Synchronization of Discrete Fractional Odd Logistic System Based on Parametric Adaptive Control Algorithm

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Keywords: Fractional odd logistic map, synchronization, difference scheme, chaos.

Abstract: In this study, a simple but efficient method for chaos synchronization of fractional difference system is proposed, which is based upon the parametric adaptive control algorithm. Using this new method, chaos synchronization for discrete fractional odd logistic system is implemented.

1 INTRODUCTION

Since Pecora and Carroll introduced an idea of achieving synchronization between the drive and response systems, chaos synchronization has been widely explored and studied due to its potential applications in secure communication, ecological systems and system identification. Recent studies show that chaos of fractional differential systems can be synchronized, see and references cited therein. Compared with the fruitful results in the chaos synchronization of continuous fractional differential equations, the fractional difference equation is a particularly new topic. The dynamical behaviors of the fractional one and two dimensional maps and the results show that the chaos does exist there. The DFC (Pyragas K., 1992 and Pyragas K., 1992) is proved to be an efficient tool to discrete the chaotic systems with a memory effect. Naturally, a question maybe put forth: how to achieve the fractional synchronization of such maps? In this paper, we investigate the chaos synchronization of the discrete fractional odd logistic map in the design the synchronized systems based upon the parametric adaptive control algorithm. The remainder of this paper is organized as follows. In section 2, introduces the definitions and the properties of the discrete fractional calculus. Section 3 presents fractional odd logistic map on time scales and shows the discrete chaotic solutions while the difference orders and the coefficients are changing. Section 4 is the conclusion.

2 FRACTIONAL ODD LOGISTIC MAP

Considering the discrete fractional calculus, we start with some necessary definitions from discrete fractional calculus theory and preliminary results so that this paper is self-contained.

Definition 1. (F.M. Atici, P.W. Eloe, 2009) Let νth fractional sum of $u(t)$ is defined by

$$\Delta_a^{-\nu} u(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \big(t-\sigma(s)\big)^{\nu-1} u(s),$$

Where a is the starting point, $\sigma(s) = s + 1$ and u is defined for $s = a \ mod$ (1) and $\Delta_a^{-\nu} u(t)$ is defined for $t = (a + \nu) \ mod$ (1). In particular $\Delta_a^{-\nu}$ maps a function defined on N_a to functions defined on $N_{a+\nu}$, where $N_a = a, a+1, a+2 \cdots$. In addition,

$$t^{\nu} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

Definition 2. (T. Abdeljawad, 2011) For $\alpha > 0$, $u: N_a \to R$ and α be given, the Caputo-like delta difference is defined by

$${}_C\Delta_\nu^\alpha u(t) := \Delta_a^{n-\nu} \Delta^n u(t)$$

$$= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} \big(\mathrm{t}-\sigma(s)\big)^{(n-\nu-1)} u(s)$$

$$t \epsilon N_{a+n-\nu}, n = [\nu] + 1$$

where ν is the difference order, and

Hu, S. and Wang, Y.
Synchronization of Discrete Fractional Odd Logistic System Based on Parametric Adaptive Control Algorithm.
In *3rd International Conference on Electromechanical Control Technology and Transportation (ICECTT 2018)*, pages 580-583
ISBN: 978-989-758-312-4

Theorem 1. (F.L. Chen, X.N. Luo, Y. Zhou, 2011) For the delta fractional difference equation

$$ {}_C\Delta_a^\alpha u(t) = f(t+\alpha-1, u(t+\alpha-1)) $$

$$ \Delta^k u(a) = u_k, n = [\alpha]+1, k = 0,1,\dots,n-1. $$

the equivalent discrete integral equation can be obtained as

$$ u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \sum_{a+n-\alpha}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} \; f\big(s + \alpha - 1, u(s+\alpha-1)\big) \qquad (1) $$

where the initial iteration reads

$$ u(0) = \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} \Delta^k u(a) \; . $$

The complex difference equation with long-term memory is obtained here. It can reduce to the classical one when the difference order $\alpha = 1$ but the integer one does not hold the discrete memory. Through a discrete fractional odd logistic map it reveals that the dynamical behavior holds discrete memory even the difference order is very small. For the famous odd logistic map

$$ u(n+1) = u(n)(\lambda - u(n)^2), n = 0,1,2 \cdots $$

The odd logistic map is based to maps of the plane with dihedral symmetry.

We can redefined it as

$$ \Delta u(n) = u(n)(\lambda - 1 - u(n)^2) $$

From the discrete fractional calculus, the fractional one can be given as

$$ {}_C\Delta_a^\alpha u(t) = u(t+\alpha-1)(\mu - u(t+\alpha-1)^2), 0 < \alpha \le 1, t \in N_{a+1-\alpha} $$

Where $\mu = \lambda - 1$. From [1], we can obtain the following discrete integral form from $0 < \alpha \le 1$.

$$ u(t) = u(a) + \frac{1}{\Gamma(\alpha)} \sum_{a+1-\alpha}^{t-\alpha} \big(t-\sigma(s)\big)^{(\alpha-1)} \; f\big(s + \alpha - 1, u(s+\alpha-1)\big) \;\; t \in N_{a+1} \qquad (2) $$

where $\frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)}$ is a discrete kernel function and $\big(t-\sigma(s)\big)^{(\alpha-1)} = \frac{\Gamma(t-s)}{\Gamma(t-s+1-\alpha)}$. As a result, the numerical formula can be presented explicitly

$$ u(n) = u(a) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; f\big(j - 1, u(j-1)\big) \, . \qquad (3) $$

For the fractional odd logistic map, an explicit numerical formula ban be given as

$$ u(n) = u(a) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; u(j-1)(\mu - u(j-1)^2) \, . \qquad (4) $$

3 CHAOS SYNCHRONIZATION OF THE FRACTIONAL ODD LOGISTIC MAP

3.1 Sufficient conditions of system synchronization

Making use of equation (4), we obtain the iteration equations of chaotic system as follows:

$$ x(n) = x(a) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; x(j-1)(\mu_c - x(j-1)^2) \, . \qquad (5) $$

$$ y(n) = y(a) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; y(j-1)(\mu_s - y(j-1)^2) \, . \qquad (6) $$

We iterate the following three equations to synchronize the two systems by controlling the initial states and parameters of the response system.

$$ \begin{cases} x(n) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{j=0}^{n-1} \dfrac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; x(j)(\mu_c - x(j)^2), \\ y(n) = y_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\sum_{j=0}^{n-1} \dfrac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; x(j)(\mu(n-1) - x(j)^2), \\ \mu(n) = \mu(n-1) + \dfrac{\mathrm{K}\Gamma(n+1)}{\Gamma(n+\alpha+1)} \big(y(n) - x(n)\big). \end{cases} \qquad (7) $$

Where, $x(n)$ is the master system, $y(n)$ is the response system, $\mu(n)$ is the adaptive control parameter, and K is the control stiffness and it is adjustable.

Lemma 1.(T. Abdeljawad, 2011)

$$ \sum_{j=0}^{n} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)\Gamma(\alpha)} = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} . $$

Theorem 2. Iteration system (7) is convergent when $-6\Gamma(\alpha+1) < K < 0$.

Proof. Let $\Delta\mu(n) = \mu(n) - \mu_c$, $e(n) = y(n) - x(n)$, where $e(n)$ is the synchronization error system, then we can get the following equation from system (7):

$$ e(n) = \frac{\Delta\mu(n-1)}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)} \; x(j). \qquad (8) $$

Subtracting μ_c from the both sides of equation

$$ \mu(n) = \mu(n-1) + \frac{\mathrm{K}\Gamma(n+1)}{\Gamma(n+\alpha+1)} \big(y(n) - x(n)\big), $$

yields

$$\Delta\mu(n) = \Delta\mu(n-1)\left(1 + \frac{K\Gamma(n+1)}{\Gamma(n+\alpha+1)}\sum_{j=0}^{n-1}\frac{\Gamma(n-j+\alpha)}{\Gamma(n-j+1)\Gamma(\alpha)}x(j)\right) \quad (9)$$

It is obvious that for any $i = 1,2,\cdots.x(i) \in (0,1)$ because of $x(0) \in (0,1)$. Then apply lemma 1 to equation (9), the following inequality holds

$$\Delta\mu(n) \le \prod_{i=1}^{n-1}\left(1 + \frac{K\Gamma(n+1)}{\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(\alpha+1)}\right)\Delta\mu(0). \quad (10)$$

When

$$\left|1 + \frac{K\Gamma(n+1)}{\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(\alpha+1)}\right| < 1,$$

also if $-6\Gamma(\alpha+1) < K < 0$, we can get

$$\lim_{n\to\infty}\Delta\mu(n) = 0,$$

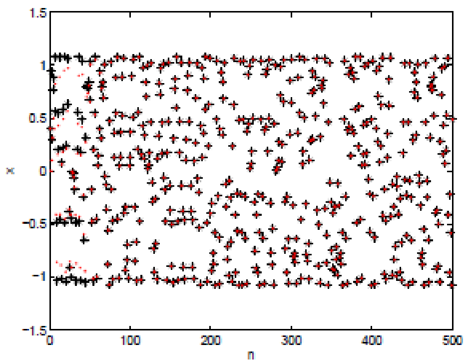
or $\lim_{n\to\infty} e(n) = 0$, the iteration system (7) is convergent.

3.2 Numerical simulations

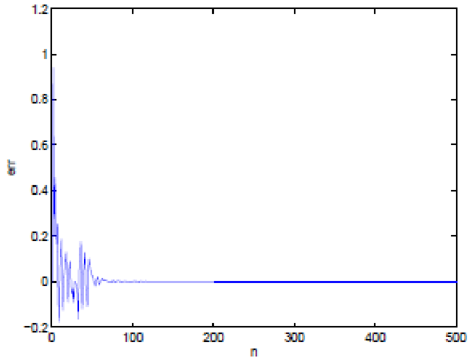
In this section, three cases for different fractional order α and different K in system (7) will be given to verify the synchronization of the odd logistic system. We assume the initial condition associated with master system and response system as $x(0) = 0.4$ and $y(0) = 0.4$.

Case1: $\alpha = 1$.

We choose $\mu(0) = 0.4$ and K=-1 as the parameters. By numerical simulation, we can see the dynamical behaviors of master system and response system as show in figure 1(a). The error system showed in figure 1(b) is stable at zero, so the synchronization can reach.



(a)

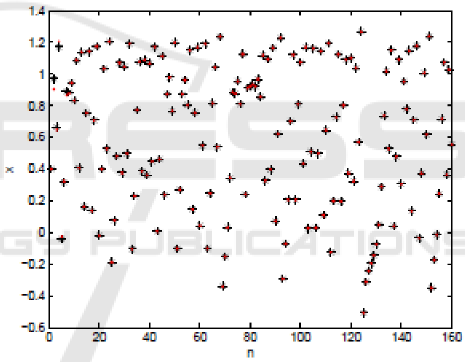


(b)

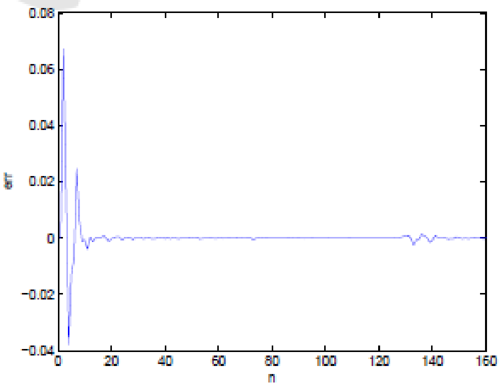
Figure 1 α=1, u(0)=0.4, K=-1

Case2: $\alpha = 0.3$.

The parameters are chosen as $\mu(0) = 0.4$ and K=-4, and the dynamical behaviors of master system, response system and error system are showed in figure 2(a), figure 2(b) respectively.



(a)

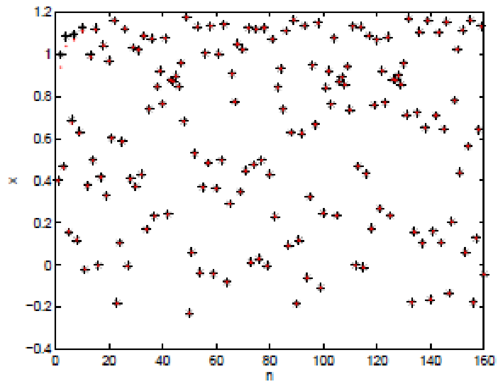


(b)

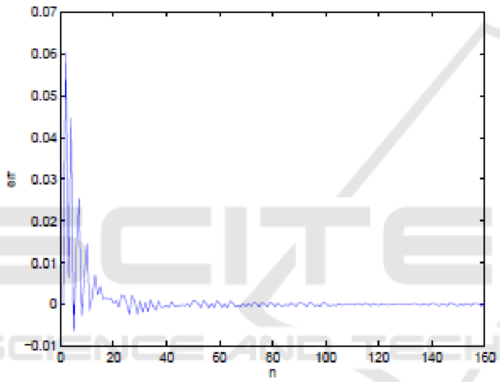
Figure 2 α=0.3, u(0)=0.4, K=-4

Case3: $\alpha = 0.1$.

In this case, we also choose $\mu(0) = 0.4$ and K=-4 the same as the parameters in case 2. Figure 3 shows the history of $x(n)$, $y(n)$ and $e(n)$.



(a)



(b)

Figure 3 α=0.1, u(0)=0.4, K=-4

4 CONCLUSION

In this paper, fractional odd logistic system is investigated, and parametric adaptive control algorithm is applied to synchronize two chaotic systems. We proved that the sufficient conditions of system synchronization is $-6\Gamma(\alpha+1)<K<0$. Moreover, numerical simulations are given and the results show that the algorithm can work efficiently for synchronization. Future works regarding this topic include varying parameters of the control system or applying the adaptive control algorithm to other systems. Also, the studies of this paper may have some referenced value for secure communication.

ACKNOWLEDGEMENTS

This work is financially supported by the Zhejiang Natural Science Foundation (Grant no. LQ12A01010), the special fund for outstanding young teachers in Shanghai universities research, and the foundation of Shanghai Normal University Tianhua College.

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