The time to calculate the four refinement steps on a standard laptop with an Intel(R) Core (TM) i5-3550 CPU @ 3.30 GHz processor was 2.1 seconds.*

Figure 2 (a) shows the Lyapunov function v_4 *constructed with the final set of collocation points obtained with the refinement algorithm. Figure 2 (b), displays different sublevel sets of* v_4*.*

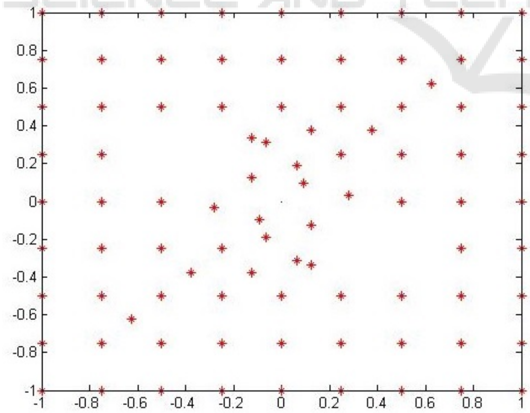


Figure 1: The grid points $N_4 = 88$ after the termination of the refinement algorithm.

We continue with the verification of the obtained Lyapunov function. We will derive and then use an estimate for the size of the checking grid to verify the negativity of the orbital derivative of v*.*

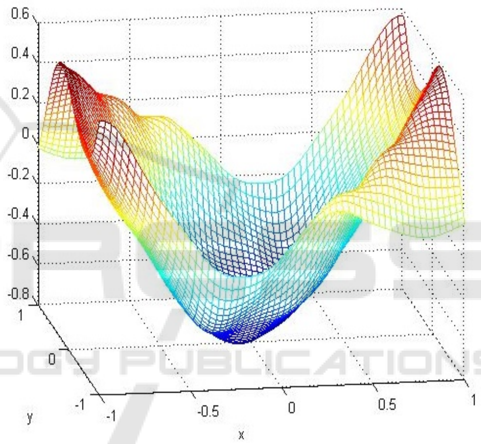
We have $F = 4.717$*,* $D_1 = 5.657$*, and* $D_2 = 9.2195$*. We calculate the value of* $\beta = \sum_{k=1}^{88} |\beta_k| = 0.4105$*.*

Hence, (16) *gives*

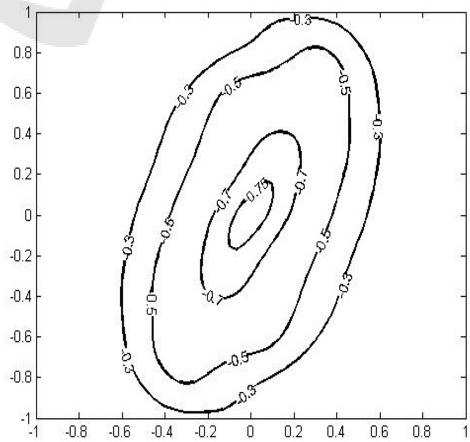
$$\left|\frac{\partial^2 v'(x)}{\partial x_i \partial x_j}\right| \leq 0.4105 \Big(157870.623 \times (4.717)^2 + 19358.745 \times 4.717 \times 5.657 + 1124.797 \times 4.717 \times 9.2195\Big) = 1.7 \times 10^6. \tag{17}$$

Since $d = 2$ *is even, we use Theorem 4 and use the estimate* (17) *in* (15)*. We obtain*

$$v'(x) \leq \max_{y \in Y_{od}} v'(y) + \left(\max_{x \in \tilde{C}} \max_{r,s=1,2} \left|\frac{\partial^2 v'(x)}{\partial x_r \partial x_s}\right|\right) h_1^2, \leq \max_{y \in Y_{od}} v'(y) + \left(1.7 \times 10^6\right) h_1^2. \tag{18}$$



(a) The constructed Lyapunov function $v_4(x,y)$ with the refinement algorithm.



(b) Different sublevel sets of v_4.

Figure 2: (a) The constructed Lyapunov function $v_4(x,y)$ with the refinement algorithm and (b) different sublevel sets of v_4.

Assume that the orbital derivative of the constructed function v' is a good approximation to V' with $V'(x) = -\|x\|_2^2$ in $[-1,1]^2 \setminus [-0.1,0.1]^2$. Then

$$\sup_{y\in[-1,1]^2\setminus[-0.1,0.1]^2} v'(y) \approx -0.1^2.$$

The right-hand side of (18) *is negative if*

$$\begin{aligned} 1.7\times 10^6\, h_1^2 &\leq 10^{-2} \\ h_1 &\leq \sqrt{\frac{1}{1.7}}\, 10^{-4} = 7.67\cdot 10^{-5}. \end{aligned}$$

The estimated value of h_1 indicates the supposed density of the checking grid Y_{od}. To allow for a small error, we have chosen the density of Y_{od} to be $h_1 = 4.17\cdot 10^{-5}$. Then checking the value of the orbital derivative over $Y_{od} = h_1\mathbb{Z}^2 \cap C$ gives $\max_{y\in Y_{od}} v'(y) = -0.0038$. These function evaluations took 47,290.4 seconds on the same laptop, note that this could be parallelized to make the computation faster. Therefore, (18) yields

$$\begin{aligned} v'(x) &\leq -0.0038 + \left(1.7\times 10^6\right)\left(4.17\times 10^{-5}\right)^2, \\ &= -0.0038 + 0.003 = -0.0008, \end{aligned}$$

which shows that $v'(x) < 0$ for all $x \in \tilde{C} = [-1,1]^2 \setminus (-0.1-h_1, 0.1+h_1)^2$, i.e. the function v, constructed by the refinement method, is a Lyapunov function.

Example 2. *We will now consider*

$$\begin{cases} \dot{x} = -x(1-x^2-y^2) - y, \\ \dot{y} = -y(1-x^2-y^2) + x, \end{cases}$$

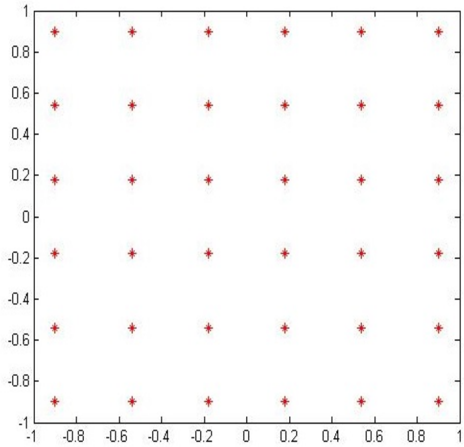
see (Giesl, 2007, Example 2.10), which has an exponentially stable equilibrium at $x_0 = (0,0)$ with domain of attraction $B_1(0,0)$. We use the refinement algorithm to construct an approximant v of the Lyapunov function satisfying $V'(x) = -1$. Note that such a function exists in $A(x_0)\setminus\{x_0\}$.

We set $K = [-0.9,0.9]^2$, $E = [-0.1,0.1]^2$ and $C = K \setminus E$. We choose the Wendland function $\phi_{6,4}$, see previous example. We start our refinement algorithm with an initial set of collocation points $X_1 = \{(x,y)\in\mathbb{R}^2 \mid x,y \in \{0,\pm h,\ldots,\pm 0.9\}\}\setminus E$, $h = 0.36$, i.e. $N_1 = 36$ points, see Figure 3 (a).

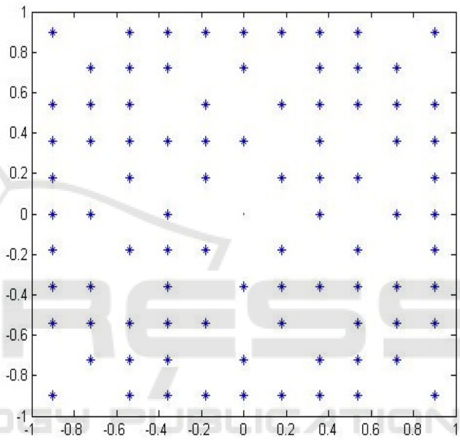
After performing four refinement steps, the algorithm terminates and constructs a function v_4 with $N_4 = 88$ points, see Figure 3 (b). The time to calculate the four refinement steps was 1.8 seconds.

Figure 4 (a) shows the Lyapunov function v_4 constructed with the final set of collocation points obtained with the refinement algorithm $N_4 = 88$ points. Figure 4 (b) displays different sublevel sets of v_4.

Let us now verify that the constructed function is indeed a Lyapunov function. We first calculate the quantities $F = 3.57$, $D_1 = 4.98$, $D_2 = 5.69$ as well as $\beta = \sum_{k=1}^{88}|\beta_k| = 1.7372$.



(a) The initial grid X_1 with $N_1 = 36$ points.



(b) The final grid X_4 with $N_4 = 88$ points.

Figure 3: (a) shows the distribution the initial grid points and (b) the distribution of final grid points after the last refinement step.

We now use the verification estimates to determine the order of the checking grid Y_{od}. We calculate the second derivative of the orbital derivative by (16)

$$\left|\frac{\partial^2 v'(x)}{\partial x_i \partial x_j}\right| \leq 1.7372\times 2379065.026 = 4.1\times 10^6.$$

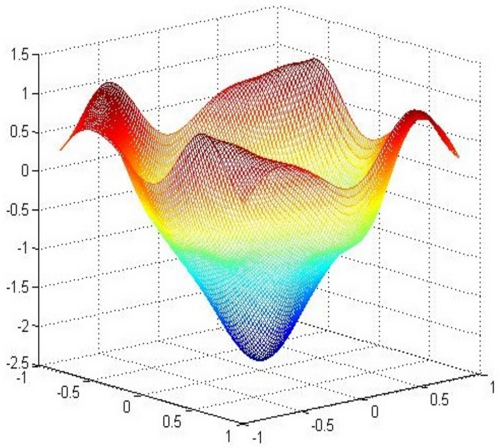
Since $d = 2$ is even, we apply Theorem 4. Thus,

$$v'(x) \leq \max_{y\in Y_{od}} v'(y) + \left(4.1\times 10^6\right) h_1^2. \tag{19}$$

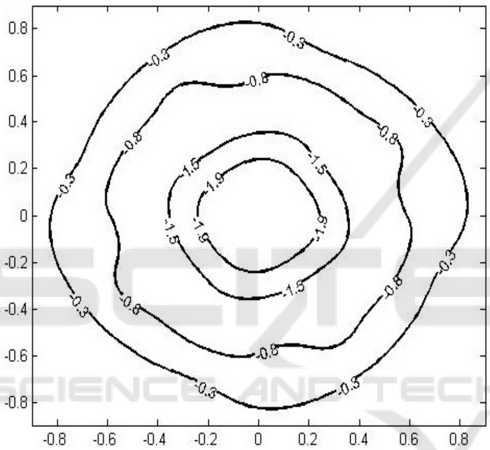
If v' is a good approximation to V', then $\max_{y\in Y_{od}} v'(y) \approx -1$. The right-hand side of (19) *is negative if*

$$h_1 \leq \sqrt{\frac{1}{4.1}}\, 10^{-3} \approx 0.494\times 10^{-3}.$$

To allow for an error, we have chosen the density of Y_{od} to be $h_1 = 4.05\times 10^{-5}$. Then checking the value of the orbital derivative over $Y_{od} = h_1\mathbb{Z}^2 \cap C$



(a) The constructed Lyapunov function $v_4(x,y)$ with the refinement algorithm.



(b) Different sublevel sets of v_4.

Figure 4: (a) The constructed Lyapunov function $v_4(x,y)$ with the refinement algorithm and (b) different sublevel sets of v_4.

gives $\max_{y \in Y_{od}} v'(y) = -0.0142$. *These function evaluations took 35,904.5 seconds on the same laptop, note that this could be parallelized to make the computation faster. Therefore, (19) yields*

$$v'(x) \leq -0.0142 + \left(4.1 \times 10^6\right) \left(4.05 \times 10^{-5}\right)^2, \\ = -0.0142 + 0.0067 = -0.0075.$$

Thus, $v'(x) < 0$ *for all* $x \in \tilde{C} = [-1,1]^2 \setminus (-0.1 - h_1, 0.1 + h_1)^2$, *and the constructed function* v *is a Lyapunov function.*

Note that K *is not a subset of the domain of attraction. As usually the domain of attraction is not known in advance, this is a realistic situation, and even in this situation, the algorithm works well.*

7 CONCLUSION

Lyapunov functions are an important tool to determine the domain of attraction of an equilibrium. They can be constructed by approximating a solution of a PDE using Radial Basis Functions. This paper combines a refinement algorithm of this method, refining the set of collocation points using Voronoi vertices, with a verification that the constructed function is indeed a Lyapunov function. The refinement algorithm results in fewer collocation points than by uniformly refining the set of collocation points. The verification provides a rigorous method to show that the constructed function has a negative orbital derivative. The combination of these two methods, which was proposed for the first time in this paper, thus provides an efficient and rigorous method to construct Lyapunov functions. This was demonstrated in two examples.

REFERENCES

Berg, M., Cheong, O., Kerveld, M., and Overmars, M. (2008). *Computational geometry: Algorithms and Applications*. Springer-Verlag, Berlin.

Buhmann, M. (2003). *Radial basis functions: theory and implementations*, volume 12 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge.

Giesl, P. (2007). *Construction of Global Lyapunov Functions Using Radial Basis Functions*. Lecture Notes in Math. 1904, Springer.

Giesl, P. (2008). Construction of a local and global Lyapunov function using Radial Basis Functions. *IMA J. Appl. Math.*, 73(5):782–802.

Giesl, P. and Hafstein, S. (2015). Review on computational methods for Lyapunov functions. *Discrete Contin. Dyn. Syst. Ser. B*, 20:2291–2331.

Giesl, P. and Mohammed, N. (2018). Verification estimates for the construction of Lyapunov functions using meshfree collocation. submitted.

Giesl, P. and Wendland, H. (2007). Meshless collocation: error estimates with application to Dynamical Systems. *SIAM J. Numer. Anal.*, 45(4):1723–1741.

Iyengar, S., Boroojeni, K., and Balakrishnan, N. (2014). *Mathematical Theories of Distributed Sensor Networks*. Springer, New York.

Kellett, C. M. (2015). Classical converse theorems in Lyapunov's second method. *Discrete Contin. Dyn. Syst. Ser. B.*, 20(8):2333–2360.

Klein, R. (1989). *Concrete and Abstract Voronoi Diagrams*. Lecture Notes in Computer Science. Springer-Verlag, Berlin.

Massera, J. (1949). On Liapounoff's conditions of stability. *Ann. of Math.*, 50(2):705–721.

Mohammed, N. and Giesl, P. (2015). Grid refinement in the construction of Lyapunov functions using radial basis functions. *Discrete Contin. Dyn. Syst. Ser. B*, 20:2453–2476.

Powell, M. J. D. (1992). The theory of radial basis function approximation in 1990. In *Advances in numerical analysis, Vol. II (Lancaster, 1990)*, Oxford Sci. Publ., pages 105–210. Oxford Univ. Press, New York.

Preparata, F. and Shamos, M. (1985). *Computational geometry*. Texts and Monographs in Computer Science. Springer-Verlag, New York.

Schaback, R. and Wendland, H. (2006). Kernel techniques: from machine learning to meshless methods. *Acta Numer.*, 15:543–639.

Wendland, H. (1998). Error estimates for interpolation by compactly supported Radial Basis Functions of minimal degree. *J. Approx. Theory*, 93:258–272.

Wendland, H. (2005). *Scattered data approximation*, volume 17 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge.