Developing the Developable Surfaces in a Space to the Plane using Some Triangle Pieces

Kusno and Nur Hardiani
Jember University, Jl. Kalimantan No. 37, Jember, Indonesia
Department of Tadris Matematika, State Islam University, Mataram, Indonesia

Keywords: Developing, Developable Surfaces, Space, Plane, Triangle.

Abstract: This paper deals with the development in the space to the plane of the polygons, the cone, the cylindrical surfaces and the developable quartic Bézier patches in which its boundary curves are respectively parallel, and the normal vectors of the surfaces must be in the same orientation. The method is as follows, we approximate the surfaces into some triangle pieces then we transform consecutively these pieces in the plane. The result of the study shows that the use of the triangle approximation method can develop effectively these surfaces in the space to the plane. In addition, it can be applied to detect all surface measures of an object that are defined by those surface types.

1 INTRODUCTION

Some methods related to the development of surface to the plane have been presented. The development of the pipeline surfaces can be carried out by enumerating of two boundary curves in some approximation polygons (Weiss and Furtner, 1998). We can develop a surface to the plane by using the techniques of interactive piecewise flattening of parametric 3-D surfaces, leading to a non-distorted (Bennis and Gagalowicz, 1991). After that, developing an arbitrary developable surface into a flattened pattern is based on the geodesic curve length preservation and linear mapping principles (Clements, 1991; Gan et al., 1996). We can simulate the physical model of transitional pipeline parts whose cross sections are plane curve and polygon and are made of unwrinkled or unstretched materials. It is based on the approximation of the boundary surface triangulation (Obradović et al., 2014). Different from the previous methods, we are interested in the discussion about the development of the convex polygons, the conic/cylindrical surfaces and the developable quartic Bezier patches in a space to the plane using the triangle pieces.

This paper is organized in the following steps. In the first, we talk about the development of the triangle and polygon plane surfaces in space to the plane. In the second, we evaluate the development of the cone and the cylinder defined by the linear interpolation of two parallel circles. In the third, the construction and the development of developable quartic Bezier patches in a space to the plane are introduced. Finally, the results will be summarized in the conclusion section.

2 DEVELOPING THE TRIANGLE AND THE POLYGON PLANE SURFACE IN A SPACE TO THE PLANE

Let a triangle plane ΔABC. The vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form an angle $x°$ in the space orthonormal coordinate $[\boldsymbol{O}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}]$. The problem is how to develop the plane ΔABC in the plane orthonormal coordinate $[\boldsymbol{O}, \boldsymbol{i}, \boldsymbol{j}]$ in which the vector $\overrightarrow{A'B'}$ of the side development $\overline{AB}$ of triangle ΔABC is align to the determined unit vector a_1 (Figure 1a).

To develop the triangle ΔABC in space to the plane $[\boldsymbol{O}, \boldsymbol{i}, \boldsymbol{j}]$ can be undertaken as follows (Gan et al., 1996).

1). Determine the vector $\overrightarrow{A'B'} = \left\|\overrightarrow{AB}\right\| \boldsymbol{a}_1$ and calculate the unit vector $\boldsymbol{a}_2 \perp \boldsymbol{a}_1$.
2). Evaluate the measure of angle

Kusno, . and Hardiani, N.
Developing the Developable Surfaces in a Space to the Plane using Some Triangle Pieces.
DOI: 10.5220/0008523104270431
In *Proceedings of the International Conference on Mathematics and Islam (ICMIs 2018)*, pages 427-431
ISBN: 978-989-758-407-7

$$x = \arccos\left(\overrightarrow{AB}\,.\frac{\overrightarrow{AC}}{|\overrightarrow{AB}|}.\,|\overrightarrow{AC}|\right).$$

3). Calculate

$$\begin{aligned}\overrightarrow{OC'} &= \overrightarrow{OA'} + \overrightarrow{A'C'}\\ &= < x_{A'}, y_{A'} > + (|\overrightarrow{AC}|\cos x)a_1 + (|\overrightarrow{AC}|\cdot\sin x)a_2.\end{aligned}$$

4). Construct the developed triangle $\Delta A'B'C'$ in the plane orthonormal coordinate $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$ by using the linear interpolation of the couple points A', B'andC' that are

$$\overrightarrow{A'B'} = (1-u)\overrightarrow{OA'} + u\,\overrightarrow{OB'}; \quad (1)$$
$$\overrightarrow{B'C'} = (1-u)\overrightarrow{OB'} + u\,\overrightarrow{OC'};$$
$$C'A' = (1-u)OC' + u\,OA';$$

with $0 \le u \le 1$.

If the positions A, B and C are $A(-1,0,1)$, $B(2,1,4)$ and $C(0,6,3)$, then we will find the development of the triangle as it is shown in Figure 1b. If the positions A, B and C are $A(-1,0,1)$, $B(4,1,2)$ and $C(0,6,3)$, its development is shown in Figure 1(c).

Consider a piece of convex polygon plane □ of the vertices □ $= [P_1, P_2, P_3, \dots, P_n]$ in space that are shown in Figure 2a. The development of the polygon □ to the plane orthonormal coordinate$[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$ can be carried out as follows.

1). Determine an initial point of the development Q_1 and two orthonormal unit vectors $a_1 \perp a_2$.

2). Determine a point Q_2 that is an image of the point P_2 such that$\overrightarrow{Q_1Q_2} = |\overrightarrow{P_1P_2}|\mathbf{a}_1$.

3). Calculate the length $d_i = |\overrightarrow{P_1P_{i+1}}|$ for $i = 2, 3, \dots, n$ and evaluate the measure of the angle

$$\theta_i = u\angle P_2P_1P_{i+1} = \arccos\left[\frac{\overrightarrow{P_1P_2}.\overrightarrow{P_1P_{i+2}}}{|\overrightarrow{P_1P_2}|.\,|\overrightarrow{P_1P_{i+2}}|}\right]$$

for $i = 1, 2, 3, \dots, (n-2)$.

4). Evaluate the points Q_i for $i = 3, 4, \dots, n$ in the plane$[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$ as the images $P_2, P_3, P_4, \dots, P_n$by using the formula

$$\overrightarrow{Q_1Q_i} = \overrightarrow{OQ_1} + d_i\,[\cos\theta_{i-2}\mathbf{a_1} + \sin\theta_{i-2}\mathbf{a_2}].$$

5). Using equation (1), construct the polygon of development $Q_1Q_2Q_3 \dots Q_n$ in the plane $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$.

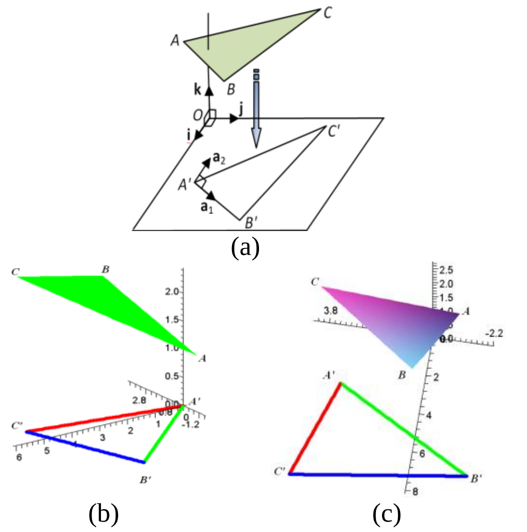


Figure 1: (a) Calculation of the angle $\angle CAB$ in the plane $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$, (b) Development of triangle plane ABC in space to the plane $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$ by starting point at $A'(0,0,0)$, and (c) by starting point at $A'(3,3,0)$.

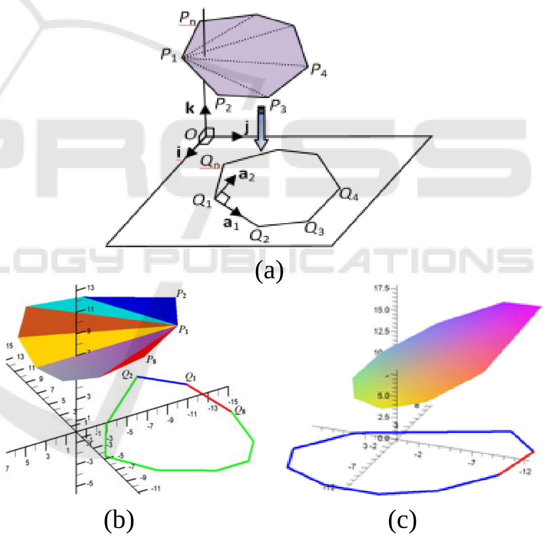
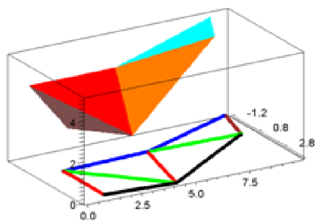


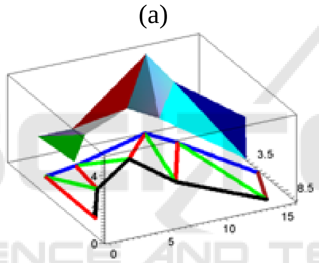
Figure 2: (a) Calculation of the angles $\angle P_iP_1P_{i+1}$ for $i = 2, \dots, n-1$ of the convex polygon $P_1P_2 \dots P_n$ in the plane $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$, (b) Decomposition of the polygon $P_1P_2 \dots P_8$ into some triangles, (c) Development of the convex polygon $P_1P_2 \dots P_8$ to the plane $[\boldsymbol{O},\boldsymbol{i},\boldsymbol{j}]$.

In case of the polygon vertices P_1(0,-0,7), P_2(4,-12,7), P_3(8,-5,7), P_4(9,0,7), P_5(5,3,7), P_6(0,4,7), P_7(-3,3,7), P_8(-4,0,7) and P_9(-3,-5,7) that are lied in the plane z =7, the development of the polygon in plane can be shown in Figure 2b. If the polygon vertices P_1(0,-10,18), P_2(4,-12,16), P_3(8,-5, 5), P_4(7,0,1), P_5(4,3,1), P_6(0,4,4), P_7(-3,3,8), P_8(-4,0,12) and P_9(-3,-5,16) are determined in the plane$x + y + z$ - 8= 0, then the result of development is shown in Figure 2c.

In the more general cases, if we are given a series of n consecutive triangles plane in space $[P_1P_2P_3, P_2P_3P_4, \ldots, P_nP_{(n+1)}P_{(n+2)}]$ that are defined by n+2 points $P_1, P_2, P_3, \ldots, P_{(n+2)}$, then the development of the triangles to the plane $[\boldsymbol{O}, \boldsymbol{i}, \boldsymbol{j}]$ can be realized respectively by equation (1). To justify the method, when we give the points data of triangle pieces in space $P_1(3,0,3)$, $P_2(0,0,4)$, $P_3(3,3,2)$, $P_4(0,4,4)$, $P_5(1,8,5)$ and $P_6(1,9,5)$, we will find its development in the plane that are shown in Figure 3a. If the triangle pieces are defined by the points $P_1(6,0,3)$, $P_2(0,1,2)$, $P_3(3,3,3)$, $P_4(0,4,2)$, $P_5(1,8,3)$, $P_6(1,11,5)$, $P_7(5,8,4)$, $P_8(1,12,4)$, $P_9(10,12,3)$ and $P_{10}(7,14,4)$, then its development in the plane are shown in Figure 3b.



(a)



(b)

Figure 3: (a) Development of four triangles plane series toplane, (b) Development of eight triangles plane series to plane.

3 DEVELOPING THE CONE AND THE CYLINDER OF TWO CIRCLES LINEAR INTERPOLATION

Consider a cone or a cylinder surfaces $\mathbf{S}(u,v) = (1-v)\,\mathbf{C}_1(u) + v\mathbf{C}_2(u)$ in which its boundary curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are the parallel circles

$$\mathbf{C}_1(u) = < r_1\cos u + a,\ r_1 \sin u + b,\ z_{C1}> \qquad (2)$$

and

$$\mathbf{C}_2(u) = < r_2\cos u + c,\ r_2 \sin u + d,\ z_{C2}>$$

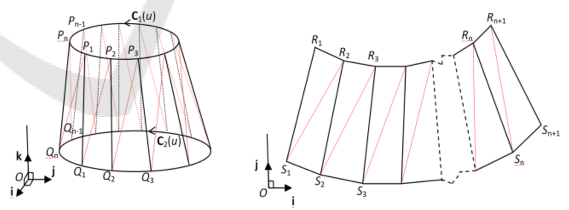
with radius r_1, r_2 and the value a, b, z_{C1}, c, d, z_{C2} real constants, $0 \le u \le 2\pi$ and $0 \le v \le 1$. Developing the surface $\mathbf{S}(u,v)$ to the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$ can be carried out by using the triangles approximation as follows (Figure 4a,b).

1) Determine the $(n+1)$ parameter values $u_i = \frac{(i-1)}{n}(2\pi)$ of $i = 1, 2, 3, \ldots, (n+1)$ to define the $2n$ triangle plane pieces $[P_1Q_1P_2, P_2Q_1Q_2, P_2Q_2P_3, \ldots, P_nQ_nP_{n+1}, P_{n+1}Q_nQ_{n+1}]$. The point P_i is defined by $\mathbf{C}_1(u_i)$ and the point Q_i is defined by $\mathbf{C}_2(u_i)$ for $i = 1, 2, 3, \ldots, n+1$ with $P_{n+1} = P_1$ and $Q_{i+1} = Q_1$.

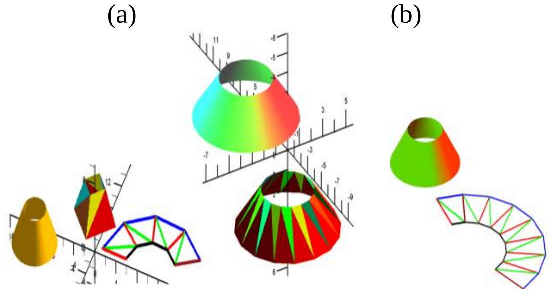
2) Calculate the length $\left|\overrightarrow{Q_iP_i}\right|$, $\left|\overrightarrow{Q_iP_{i+1}}\right|$, $\left|\overrightarrow{Q_iQ_{i+1}}\right|$ and the measure of consecutive angles $\angle P_iQ_iP_{i+1}$ and $\angle P_{i+1}Q_iQ_{i+1}$ for $i = 1, 2, 3, \ldots, n$.

3) In the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$, determine an initial point of development S_1 and two orthonormal vectors $\mathbf{a}_1 \perp \mathbf{a}_2$. Using the triangle development method in section 2 and the determined initial point S_1 can be developed respectively and consecutively the triangles $[P_1Q_1P_2, P_2Q_1Q_2, P_2Q_2P_3, \ldots, P_nQ_nP_{n+1}, P_{n+1}Q_nQ_{n+1}]$ in space to the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$ that are $[R_1S_1R_2, R_2S_1S_2, R_2S_2R_3, \ldots, R_nS_nR_{n+1}, R_{(n+1)}S_nS_{(n+1)}]$.

If $\mathbf{C}_1(u) = <2\cos u + 1, 2 \sin u - 3, 4>$ and $\mathbf{C}_2(u) = <4\cos u + 1, 4 \sin u - 3, 2>$, then the development of the conic surface $\mathbf{S}(u,v)$ into 8 triangles to the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$ is shown in Figure 4c. In the Figure 4d, we present a cone approximated by 32 triangles. The Figure 4e show the development of the conic surface into 16 triangles to the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$.



(a) (b)



(c)

Figure 4: (a) Decomposition a cone into some triangles, (b) Development of the cone to the plane $[\mathbf{O},\mathbf{i},\mathbf{j}]$, (c) Development examples of the pyramid and the cone to plane.

4 DEVELOPING THE DEVELOPABLE QUARTIC BEZIER PATCHES IN A SPACE TO THE PLANE

The developable surfaces are local properties of the surface. It is not only in the form of the plane surface but also in the form of the conic, the cylindrical and the tangent lines surfaces (Lipschultz, 1969; Kusno, 1998). This section discusses about the definition of developable quartic Bézier patches supported by two parallel plane and then talk about the developing of the patches in the plane using the method of the triangle approximation. We analyze as follows.

Let the quartic Bézier curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ in the form

$$\mathbf{C}_1(u) = \sum_{i=0}^{4} \mathbf{P}_i\, B_i^4(u), \qquad (3)$$

and

$$\mathbf{C}_2(u) = \sum_{i=0}^{4} \mathbf{q}_i\, B_i^4(u),$$

with $B_i^4 = \frac{4!}{i!(4-i)!}(1-u)^{4-i}.\ u^i$ and $0 \leq u \leq 1$.

Because of the application reason, the curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are lied respectively in two parallel planes $[\Psi_1, \Psi_2]$. So, by using the developable condition of regular developable surfaces, it must be formulated in the form (Frey and Bindschadler, 1993; Kusno, 1998)

$$\mathbf{C}'_2(u) = \rho(u)\, \mathbf{C}'_1(u), \qquad (4)$$

with the real scalar $\rho(u) > 0$. In order to simplify the calculation, we choose the scalar $\rho(u)$ positive constant i.e. $\rho(u) = \alpha \in R^+$. From the condition (4), we will find

$$\sum_{i=0}^{3}[(\mathbf{q}_i - \mathbf{q}_{i+1}) + \alpha(\mathbf{p}_{i+1} - \mathbf{p}_i)]B_i^3(u) = \mathbf{0}. \qquad (5)$$

The polynomials $B_i^3(u)$ are not zero for $i = 0, 1, 2, 3$ thus

$$[(\mathbf{q}_i - \mathbf{q}_{i+1}) + \alpha.(\mathbf{p}_{i+1} - \mathbf{p}_i)] = \mathbf{0}, \qquad (6)$$

for all $i = 0, 1, 2, 3$. When we add those equations, we will find an equation of the Bézier polygon control points of the curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ as follows

$$[(\mathbf{q}_4 - \mathbf{q}_0) = \alpha.(\mathbf{p}_4 - \mathbf{p}_0)]. \qquad (7)$$

So, to construct a regular developable Bézier patch which is supported by two curves $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ of degree 4 and conditioned by $\rho(u)$ positive constant must be verify the equation (4) and (5), namely

1. the two vectors parallel $(\mathbf{q}_4 - \mathbf{q}_0)$ and $(\mathbf{p}_4 - \mathbf{p}_0)$ must be in the same direction to calculate α value such that

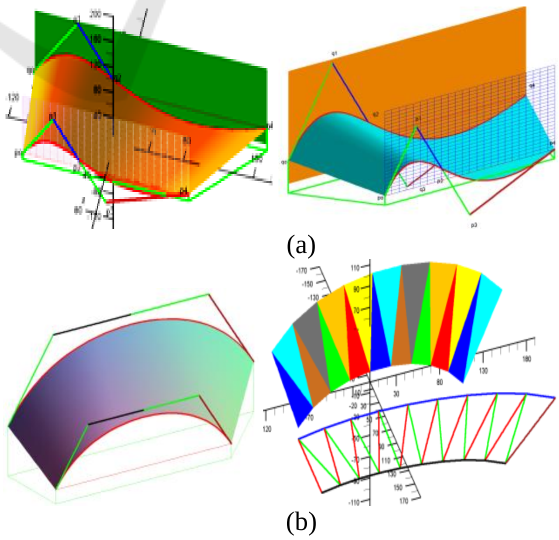
$$\alpha = \frac{\|\mathbf{q}_4 - \mathbf{q}_0\|}{\|\mathbf{p}_4 - \mathbf{p}_0\|}; \qquad (8)$$

2. every $i = 0, 1, 2, 3$ the vector $(\mathbf{q}_{i+1} - \mathbf{q}_i)$ and $(\mathbf{p}_{i+1} - \mathbf{p}_i)$ must be parallel and proportional to α.

Furthermore, to facilitate the continuous connection of two adjacent patches, in equation (4), must be necessary to determine four boundary control points $[\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_4, \mathbf{q}_4]$ of the Bézier curve $\mathbf{C}_1(u)$, $\mathbf{C}_2(u)$ and two control points $[\mathbf{p}_1, \mathbf{p}_3]$ of the Bézier curve $\mathbf{C}_1(u)$. Therefore, if we determine the control points $[\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4, \mathbf{q}_0, \mathbf{q}_4]$, then from the equations (4) we will find respectively four equations to calculate the control points $[\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2, \mathbf{p}_2]$ in the form

$$\mathbf{q}_1 = \mathbf{q}_0 + \alpha.(\mathbf{p}_1 - \mathbf{p}_0); \quad \mathbf{q}_3 = \mathbf{q}_4 - \alpha.(\mathbf{p}_4 - \mathbf{p}_3); \qquad (9)$$
$$\mathbf{q}_2 = \tfrac{1}{2}(\mathbf{q}_1 + \mathbf{q}_3); \qquad \mathbf{p}_2 = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_3).$$

Thus, using the data control points $[\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_4, \mathbf{q}_0, \mathbf{q}_4]$, equation system (6) and equation (8) can determine the control points $[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_2]$. All these control points can define the developable quartic Bézier patch of equation (4). In addition, if we fix in equation (8) the value $\alpha = 1$ and $\alpha > 1$, then we will find respectively the developable cylindrical patches and the developable conic patches.



(a) (b)

Figure 5: (a) Two examples of the developable quartic Bézier patches, (b) development of the developable quartic Bézier patch in a space to plane by using some triangles.

To justify this method and develop its result surface in a space to the plane, we simulate as follows. If we substitute in equation system (6) the control points data $\mathbf{p}_0,=<35,-95,0>$, $\mathbf{p}_1 =<35,-60,65>$, $\mathbf{p}_3 =<35,0,-45>$,$\mathbf{p}_4 = <35,95,0>$,$\mathbf{q}_0 =<-40,-100,25>$ and$\mathbf{q}_4 =<-40,116,25>$, then the solution will find the control points $\mathbf{q}_1 =<-40,-51,116>$, $\mathbf{q}_2 = <-40,-9,39>$,$\mathbf{q}_3 =<-40,33,-38>$,$\mathbf{p}_2=<35,-30,10>$ that are define the developable quartic Bézier patch in Figure 5a,b. Furthermore, the Figure 5c,d represent the development of the developable quartic Bézier patch to the plane using the triangles approximation method.

5 CONCLUSIONS

By using the triangles approximation method, we presented the development in the plane of some developable surfaces in which its boundary curves are respectively parallel, and the normal vectors of the surface must be in the same orientation. It is very useful to develop the polygon plane, the conic, the cylindrical surfaces or the developable quartic Bézier patches. Therefore, it can be applied to detect all surface measures of an object that are defined by those surface types.

The development of the polygon plane, the conic, the cylindrical surfaces or the developable quartic Bézier patches have been introduced. The interesting thing to discuss ahead is how to define and develop in planethe developable Bézier patches of high degrees.

REFERENCES

Bennis, C., Gagalowicz, A., 1991. Piecewise Surface Ftattening for Non-Distorted Texture Mapping. *Computer Graphics*, vol. 25, pp. 237-246.

Clements, J. A., 1981. Computer System to Derive Developable Hull Surface and Tables of Offsets. *Marine Technology*, vol. 8, pp. 221-233.

Frey W. H., Bindschadler, D., 1993. *Computer-Aided Design of a Class of Developable Bézier Surfaces* (USA: R and D Publication 8057, General Motors).

Gan, M. C., Tan S. T. &Chan, K. W., 1996. Flattening Developable Bi-parametric Surfaces. *Computers & Structures*, vol. 58, pp.703-708.

Kusno, 1998. *Contribution a la Solution du Problèm de Construction et de Raccordement Géométrique de Surfaces Développables Réguliers a l'Aides de Carreaux de Bézier*, Thèse de Docteur-Université de Metz, France.

Kusno, 2010. *Geometri Rancang Bangun*. Jember University Press.

Lipschultz, M.1969. *Theory and Problems of Differential Geometry*. Schaum's Outline Series McGraw-Hill New York.

Obradović, R., Beljin B. & Popkonstantinović B., 2014. Approximation of Transitional Developable Surfaces between Plane Curve and Polygon. *Acta Polytechnica Hungarica*, vol. 11, pp. 217-238.

Weiss G., Furtner, P., 1988. Computer-Aided Treatment of Developable Surfaces. *Computer & Graphics*, vol. 12, pp. 39-51.