On Some Probability Results Characterizing the Distribution in Micro-economic Structures using the Formula of Faà Di Bruno

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Abstract: In this article we will treat the distribution of individuals in micro-economic structures in two different cases. In the first one, no constraints on the types are imposed on individuals or structures, in the second one, three types of constraints are imposed. This work is finished by an application that combines between two techniques, one purely theoretical based on the formula of Faà Di Bruno and the other is very practical having an economic aspect, in order to help and to facilitate the management of the distributions of the individuals in micro-economic structures in the discrete frame.

1 INTRODUCTION

When we have m micro-economic (Avinash, 2014) structures and we want to engage n individuals in these projects, we have two cases to consider:

Assignment without constraints: In this case we will deal with two types of assignment:

- Assign the individuals to these projects without placing constraints on their skills or the type of micro-economic structures.
- All the individuals have the same skills and all the projects are of the same type.

Assignment with constraints: In this case we will have three types of assignment:

- Random distribution of the individuals.
- Distribution of the individuals in an ascending order.
- Distribution of the individuals with conditioning.

2 COMBINATORIAL RESULTS LINKING MICRO-ECONOMIC STRUCTURES AND INDIVIDUALS

2.1 Possible Surjections

This part is dedicated to finding the number of possibilities to assign the individuals to the m projects in the first kind of assignment, i.e. the one without constraints on the type. The two assignments above are identical, since the assignment in both cases is unconstrained.

Each possibility represents a surjection from the set of all individuals to the set of all micro-economic structures. By nature the number of individuals is greater than the number of projects, so we work under the assumption $m \leq n$.

We define:

$P_n = \{e_1, e_2, ..., e_n\}$ the set of individuals.

$A_m = \{a_1, a_2, ..., a_m\}$ the set of micro-economic structures.

$S_{n,m}$ the number of surjections from P_n to A_m.

Calculating the number of possible assignments of n individuals in m micro-economic structures is

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equivalent to calculating $S_{n,m}$ the number of surjections from P_n to A_m.

Proposition. The number of surjections $S_{n,m}$ from P_n to A_m verifies the following equalities:

$$S_{n,m} = m^n - \sum_{k=1}^{m-1} C_m^k S_{n,k} \quad (1)$$

$$S_{n,m} = m(S_{n-1,m} + S_{n-1,m-1}) \quad (2)$$

$$S_{n,m} = \sum_{k=1}^{m} C_m^k (-1)^{m-k} k^n \quad (3)$$

Proof. See (Robert, 2003), (Rowan, 2009) and (Martin, 2007).

Remark. the terms $(-1)^{m-k}$ change signs, but $S_{n,m}> 0$ for all n and $m \leq n$, indeed $S_{1,1}$=1 and $S_{n,n}$= n !, the other cases are justified by the principle of recurrence of the decomposition of $S_{n,m}$.

2.2 Possible Partitions

Keep in mind that the set of individuals is P_n which contains *n* elements. To calculate the possible partitions of this set we introduce the *n*-th number of Bell (Jean, 2017) that presents the number of partitions of a set with *n* distinct elements, using the following formula

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad (4)$$

There is a recursive formula to compute this number step by step which is the formula of Aitken, and it is represented as following

$$B_{n+1} = \sum_{i=0}^{n} C_n^i B_i \quad (5)$$

There is also a close connection between the number of possible surjections and the number of possible partitions, defined as follows:

$$B_n = \sum_{i=1}^{n} \frac{S_{n,i}}{i!} \quad (6)$$

3 COMBINATORICS AND SET OF STRUCTURES

Assuming that we have *n* individuals that we distribute in *m* microeconomic structures classified by type t_1, t_2,…, t_d. For exemple we take the poor as the individuals and the Income Generating Activities (Shiree, 2011) and (Carletto, 2007) as the micro-economic structures. For their types we can have: livestock and animal production, sale of artisanal or agricultural products…

We define Ω as the universe made up of ways to place *n* individuals in *m* microeconomic structures.

3.1 Random Distribution

When we want to distribute the individuals in the structures in a random fashion, the way to differentiate between two structures will be only by the number of individuals in each one of them. However there can have zero or more than one individual in each structure.

In this case the number of ways to allocate our *n* individuals in the *m* microeconomic structures will be the number of ways to write the integer *n* as the sum of *m* natural integers. It is equal to

$$Card\ \Omega = C_{n+m-1}^n \quad (7)$$

3.2 Distribution of the Individuals in an Ascending Order

If we build distributions of *n* individuals numbered from 1 to *n*, the difference between two structures will be only by the individual's number that contains each one of them. It is the most obvious coding of the individuals; it's all about ordering them, using the date of engagement for example.

Each individual have *m* ways to choose their respective micro-economic structure. So

$$Card\ \Omega = m^n \quad (8)$$

3.3 Distribution of the Individuals with Conditioning

In this last case, each micro-economic structure can either be empty or containing one and only one individual, as a result: $n \leq m$. Therefore

$$Card\ \Omega = A_m^n \quad (9)$$

4 APPLICATION: DISTRIBUTION IN SEVERAL REGIONS

4.1 Application 1

Assuming we have q regions R_1, R_2,..., R_q and we classify the micro-economic structures according to d types t_1, t_2,..., t_d. We put:

- X_{ij} the random variable (Walter, 2008) which denotes the number of individuals in the structure of type t_i in the region R_j.
- $S_{i.}$ the random variable that designates the number of individuals in the structure of type t_i in all the regions such as

$$S_{i.} = \sum_{j=1}^{q} X_{ij} \tag{10}$$

As an example of the type t_1 :

X_{11} the number of individuals in the structure of the type t_1 in the region 1.

X_{12} the number of individuals in the structure of the type t_1 in the region 2.

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X_{1q} the number of individuals in the structure of the type t_1 in the region q.

$$\Longrightarrow \quad S_{1.} = \sum_{j=1}^{q} X_{1j}$$

Theorem 1. The probability law (Sharma, 2009) of the random variable $S_{i.}$ is given by the following expression:

$$\mathrm{P}[S_{i.}=k] = \sum_{\substack{\sum_{l=1}^{k} l a_l = k \\ \sum_{l=1}^{k} a_l = p}} A_q^p \, (P[X_{i.}=0])^{q-p} \times \prod_{l=1}^{k} \frac{(P[X_{i.}=l])^{a_l}}{a_l!} \tag{11}$$

Proof. Keep in mind that X_{i1}, X_{i2},..., X_{iq} are an independent and identical random variables in □. for $S_{i.}=\sum_{j=1}^{q} X_{ij}$

the generating function (Henk, 2012) and (Geoffrey, 2014) in this case can be calculated as :

$$g_{S_{i.}}(t)=(g_{X_{i.}}(t))^q \tag{12}$$

If we take a function $f(t)= t^q$, we will have $g_{S_{i.}}(t) = \left(fog_{X_{i.}}\right)(t)$.

We can also calculate the probability law using the generating function by using the following equality (Norman, 2005),

$$g_{S_{i.}}{}^{(k)}(0)= k!\ \mathrm{P}[S_{i.} = k] \tag{13}$$

Which means that

$$g_{S_{i.}}{}^{(k)}(0)= k!\ \mathrm{P}[S_{i.} = k]=\left(fog_{X_{i.}}\right)^{(k)}(0) \tag{14}$$

Then we introduce the formula of Faà Di Bruno (Johnson, 2002) and (El Khomssi, 2016) that gives the m-th derivative of a composite function

$$(fog)^{(m)}=\sum_{\substack{\sum_{i=1}^{m} i a_i = m \\ \sum_{i=1}^{m} a_i = p}} f^{(p)}og\ \frac{m!}{\prod_{i=1}^{m} a_i!(i!)^{a_i}} \times \prod_{i=1}^{m}(g^{(i)})^{a_i} \tag{15}$$

In our case $f(t)=t^q$ and $g(t)= g_{X_{i.}}(t)$, so we have

$$\left(fog_{X_{i.}}\right)^{(k)}(t)= \sum_{\substack{\sum_{l=1}^{k} l a_l = k \\ \sum_{l=1}^{k} a_l = p}} f^{(p)}og_{X_{i.}}(t) \times \frac{k!}{\prod_{l=1}^{k} a_l!(l!)^{a_l}} \prod_{l=1}^{k}(g_{X_{i.}}{}^{(l)}(t))^{a_l} \tag{16}$$

We choose to take t=0 in order to apply (14). Therefore

$$\begin{aligned} g_{S_{i.}}{}^{(k)}(0) &= \sum_{\substack{\sum_{l=1}^{k} l a_l = k \\ \sum_{l=1}^{k} a_l = p}} A_q^p (P[X_{i.}=0])^{q-p} \frac{k!}{\prod_{l=1}^{k} a_l!(l!)^{a_l}} \times \prod_{l=1}^{k}(l!\, P[X_{i.}=l])^{a_l} \\ &= \sum_{\substack{\sum_{l=1}^{k} l a_l = k \\ \sum_{l=1}^{k} a_l = p}} A_q^p (P[X_{i.}=0])^{q-p} \frac{k!}{\prod_{l=1}^{k} a_l!} \times \prod_{l=1}^{k}(P[X_{i.}=l])^{a_l} \\ &= \sum_{\substack{\sum_{l=1}^{k} l a_l = k \\ \sum_{l=1}^{k} a_l = p}} A_q^p (P[X_{i.}=0])^{q-p}\ k! \times \prod_{l=1}^{k} \frac{(P[X_{i.}=l])^{a_l}}{a_l!} \end{aligned}$$

And then we divide by $k!$ to have the desired result.

Theorem 2. When q is also a random variable, the probability law of the random variable, that we denote in this case $S_{I.}$, is given by

$$\mathrm{P}[S_{I.}=k]=\sum_{\substack{\sum_{l=1}^{k} l a_l=k \\ \sum_{l=1}^{k} a_l=p}} g_q{}^{(p)}\,(P[X_{i.}=0]) \times \prod_{l=1}^{k} \frac{(P[X_{i.}=l])^{a_l}}{a_l!} \tag{17}$$

Proof. This time q is also a random variable and X_{i1}, X_{i2},..., X_{iq} an independent and identical random variables in the set □. We still have $S_{I.}=\sum_{j=1}^{q} X_{ij}$.

in this case the generating function (Geoffrey, 2014) is defined as

$$g_{S_{I.}}(t)=(g_q\, o g_{X_{i.}})(t) \tag{18}$$

We directly apply the formula of Faà Di Bruno with $f(t)= g_q(\mathrm{t})$ and $g(t)= g_{X_{i.}}(t)$, so

$$(g_q\, o g_{X_{i.}})^{(k)}(t)=\sum_{\substack{\sum_{l=1}^{k} l a_l=k \\ \sum_{l=1}^{k} a_l=p}} g_q{}^{(p)}\left(g_{X_{i.}}(t)\right) \times \frac{k!}{\prod_{l=1}^{k} a_l!(l!)^{a_l}} \prod_{l=1}^{k} (g_{X_{i.}}{}^{(l)}(t))^{a_l} \tag{19}$$

For $t=0$,

$$(g_q\, o g_{X_{i.}})^{(k)}(0)=\sum_{\substack{\sum_{l=1}^{k} l a_l=k \\ \sum_{l=1}^{k} a_l=p}} g_q{}^{(p)}\,(P[X_{i.}=0]) \times \frac{k!}{\prod_{l=1}^{k} a_l!(l!)^{a_l}} \prod_{l=1}^{k} (l!\,P[X_{i.}=l])^{a_l}$$
$$=\sum_{\substack{\sum_{l=1}^{k} l a_l=k \\ \sum_{l=1}^{k} a_l=p}} g_q{}^{(p)}\,(P[X_{i.}=0]) \times \frac{k!}{\prod_{l=1}^{k} a_l!} \prod_{l=1}^{k} (P[X_{i.}=l])^{a_l}$$

Then we divide by $k!$ to find the final expression of $\mathrm{P}[S_{I.}=k]$.

Theorem3. The k-th moment (Geoffrey, 2014) of the random variable $S_{i.}$ can be written as

$$E[(S_{i.})^{k}]=\sum_{r=0}^{k} \left\{ {k \atop r} \right\} E[(S_{i.})_r]$$
$$=\sum_{r=0}^{k} \left\{ {k \atop r} \right\} \sum_{\substack{\sum_{l=1}^{r} l a_l=r \\ \sum_{l=1}^{r} a_l=p}} \frac{q!}{(q-p)!} \times \frac{r!}{\prod_{l=1}^{r} a_l!(l!)^{a_l}} \prod_{l=1}^{r} (E[(X_{i.})_l])^{a_l} \tag{20}$$

And for the random variable $S_{I.}$ we have

$$E[(S_{I.})^{k}]=\sum_{r=0}^{k} \left\{ {k \atop r} \right\} E[(S_{I.})_r]$$
$$=\sum_{r=0}^{k} \left\{ {k \atop r} \right\} \sum_{\substack{\sum_{l=1}^{r} l a_l=r \\ \sum_{l=1}^{r} a_l=p}} g_q{}^{(p)}\,(1) \times \frac{r!}{\prod_{l=1}^{r} a_l!(l!)^{a_l}} \prod_{l=1}^{r} (E[(X_{i.})_l])^{a_l} \tag{21}$$

In order to prove both cases, we calculate the high order moment $E[(S_{i.})^{k}]$ and $E\left[(S_{I.})^{k}\right]$ of the two random variables by following exactly the same steps of the two proofs above (depending on the case) using this time the link between the generating function and the high order moment (Norman, 2005) presented as

$$E[(S_{i.})_k]=g_{S_{i.}}{}^{(k)}(1) \tag{22}$$

And using the formula that link the high order moment to the factorial moment (we use the Stirling number of the second kind properties (Weisstein, 2002)

$$E[(S_{i.})^{k}]=\sum_{r=0}^{k} \left\{ {k \atop r} \right\} E[(S_{i.})_r] \tag{23}$$

4.2 Application 2

In this application we still have q regions R_1, R_2,..., R_q and we classify the micro-economic structures according to d types t_1, t_2,..., t_d. We also put :

- X_{ij} the random variable (Walter, 2008) which denotes the number of individuals in the structure of type t_i in the region R_j.

But this time we have:

- $\mathrm{S}_{.\mathrm{j}}$ the random variable that designates the number of individuals in the region $\mathrm{R_j}$ in all the structures, such as

$$S_{.j}=\sum_{i=1}^{d} X_{ij} \tag{24}$$

As an example of the region R_1 :

X_{11} the number of individuals in the structure of the type t_1 in the region 1.

X_{21} the number of individuals in the structure of the type t_2 in the region 1.

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X_{d1} the number of individuals in the structure of the type $\mathrm{t_d}$ in the region 1.

$$\Longrightarrow \quad S_{.1}=\sum_{i=1}^{d} X_{i1}$$

The same results of the application 1 can be obtained by following exactly the same calculation procedure with $X_{1j}, X_{2j}, \ldots, X_{dj}$ an independent and identical random variables in □.

This time we will have:

$$g_{S_{.j}}(t) = (g_{X_{.j}}(t))^{d} \qquad (25)$$

When d is also a random variable we will have:

$$g_{S_{.J}}(t) = (g_d \, o g_{X_{.j}})(t) \qquad (26)$$

5 CONCLUSIONS

As a conclusion, we have treated a method that simplifies the calculations and helps to facilitate the management of the distributions of the individuals in the micro-economic structures in the discrete frame. A study of the continuous case with the proper application will be the object of the next paper.

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