On Distance Irregularity Strength of Lollipop, Centipede, and Tadpole Graphs

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Abstract: Let *G* be a simple graph. A distance irregular vertex *k*-labelling of a graph *G* is defined as a labelling $\lambda: V(G) \longrightarrow \{1,2,\ldots,k\}$ which is every two distinct vertices $x,y \in V(G)$ have different weights, $wt(x) \neq wt(y)$. The weight of a vertex *x* in *G*, denoted by $wt(x)$, is the sum of the labels of all the vertices adjacent to *x* (distance 1 from *x*), namely, $wt(x)= \sum_{y\in N(x)}\lambda(y)$, where $N(x)$ is the set of all the vertices adjacent to *x*. The minimum *k* for which the graph *G* has a distance irregular vertex *k*-labelling is called the distance irregularity strength of *G* and denoted by dis(*G*). In this paper, we determine the exact value of the distance irregularity strength of lollipop, tadpole, and centipede graphs.

1 INTRODUCTION

A graph labelling is a pairing of the vertices or edges to a label represented by integers (usually) satisfying a certain condition. Graph labelling was introduced in 1960s. There are about 2500 papers about graph labelling (Gallian, 2016).

The concept of distance irregular vertex labelling of graphs was introduced in (Slamin, 2017). A *distance irregular vertex k-labelling* of graphs *G* is an assignment of positive integers to vertex set, $\lambda: V(G) \to \{1,2,\cdots,k\}$ so that the weights calculated at vertices are distinct. The weight of a vertex $v \in V(G)$ under assignment λ is the sum of the labels of all vertices adjacent to v, that is

$$wt(v) = \sum_{u \in N_G(v)} \lambda(u)$$

where $N_G(v)$ is a set of all neighbors of vertex v. A *distance irregularity strength* of *G* is the minimum k for which the graph *G* having a distance irregular vertex k-labeling, denoted by $\mathrm{dis}(G)$.

Not all graphs can be labelled with a distance irregular vertex k-labeling. (Slamin, 2017) gave the following observation.

Observation 1. Let *G* be a connected graph. Suppose $x, y \in V(G)$. If $N_G(x) = N_G(y)$, then *G* has no distance irregular vertex *k*-labelling.

Slamin (2017) determined the distance irregularity strength of complete, path, cycle, and wheel graph as follows:

$$\mathrm{dis}(K_n) = n \text{ for } n \geq 3,$$

$$\mathrm{dis}(P_n) = \left\lceil \frac{n}{2} \right\rceil \text{ for } n \geq 4,$$

$$\mathrm{dis}(C_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{ for } n \equiv 0,1,2,5 \bmod (8),$$

and

$$\mathrm{dis}(W_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{for } n \equiv 1, 2, 5 \bmod (8).$$

Next, (Novindasari et al., 2016) determined the distance irregularity strength of ladder graph and triangular ladder graph, that is $\mathrm{dis}(L_n) = n + 2$ for $n \geq 3$ and $\mathrm{dis}(\mathbb{L}_n) = n$ for $n \geq 3$.

A lower bound of a distance irregularity strength can be seen as follows:

Lemma 1. Let G be a connected graph on n vertices with minimum degree δ and maximum degree Δ. If there is no vertex having identical neighbors, then

$$\mathrm{dis}(G) \geq \left\lceil \frac{n+\delta-1}{\Delta} \right\rceil. \text{ (Slamin, 2017)}$$

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Definition 1.A lollipop graph $L_{n,m}$ is a graph obtained by joining one vertex of a complete K_n to a vertex of degree one of a path P_m.

So, the lollipop graph $L_{n,m}$ has $m+n$ vertices.

Definition 2.A centipede graph c_n is a graph obtained by taking of path P_n and n copies $K_1$and then joining the ith vertex of P_n with an edge to every vertex in the ith copy of K1.

Definition 3.A tadpole graph $T_{n,m}$ is obtained by joining one vertex of a cycle C_n to a vertex of degree one of a path P_m.

In this paper, we discuss about a distance irregularity strength of a lollipop $L_{n,m}$, centipede c_n and tadpole $T_{n,m}$, for each natural number n and especially m = 1.

2 MAIN RESULT

In this section, we determine the exact value of a distance irregularity strength of a lollipop $L_{n,1}$.

Theorem 1. Let $n \geq 3$ be a natural number and $L_{n,1}$ be a lollipop graph. Then $\text{dis}(L_{n,1}) = n-1$.

Proof. Suppose the vertex set of a lollipop $L_{n,1}$is $V(L_{n,1}) = \{x_i | 1 \leq i \leq n+1\}$, where $d(x_1) = 1$, $d(x_2) = n$, and $d(x_i) = n-1$, for each $i \in [3, n+1]$. First, we prove that $dis(L_{n,1}) \geq n-1$. Suppose $x_i, x_j \in V(L_{n,1})$, for each $i, j \in [3, n+1]$ and $i \neq j$. Then, $N(x_i) - x_j = N(x_j) - x_i$. By Observation 1, $\lambda(x_i) \neq \lambda(x_j)$for each $i, j \in [3, n+1]$and $i \neq j$. Hence, the labels of all vertices $x_3, x_4, \ldots, x_{n+1}$ must be different. So, $\text{dis}(L_{n,1}) \geq n-1$.

Next, we show that $\text{dis}(L_{n,1}) \leq n-1$. We define a distance irregular vertex $(n-1)$-labeling λ of a lollipop $L_{n,1}$as follows.

$$\lambda(x_i) = \begin{cases} 1 & ,\text{for } i = 1,2,4 \\ 2 & ,\text{for } i = 3 \\ i-2 & ,\text{for } 5 \leq i \leq n+1 \end{cases}$$

Under the labelling λ, we obtain the vertex weights of a lollipop $L_{n,1}$ as follows.

$$wt(x_i) = \begin{cases} 1 & ,\text{for } i = 1 \\ (n^2-n-6)+4 & ,\text{for } i = 2 \\ ½(n^2-n)-1 & ,\text{for } i = 3 \\ ½(n^2-n) & ,\text{for } i = 4 \\ ½(n^2-n)+3-i & ,\text{for } 5 \leq i \leq n+1 \end{cases}$$

The labelling λ provide different weights for each vertex and the largest label is $n-1$ which leads to $\text{dis}(L_{n,1}) \leq n-1$. We conclude that$\text{dis}(L_{n,1}) = n-1$ for $n \geq 3$. ■

Figure 1 illustrates a distance irregular vertex labelling of the lollipop graph $L_{5,1}$ with distance irregularity strength 4. The number in the circle is vertex weight and number outside the circle is vertex label.

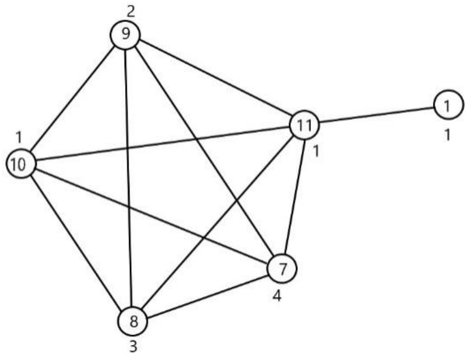

Figure 1. A distance irregular vertex labelling of $L_{5,1}$ with $\text{dis}(L_{5,1}) = 4$

Now, we discuss a distance irregularity strength of a centipede graph c_n by the following theorem.

Theorem 2. Let c_n be a centipede graph with $n \geq 3$.Then$\text{dis}(c_n) = n$.

Proof. Suppose $n \geq 3V(c_n) = \{x_i, y_i | 1 \leq i \leq n\}$, where y_i is a leaf for $i \in [1, n]$, $d(x_1) = d(x_n) = 2$, and $d(x_i) = 3$ for $i \in [2, n-1]$ and $E(c_n) = \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_i y_i | 1 \leq i \leq n\}$.

A vertex notation of centipede c_n can be seen in Figure 2.

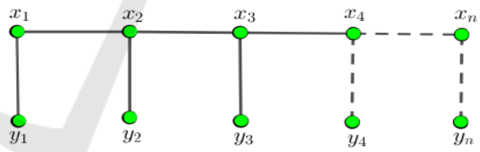

Figure 2. The notation of vertices in centipede c_n.

First, we show that $\text{dis}(c_n) \geq n$. Since all leaves y_i must have distinct weight, label of vertices x_i must be different for each $i \in [1, n]$. So, $dis(c_n) \geq n$.

Now, we show the upper bound of distance irregularity strength of centipede. We consider 2 cases.

Case 1. For n = 3.

A distance irregular vertex labeling for centipede c_3 and the weights of its vertices can be depicted in Figure 3.

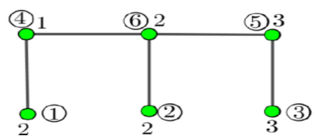

Figure 3. A distance irregular vertex labelling of c_3.

Case 2. For $n \geq 4$

Define a distance irregular vertex labeling λ of centipede c_n for $n \geq 4$ as follows.

$$\lambda(y_i) = i, \text{for } 1 \leq i \leq n$$

$$\lambda(x_i) = \begin{cases} n-1, & \text{for } i = 1 \\ n-2, & \text{for } i = 2,3 \\ 4, & \text{for } i = n \\ n+1-i, & \text{for } 4 \leq i \leq n-1 \end{cases} \quad, n \geq 5$$

Under a labelling λ, we get the weights of the vertices of c_n as follows.

$$wt(y_i) = i, \text{for } 1 \leq i \leq n$$

$$wt(x_i) = \begin{cases} n+1, & \text{for } i = 1 \\ n+2, & \text{for } i = 2 \\ n+4, & \text{for } i = 3 \\ n+1+i, & \text{for } 4 \leq i \leq n-1 \\ n+3, & \text{for } i = n \end{cases}$$

It is clear that every vertex of c_n has different weight. This shows that $\text{dis}(c_n) \leq n$. Therefore, $\text{dis}(c_n) = n$. ■

Slamin (2017) was proved that $\text{dis}(C\_n) = \lceil (n+1)/2 \rceil$ for $n \equiv 0,1,2,5 \bmod (8)$. A tadpole $T_{n,1}$ is a graph formed from cycle C_n by connecting one vertex to a leaf y. So, we can prove that a tadpole $T_{n,1}$ has a distance irregular vertex k-labelling based on a distance irregular vertex k-labelling cycle C_n. So, we have the following corollary.

Corollary 1. Let $T_{n,1}$ be a tadpole graph with $n \geq 5, n \equiv 1 \bmod (4)$. Then $\text{dis}(T_{n,1}) = \frac{n+1}{2}$.

Proof. (Slamin, 2017) proved that $\text{dis}(C_n) = \lceil (n+1)/2 \rceil$, for $n \equiv 1 \bmod (4)$ with the vertex label $\lambda: V(C_n) \to \{1,2,\cdots,\text{dis}(C_n)\}$:

$$\lambda(x_i) = \begin{cases} \frac{n+1}{2} - 2\left\lfloor \frac{i}{4} \right\rfloor, & \text{for } i = 1,3,\cdots,n-2 \\ \left\lceil \frac{i}{2} \right\rceil, & \text{for } i = 2,4,\cdots,n-1 \\ 1, & \text{for } i = n \end{cases}$$

By connecting a leaf y to the vertex having the greatest weight x_2 in a cycle C_n, and giving label of a leaf is 1, we get

$$\text{dis}(T_{n,1}) = \frac{n+1}{2}.$$

We illustrate distance irregularity vertex labelling of the tadpole graph in Figure 4.

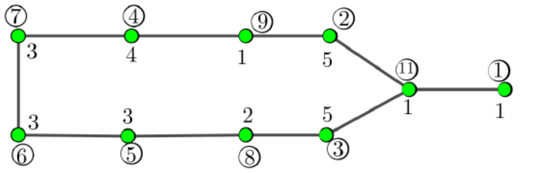

Figure 4. A distance irregular vertex labelling of $T_{9,1}$ with $dis(T_{9,1})$ =5.

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