On inclusive 1-Distance Vertex Irregularity Strength of Firecracker, Broom, and Banana Tree

Ikhsanul Halikin, Ade Rizky Savitri, and Kristiana Wijaya
Graphs, Combinatorics, and Algebra Research Group
Department of Mathematics, Faculty of Mathematics and Natural Sciences
University of Jember (UNEJ), Jl. Kalimantan 37 Jember, Indonesia 68121

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Abstract: Let k be a natural number and G be a simple graph. An *inclusive d-distance vertex irregular labelling* of a graph G is a function $\lambda: V(G) \longrightarrow \{1,2,\ldots,k\}$ so that the weights at each vertex are different. Let v be a vertex of G. The weight of $v \in V(G)$, denoted by $wt(v)$, is the sum of the label of v and all vertex labels up to distance 1 from v. An *inclusive* 1-*distance vertex irregularity strength* of G, denoted by $\widehat{dis}(G)$ is the minimum k for the existence of an inclusive 1-distance vertex irregular labelling of a G. Here, we find the exact value of an inclusive 1-distance vertex irregularity strength of a firecracker, a broom, anda banana tree.

1 INTRODUCTION

Suppose that G is an undirected and finite graph without loop and parallel edges. For a vertex v in a graph G, the degree of v with notation $d(v)$ is the number of edges in G that are incident to v. For two vertices u and v in a graph G (not necessarily distinct), a $u - v$ walk in G is defined as a sequence of vertices and edges in G, starting with u and ending at v such that consecutive vertices are connected by an edge. A path defined as a $u - v$ walk with different vertices. The length of the shortest path from vertex u to vertex v is said to be a distance from u to v and denoted by $d(u,v)$ (see Chartrand, Lesniak & Zhang, (2011) for another terminology).

The labelling in graph is one of research topics introduced in the 1960s. The labelling of a graph is a function from a set of graph elements (vertices or edges or both) onto a set of numbers (usually natural numbers) with certain condition. There are many kinds of graph labelling that have been introduced (see Gallian (2016) for a complete survey). Chartrand et al. suggested the concept of an irregular labelling in 1988. The problem of this labelling is how to assign natural numbers label to the edges of a graph so that the sum of edge labels at each vertex is different. In this labelling also introduced a notion, called irregularity strength, i.e. the minimum largest label among all of the possible irregular assignments of a graph (Chartrand et al., 1988).

In 2007, Bačá et al. introduced the similar assignment but apply to both edges and vertices of a graph. This labelling is called the irregular total k-labelling. A total k-labelling is a mapping from the vertex set and edge set to the set of natural numbers $\{1,2,\ldots,k\}$. The minimum k for such labelling is said to be the total irregularity strength. Furthermore, Mirka, Rodger & Simanjuntak (2003) introduced another kind labelling, which is called distance magic labelling.

Motivated by Mirka and Bačá, Slamin (2017) introduced a distance vertex irregular labelling of graphs. A *distance vertex irregular labelling* of a graph G is a function $\lambda: V(G) \longrightarrow \{1,2,\ldots,k\}$ such that the weight of every vertex v in G is different. The weight of a vertex$v \in V(G)$, denoted by $wt(v)$,is the sum of the labels of all the vertices of distance1 from v. Moreover, Bong, Lin & Slamin (2017), generalized concept of a distance irregular vertex labelling to *inclusive* vertex irregular d-distance vertex labelling. Inclusive in this labelling means that the weight of the vertex v included the label of a vertex v. The minimum k for the existence of this labelling is said to bea distance irregularity strength of G and denoted by $\widehat{dis}_d(G)$. Furthermore, Bong, Lin & Slamin (2017) obtained $\widehat{dis}(G)$, for G are a path P_n for n= 3k, $k \in \mathbb{N}$, a star $K_{1,n}$, and a double

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star $S(m, n)$ with $m \leq n$. In the same paper, they gave the lower bound for caterpillar, cycle and wheel. In 2018, Bačá et al. determined the exact value of the inclusive distance vertex irregularity strength of a complete graph, complete bipartite graph, path, fan, and cycle.

In this paper, we discuss an inclusive 1-distance vertex irregular labelling and find the exact value of an inclusive 1-distance vertex irregularity strength of a firecracker, broom, and banana tree.

2 DEFINITION AND USEFUL PROPERTIES

Before we start the further discussion, we will present the definition and some useful properties of an inclusive 1-distance vertex irregular labelling.

Definition 1. Let *k* be a natural number. An *inclusive d-distance vertex irregular labelling* of a graph *G* is a function $\lambda: V(G) \longrightarrow \{1,2, \ldots, k\}$ so that the weights of two vertices *u* and *v* are different for each $u, v \in V(G)$. The weight of a vertex $v \in V(G)$, denoted by $wt(v)$, is defined as the sum of the label of *v* and all vertex labels up to distance *d* from *v*, namely

$$wt(v) = \lambda(v) + \sum_{1 \leq d(u,v) \leq d} \lambda(u),$$

where $d(u, v)$ is distance from vertex *u* to *v*.

The smallest *k* for the largest labelling this labelling is called an *inclusive d-distance irregularity strength* of *G* and denoted by $\widehat{dis}_d(G)$. Since in this paper we take $d = 1$, we denote it with $\widehat{dis}(G)$. Not all graphs *G* have an inclusive 1-distance irregularity strength of *G*, and we say that $\widehat{dis}(G) = \infty$.

Bong, Lin & Slamin (2017), gave the lower bound of the inclusive 1-distance irregularity strength of *G*, by the following lemma.

Lemma 1. For a connected graph *G* with *n* vertices, δ, Δ as minimum and maximum degree, respectively then $\widehat{dis}(G) \geq \left\lceil \frac{n+\delta}{\Delta+1} \right\rceil$.

Next, Bačá et al. (2018) proved the sufficient and necessary condition for $\widehat{dis}(G) = \infty$.

Lemma 2. For a connected graph G, $\widehat{dis}(G) = \infty$ if and only if there exist two different vertices $u, v \in V(G)$ such that $\{u\} \cup N(u) = \{v\} \cup N(v)$, where $N(u)$ is the set of all neighborhood of *u*(distance 1 from *u*).

As the firecracker, broom, and banana graphs are the kind of the tree graph, that clearly not satisfy the Lemma 2, so we can find the inclusive 1-distance vertex irregular labelling of them. The definition of firecracker, broom, and banana tree graphs are as follow:

Definition2. A firecracker graph $F_{n,m}$ is a graph formed by connecting one vertex of degree one from each of *n* copies of a star $K_{1,m}$.

Definition3. A broom $Br_{n,m}$ is a graph formed from identifying one end leaf of a path P_n with the center of a star $K_{1,m}$.

Definition4. A banana tree $B_{n,m}$is a graph obtained from connecting one vertex of degree one from each of *n* copies of a star $K_{1,m}$ with a new vertex.

In this paper, we determine an inclusive 1-distance vertex irregularity strength of a firecracker $F_{n,3}$, a broom $Br_{3,m}$, and a banana tree $B_{2,m}$.

3 MAIN RESULTS

In this section, we discuss an inclusive 1-distance irregularity strength of firecracker$F_{n,3}$, broom $Br_{3,m}$, and banana tree $B_{2,m}$.

Theorem 1. Let $F_{n,3}$ be a firecracker graph with $n \geq 3$. Then $\widehat{dis}(F_{n,3}) = n + 1$.

Proof. Suppose $V(F_{n,3}) = \{v_{ij} | 1 \leq i \leq 4, 1 \leq j \leq n\}$ where $d(v_{1j}) = 3$, $d(v_{2j}) = d(v_{3j}) = 1$, and $d(v_{41}) = d(v_{4n}) = 2$, and for $j \neq 1,2$, $d(v_{4j}) = 3$. As illustration, the vertex notation of $F_{n,3}$ can be seen in Figure 1.

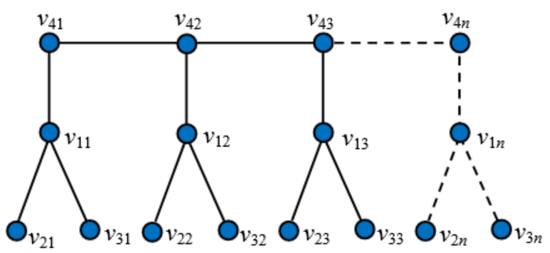

Figure 1: The notation of vertices of a firecracker $F_{n,3}$.

We know that a firecracker $F_{n,3}$ has $4n$ vertices, $\Delta(F_{n,3}) = 3$ and $\delta(F_{n,3}) = 1$. Based on Lemma 1, we get

$$\widehat{dis}(F_{n,3}) \geq \left\lceil \frac{4n+1}{3+1} \right\rceil = n + 1.$$

To show that $\widehat{dis}(F_{n,3}) \leq n + 1$, we define an inclusive irregular 1-distance vertex labelling λ of $F_{n,3}$ with label $1, 2, \ldots, n + 1$ as follow:

$$\lambda(v_{ij}) = \begin{cases} j + 1, & \text{for } i = 1; 1 \leq j \leq n, \\ 1, & \text{for } i = 2; 1 \leq j \leq n - 2, \\ 2, & \text{for } i = 2; \mathrm{n} - 1 \leq j \leq n, \\ n - 1, & \text{for } i = 3; j = 1, \\ n + 1, & \text{for } i = 3; 2 \leq j \leq n, \\ n + 1, & \text{for } i = 4; 1 \leq j \leq n. \end{cases}$$

So, the vertices weight of $F_{n,3}$ are

$$wt(v_{ij}) \begin{cases} 2n + 3, & \text{for } i = 1; j = 1, \\ 2n + 6, & \text{for } i = 1; j = 2, n \geq 4, \\ 2n + 4 + j & \text{for } i = 1; 3 \leq j \leq n - 2, n \geq 5, \\ 2n + j + 5, & \text{for } i = 1; n - 1 \leq j \leq n, \\ 3, & \text{for } i = 2; j = 1, \\ j + 2, & \text{for } i = 2; 2 \leq j \leq n - 2, n \geq 4, \\ j + 3, & \text{for } i = 2; n - 1 \leq j \leq n, \\ n + 1, & \text{for } i = 3; j = 1, \\ n + j + 2, & \text{for } i = 3; 2 \leq j \leq n, \\ 2n + 4, & \text{for } i = 4; j = 1, \\ 3n + j + 4, & \text{for } i = 4; 2 \leq j \leq n - 1, \\ 3n + 3, & \text{for } i = 4; j = n. \end{cases}$$

We obtain that all vertices of a graph $F_{n,3}$ have distinct weight. Hence, $\widehat{dis}(F_{n,3}) \leq n + 1$. Therefore, we can conclude that $\widehat{dis}(F_{n,3}) = n + 1$. ■

Theorem 2. Let $Br_{3,m}$ be a broom with $m \geq 2$, then $\widehat{dis}(Br_{3,m}) = m$.

Proof. Suppose that $V(Br_{3,m}) = \{u_i, v_j | 1 \leq i \leq 3, 1 \leq j \leq m\}$ is the vertex set of a broom $Br_{3,m}$, where the vertices u_1 and v_j are leaves of a broom $Br_{3,m}$ for each $j \in [1, m]$ and u_3 is the vertex of degree $m + 1$ (see Figure 2). Then, the broom $Br_{3,m}$ has $m + 1$ leaves. So, all leaves of a broom $Br_{3,m}$ must have distinct weight, where $wt(u_1) = \lambda(u_1) + \lambda(u_2)$ and $wt(v_j) = \lambda(u_3) + \lambda(v_j)$. Obviously that the smallest weight of a leaf of a broom $Br_{3,m}$ is at least 2 and minimum of the largest weight of a leaf of a broom $Br_{3,m}$ is at least $m + 2$. To obtain distinct weight of leaves v_j, the leaves v_j must have different label for each $j \in [1, m]$. Hence, minimum of the largest label of leaves from a broom $Br_{3,m}$ is at least m. It means that $\widehat{dis}(Br_{3,m}) \geq m$.

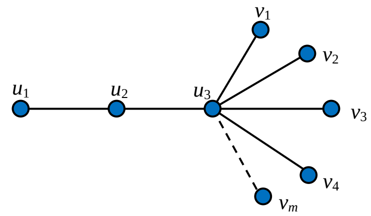

Figure 2: The notation of vertices of a broom $Br_{3,m}$.

Now, we show that $\widehat{dis}(Br_{3,m}) \leq m$. We define the inclusive irregular 1-distance vertex labelling λ as follow,

$$\lambda(v_j) = j, \text{for } 1 \leq j \leq m,$$
$$\lambda(u_i) = \begin{cases} m, & \text{for } i = 1, \\ 4 - i, & \text{for } 2 \leq i \leq 3. \end{cases}$$

So, the corresponding weights of each vertex of a broom $Br_{3,m}$ are

$$wt(v_j) = j + 1, \text{for } 1 \leq j \leq m,$$

$$wt(u_i) = \begin{cases} m + 1 + i, & \text{for } 1 \leq i \leq 2, \\ \frac{1}{2}(m^2+m+6), & \text{for } i = 3. \end{cases}$$

The differences of every vertex weight in a broom graph $Br_{3,m}$ can be verified easily. Since the largest label of a vertex of a broom $Br_{3,m}$ is at most m, $\widehat{dis}(Br_{3,m}) \leq m$. Therefore, we can conclude that $\widehat{dis}(Br_{3,m}) = m$. ■

Theorem 3. Let $B_{2,m}$ be a banana tree with $m \geq 3$, then

$$\widehat{dis}(B_{2,m}) = \begin{cases} 4, \text{for } m = 3, \\ m, \text{for } m \geq 4. \end{cases}$$

Proof. Let $V(B_{2,m}) = \{z, x_i, y_i | 0 \leq i \leq m\}$ be the vertex set of a banana tree $B_{2,m}$, where the only two vertices adjacent to z are x_1 and y_1, $d(x_0) = d(y_0) = m$, and the others are leaves. The notation of vertices of a banana tree $B_{2,m}$ as depicted in Figure 3. First, we will find the lower bound of the inclusive 1-distance irregularity strength for a banana tree $B_{2,m}$. To find this, we consider 2 cases.

Case1. For $m = 3$

Suppose the vertex set of a banana tree $B_{2,3}$ is $V(B_{2,3}) = \{z, x_i, y_i | i = 0,1,2,3\}$. A banana tree $B_{2,3}$

has 4 leaves, namely x_1, x_2, y_1, y_2. The smallest weight of a leaf of a banana tree $B_{2,3}$ is at least 2, and minimum of the largest weight of a leaf of a banana tree $B_{2,3}$ is at least 5. So, the label of each leaf is at least $\left\lceil \frac{5}{2} \right\rceil = 3$. Without loss of generality, it causes $\lambda(x_0) = 1$ and $\lambda(y_0) = 2$. However, minimum of the largest weight of all vertices of a banana tree $B_{2,3}$ is at least 10. If the largest vertex label of a banana tree $B_{2,3}$ is 3, then the vertex with weight 10 should be y_0. It cause $\lambda(y_1) = 3$ and the possibility of weight of y_1 is either 6, 7, or 8. On the other hand, the possibility of weight of x_0 is either 6 or 7. Two possibilities of weight of x_0 will cause two of vertices z, x_0, x_1, and y_1 have the same weight. Hence, the largest label of each vertex of a banana tree $B_{2,3}$ is at least 4. So, $\widehat{dis}(B_{2,3}) \geq 4$.

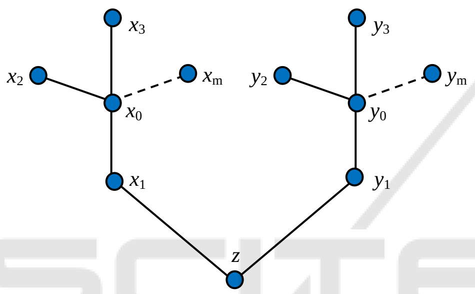

Figure 3: The notation of vertices of a banana tree $B_{2,m}$.

To show that $\widehat{dis}(B_{2,3}) \leq 4$, we can label of a banana tree $B_{2,3}$ as depicted in Figure 4.

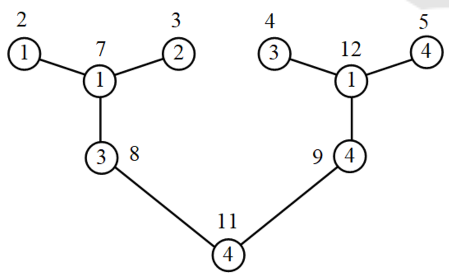

Figure 4: The labelling of banana tree $B_{2,3}$.

Figure 4 shows the inclusive irregular 1-distance vertex labelling, where the number outside the cycle shows the weight of the given vertex.

Case2. For $m \geq 4$

A banana tree $B_{2,m}$ has $(2m-2)$ leaves. The smallest weight of a leaf of a $B_{2,m}$ is at least 2 and minimum of the largest weight of a leaf of a $B_{2,m}$is at least $2m-1$. So, minimum of the largest leaf label of a banana tree $B_{2,m}$ is at least $\left\lceil \frac{2m-1}{2} \right\rceil = m$. Meanwhile, minimum of the largest weight for every vertex of a graph $B_{2,m}$ is at least $2m+4$. Therefore, minimum of the largest vertex label of a banana tree $B_{2,m}$ is at least $\min\left\{\left\lceil \frac{2m-1}{2} \right\rceil, \left\lceil \frac{2m+4}{2} \right\rceil\right\} = m$. So, $\widehat{dis}(B_{2,m}) \geq m$.

To show that $\widehat{dis}(B_{2,m}) \leq m$, let the inclusive irregular1-distance vertex labelling λ is defined in the following way:

$$\lambda(z) = m$$

$$\lambda(y_i) = \begin{cases} m-1, & \text{for } i = 0 \\ m, & \text{for } i = 1 \\ i, & \text{for } 2 \leq i \leq m \end{cases}$$

So, the corresponding weights of each vertex of a banana tree $B_{2,m}$ are as follows.

$$wt(z) = 3m$$

$$wt(x_i) = \begin{cases} \frac{1}{2}(m^2+m+2), & \text{for } i = 0 \\ 2m+1, & \text{for } i = 1 \\ i, & \text{for } 2 \leq i \leq m \end{cases}$$

$$wt(y_i) = \begin{cases} \frac{1}{2}(m^2+5m-4), & \text{for } i = 0 \\ 3m-1, & \text{for } i = 1 \\ m+i-1, & \text{for } 2 \leq i \leq m \end{cases}$$

The differences of every vertex weight can be verified easily, and the largest label is m. So, $\widehat{dis}(B_{2,m}) \leq m$. Therefore, we can conclude that $\widehat{dis}(B_{2,m}) = m$. ■

For example, the inclusive irregular 1-distance vertex labelling of a banana tree $B_{2,4}$ can be seen in Figure 5.

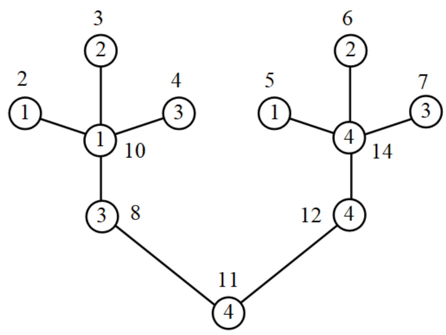

Figure 5: The labelling of banana tree $B_{2,4}$.

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REFERENCES

Arumugam, S.,& Kamatchi, N., 2012. On (a, d)-distance antimagic graphs. *Australasian Journal of Combinatorics* 54, pp. 279-287.

Bačá, M., Jendrol, S., Miller, M. & Ryan, J., 2007. On irregular total labelling. *Discrete Math*. 307, pp. 1378-1388.

Bačá, M., Fenovcıková, A. S., Slamin, & Sugeng, K. A., 2018. On inclusive distance vertex irregular labelling. *EJGTA*Vol 6 No. 1, pp 61-83.

Bong, N.H., Lin, Y., & Slamin, 2017.On inclusive and non-inclusive vertex irregular d-distance vertex labelling, submitted to *JCMCC*.

Chartrand, G., Jacobson, M. S., Lehel, J., Oellermann, O. R.,Ruiz, S. & Saba, F., 1988. Irregular networks. *Congr. Numer*. 64, pp. 187-192.

Chartrand, G., Lesniak, L. & Zhang, P., 2011. *Graphs and Digraphs*, Taylor & Francis Group.

Gallian, J. A., 2016. A dynamic survey of graph labelling. *Electron. J. Combin*, #DS6.

Miller, M., Rodger, C. & Simanjuntak, R., 2003. Distance magic labellings of graphs. *Australasian Journal of Combinatorics* 28, pp. 305-315.

Slamin, 2017. On distance irregular labellings of graphs. *Far East Journal of Mathematical Sciences (FJMS)* 102 (5), pp. 919-932.