The Finite Volume Method Applied to The Patlak-Keller-Segel Chemotaxis Model in a General Mesh

Ouafa Soualhi[1] and Mohamed Rhoudaf[2]
[1]Department of Mathematics, Moulay Ismail University, Meknes, Morocco
[2]Department of Mathematics, Moulay Ismail University, Meknes, Morocco

Keywords: chemotaxis; finite volume; numerical simulation.

Abstract: In this paper, we present the discrete duality finite volume method (DDFV) applied to a model of (Patlak) Keller-Segel modeling chemosensitive movements, this model consists of a coupled system of elliptic and parabolic equations. Firstly, we prove the existence and uniqueness of the numerical solution to the proposed scheme. Next, numerical simulations are performed to verify accuracy.

1 INTRODUCTION

Chemotaxis is the characteristic movement or orientation of cells, organisms or bacteria along chemical concentration gradient towards chemoattractant or away from chemorepellant, it is very essential for organisms to search food around them. Well-known examples, the first is the bacteria Escherichia Coli such that there cells are known to swim towards the amino acids serine and aspartic acid and towards sugars such as maltose, ribose, galactose and glucose, the second is the amoeba Dycliostelium discoideum ,where it has been used as a model organism in molecular biology and genetics, and is studied as an example of cell communication, differentiation, and programmed cell death.
There are two types of chemotaxis:
1) Positive chemotaxis: the movement of organisms towards a chemical.
2) Negative chemotaxis: the movement of organisms away from a chemical.

Patlak in 1953 (Patlak, 2953) and Keller and Segel in 1970 (Keller and Segel, 1970), were created as a classical model to describe the evolution over time of the cell density $n(x,t)$ and the chemical signal concentration variable $S(x,t)$ assuming that the cells emit directly the chemoattractant which is directly diffused. A lot of theoretical and mathematical model chemotaxis phenomena but the most famous model is the following the classical Paltak-Keller-Segel(PKS) system:

$$\begin{cases} \frac{\partial n}{\partial t} - div(\nabla n - \chi n \nabla S) = 0, & \text{on } \Omega \times [0,T], \\ -div(\nabla S) - \mu n + S = 0, & \text{on } \Omega \times [0,T], \end{cases} \tag{1}$$

where
χ : The chemotactic sensitivity function.
μ : The secretion rate at which the chemical substance is emitted by the cells, let $\mu > 0$.
Ω is a Convex, bounded and open set of $\mathbb{R}^2$ and $T > 0$.

The initial conditions on Ω are given by

$$n(x,0) = n_0(x), \text{ in } \Omega. \tag{2}$$

Therefore, the system (1) is supplemented by the following boundary conditions on $\partial\Omega \times [0,T]$.

$$\nabla n.\nu = 0, \text{ in } \partial\Omega \times [0,T], \tag{3}$$

$$\nabla S.\nu = 0, \text{ in } \partial\Omega \times [0,T], \tag{4}$$

with ν is the unite vector.

This model is very successful for describing the aggregation of the population in a finite time point-wise blowup (in a single point).

In the literature, there exist several works present some numerical method to solve the classical Keller-Segel system, let us set: F. Filbet prove the existence and uniqueness of a numerical solution to the scheme finite volume schemes in (G.Chamoun and R.TalhoukF.Filbet, 2006) and the authors present the finite volume scheme for a Keller-Segel model with additional cross-diffusion in (Bessemoulin-Chatard and Jungel, 2014). In (A.J.Carrillo, 2012) the authors present the cross diffusion and nonlinear diffusion preventing blow up in the Keller-Segel model. A second-order positivity preserving central-upwind scheme is presented by A. Chertock and A. Kurganov in (A.Chertock and A.Kurganov, 2008) for chemotaxis and haptotaxis models. Noted that, the fully discrete analysis of a discontinuous finite element method in (Ref, a) and the new interior penalty discontinuous Galerkin methods in (Ref, b). Moreover, (Haŝkovec and Schmeiser, 2009; Haŝkovec and Schmeiser, 2011) propose the numerical and theoretical study of the stochastic particle approximation and the paper (A.Marrocco, 2003) concerned the numerical simulation of chemotactic using the mixed finite elements method. Finite-element method for a simplified Keller-Segel system in (N.Saito, 2007; N.Saito, 2012) and finite difference schemes to a parabolic-elliptic system modelling chemotaxis in (N.Saito and T.Suzuki, 2005). An

Soualhi, O. and Rhoudaf, M.
The Finite Volume Method Applied to The Patlak-Keller-Segel Chemotaxis Model in a General Mesh.
DOI: 10.5220/0009775800420049
In *Proceedings of the 1st International Conference of Computer Science and Renewable Energies (ICCSRE 2018)*, pages 42-49
ISBN: 978-989-758-431-2

implicit flux-corrected transport (FCT) algorithm has been developed for a class of chemotaxis models in (R. Strehl and Turek, 2010). Fractional step methods applied to a chemotaxis model in (Ref, c).

Four-point scheme on triangles are not easily adapted to obtain consistent diffusive flow in case in unstructured meshes.
In what follows, we are interested in a finite volume method, called the discrete duality finite volume (DDFV) method the interest of this method is its ability to deal with arbitrary polygonal meshes such as nonconforming meshes or unstructured meshes without constraints of orthogonality.

The DDFV (Discrete Duality Finite Volume) method presented by Hermeline (F.Hermeline, 2000), Domelevo, Omnes (AK.Domelevo, 2005) and Andreianov, Boyer, Hubert (B.Andreianov and F.Hubert, 2007), the DDFV method was extended to convection-diffusion (Y.Coudière and G.Manzini, 2010), nonlinear diffusion (Y.Coudière and F.Hubert, 2011; Boyer and Hubert, 2008; B.Andreianov and F.Hubert, 2007), electro and magnetostatics (S.Delcourte and P.Omnes, 2007), miscible fluid flows in porous media (C. Chainais-Hillairet and Mouton, 2013; C.Chainais-Hillairet and Mouton, 2015), drift-diffusion and energy-transport models (C.Chainais-Hillairet, 2009), Stokes flows (Krell, 2011; Krell, 2012; Krell and Manzini, 2012; Delcourte, 2007), electromagnetism (F. Hermeline and Omnes, 2008).

Our purpose is to introduce and analyse the finite volume scheme DDFV for the classical model of PKS in general triangular mesh (without orthogonality condition), we demonstrate the existence and uniqueness of the solution of the DDFV schemes using Brouwer's fixed point theorem, and also we presented numerical tests to show the efficiency of the schemes and to observe the blow-up phenomenon.

The paper is organized as follows : In Section 2 we detail the DDFV formulation. The demonstrate of the existence and uniqueness of the DDFV solutions and number of numerical results obtained on different two-dimensional meshes are realized in section 3 .

2 DISCRETE DUALITY FINITE VOLUME SCHEMES FOR MODIFIED KELLER-SEGEL MODEL

2.1 Meshes and Notations

Let Ω be a polygonal open bounded connected subset of $\mathbb{R}^d$ with $d \in \mathbb{N}^*$, and $\partial\Omega = \overline{\Omega}\backslash\Omega$ its boundary .

Following Hermeline (F.Hermeline, 2000), Domelevo, Omnes (AK.Domelevo, 2005) and Andreianov, Boyer, Hubert (B.Andreianov and F.Hubert, 2007), we consider a DDFV mesh which is a triple $\mathcal{T} = (\mathcal{M}, \mathcal{M}^*, \mathfrak{D})$ described below.

The primal mesh $\mathcal{M}$ is defined as the triplet $(\mathfrak{M}, \mathcal{E}, P)$, where $\mathfrak{M}$ is a finite family of nonempty open disjoint subset $\mathcal{K}$ of Ω (the control volume primal) such that $\overline{\Omega} = \cup_{\mathcal{K}\in\mathfrak{M}}\overline{\mathcal{K}}$, with $\partial\mathcal{K} = \overline{\mathcal{K}}\backslash\mathcal{K}$ be the boundary of $\mathcal{K}$, let $m_{\mathcal{K}} = |\mathcal{K}| > 0$ is the measure of $\mathcal{K}$ and let $d_{\mathcal{K}}$ the diameter of $\mathcal{K}$, $\mathcal{E}$ is the set of edges σ of the mesh, m_σ is the measure of σ, $\mathcal{E}_{int}$ is the subset of interior edges of Ω. For all $\mathcal{K} \in \mathfrak{M}$ and $\sigma \in \mathcal{E}_{\mathcal{K}}$ (subset of edges of $\mathcal{K}$) , we denote by $\nu_{\mathcal{K},\sigma}$ the unite vector normal to σ outward to $\mathcal{K}$. P is the subset of points of Ω indexed by $\mathfrak{M}$, we denote $P = \{(x_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}; x_{\mathcal{K}} \in \mathcal{K}\}$, ($x_{\mathcal{K}}$ is the barycentre of $\mathcal{K}$) we than denote by $D_{\mathcal{K},\sigma}$ the cone with vertex $x_{\mathcal{K}}$ and basis $\mathcal{K}$.

Then, the dual mesh $\mathcal{M}^*$ is defined as the triplet $(\mathfrak{M}^*, \mathcal{E}^*, P^*)$, with $\mathfrak{M}^*$ is a finite family of nonempty open disjoint subset $\mathcal{K}^*$ of Ω (the control volume dual) such that $\overline{\Omega} = \cup_{\mathcal{K}^*\in\mathfrak{M}^*}\overline{\mathcal{K}^*}$, for all $\mathcal{K}^* \in \mathfrak{M}^*$,with $\partial\mathcal{K}^* = \overline{\mathcal{K}^*}\backslash\mathcal{K}^*$ be the boundary of $\mathcal{K}^*$, let $m_{\mathcal{K}^*} = |\mathcal{K}^*| > 0$ is the measure of $\mathcal{K}^*$ and let $d_{\mathcal{K}^*}$ the diameter of $\mathcal{K}^*$, $\mathcal{E}^*$ is the set of the edges σ^* of this mesh, m_{σ^*} is the measure of σ^*, $\mathcal{E}^*_{int}$ is the subset of interior edges of Ω. For all $\mathcal{K}^* \in \mathfrak{M}^*$ and $\sigma^* \in \mathcal{E}_{\mathcal{K}^*}$ (subset of edges of $\mathcal{K}^*$) , we denote by $\nu_{\sigma^*,\mathcal{K}^*}$ the unite vector normal to σ^* outward to $\mathcal{K}^*$. P^* is the subset of points of Ω indexed by $\mathfrak{M}^*$, we denote $\{P^* = (x_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}; x_{\mathcal{K}^*} \in \mathcal{K}^*\}$, we than note by $D_{\mathcal{K}^*,\sigma^*}$ the cone with vertex $x_{\mathcal{K}^*}$ and basis $\mathcal{K}^*$

Finally, We denote by $\mathfrak{D}$ the sets of all diamonds $\mathcal{D}$, let:

- $\mathfrak{D}_{\mathcal{K}} = \{\mathcal{D} \in \mathfrak{D}/\sigma \in \mathcal{E}_{\mathcal{K}}\}$.
- $\mathfrak{D}_{\mathcal{K}^*} = \{\mathcal{D} \in \mathfrak{D}/\sigma^* \in \mathcal{E}_{\mathcal{K}^*}\}$.
- $\mathfrak{D}_{int} = \{\mathcal{D} \in \mathfrak{D}/\sigma \in \mathcal{E}_{int}\}$.
- $\mathfrak{D}_{ext} = \{\mathcal{D} \in \mathfrak{D}/\sigma \in \mathcal{E}_{ext}\}$.
- $\mathcal{M}_{\mathcal{D}} = \{\mathcal{K} \in \mathfrak{M}$ such that $\sigma \in \mathcal{E}_{\mathcal{K}}\}$.
- $\mathcal{M}^*_{\mathcal{D}} = \{\mathcal{K}^* \in \mathfrak{M}^*$ such that $\sigma^* \in \mathcal{E}_{\mathcal{K}^*}\}$.
- $m_{\mathcal{D}}$ measure of the diamond.
- For a diamond cell $\mathcal{D}$ recall that $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ are the vertices of $\mathcal{D}_{\sigma,\sigma^*}$.
- τ the unite vector parallel to σ, oriented from $\mathcal{K}^*$ to $\mathcal{L}^*$.
- τ^* the unite vector parallel to σ^*, oriented from $\mathcal{K}$ to $\mathcal{L}$.
- $\alpha_{\mathcal{D}}$ the angle between τ and τ^*.
- $\nu_{\mathcal{K},\sigma} = -cos\alpha_{\mathcal{D}}\nu_{\sigma^*,\mathcal{K}^*} + sin\alpha_{\mathcal{D}}\tau_{\mathcal{K},\sigma}$.
- $d_{\mathcal{D}}$ the diameter of $\mathcal{D}_{\sigma,\sigma^*}$.

We consider the following property:

$$\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \leq \frac{mes(D_{\mathcal{K},\sigma})}{3}. \tag{5}$$

Finally, the size of the mesh: $size(\mathcal{T}) = \max_{\mathcal{D}\in\mathfrak{D}} d_{\mathcal{D}}$.

2.2 Discrete Operators and Duality Formula

We define the spaces:

- $\mathbb{R}^{\mathcal{T}}$ is a linear space of scalar fields constant on the cells of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}^*}$.

$$\mathbb{R}^{\mathcal{T}} = \{u_{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}),$$
$$\text{with } u_{\mathcal{K}} \in \mathbb{R}, \text{ for all } \mathcal{K} \in \overline{\mathfrak{M}}$$
$$\text{and } u_{\mathcal{K}^*} \in \mathbb{R}; \text{ for all } \mathcal{K}^* \in \overline{\mathfrak{M}^*}\}.$$

- $(\mathbb{R}^2)^{\mathcal{D}}$ is a linear space of vector fields constant on the cells of $\mathcal{D}$.

$$(\mathbb{R}^2)^{\mathcal{D}} = \{\xi_{\mathfrak{D}} = (\xi_{\mathcal{D}})_{\mathcal{D}\in\overline{\mathfrak{D}}};\text{ with } \xi_{\mathcal{D}} \in \mathbb{R}^2;\ \text{for all } \mathcal{D}\in\mathfrak{D}\}.$$

Now, we recall the definition of the discrete gradient and the discrete divergence have been introduced respectively in (Y. Coudiere and Villedieu, 1999) and (AK.Domelevo, 2005). We also introduce some trace operators and scalar products

Definition 2.1. Let

$$\nabla^{\mathfrak{D}} : \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^2)^{\mathfrak{D}},$$
$$u_{\mathcal{T}} \to \nabla^{\mathfrak{D}} u_{\mathcal{T}} = (\nabla^{\mathcal{D}} u_{\mathcal{T}})_{\mathcal{D}\in\mathfrak{D}},$$

the discrete gradient, such that for all $\mathcal{D}\in\mathfrak{D}$

$$\begin{cases} \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K}^*,\mathcal{L}^*} = \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{m_\sigma}, \\ \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K},\mathcal{L}} = \frac{u_{\mathcal{L}}-u_{\mathcal{K}}}{m_{\sigma^*}}, \end{cases}$$

equivalent to

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{sin(\alpha_{\mathcal{D}})}\left[\frac{u_{\mathcal{L}}-u_{\mathcal{K}}}{m_{\sigma^*}}\nu_{\sigma,\mathcal{K}} + \frac{u_{\mathcal{L}^*}-u_{\mathcal{K}^*}}{m_\sigma}\nu_{\sigma^*,\mathcal{K}^*}\right],$$

using the propriety $m_{\mathcal{D}} = \frac{1}{2} m_\sigma m_{\sigma^*} sin(\alpha_{\mathcal{D}})$, we have

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}}\left[(u_{\mathcal{L}}-u_{\mathcal{K}})m_\sigma \nu_{\sigma,\mathcal{K}} + (u_{\mathcal{L}^*}-u_{\mathcal{K}^*})m_{\sigma^*}\nu_{\sigma^*,\mathcal{K}^*}\right].$$

Than the discrete divergence $div^{\mathcal{T}}$ is defined by

Definition 2.2. The discrete divergence operator $div^{\mathcal{T}}$ is a mapping from $(\mathbb{R}^2)^{\mathfrak{D}}$ to $\mathbb{R}^{\mathcal{T}}$ defined for all $\xi \in (\mathbb{R}^2)^{\mathfrak{D}}$ by

$$div^{\mathcal{T}}\xi_{\mathfrak{D}} = \left(div^{\mathfrak{M}}\xi_{\mathfrak{D}}, 0, div^{\mathfrak{M}^*}\xi_{\mathfrak{D}}, div^{\partial\mathfrak{M}^*}\xi_{\mathfrak{D}}\right),$$

such that

$$\begin{cases} div^{\mathfrak{M}}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}}(\xi_{\mathfrak{D}}))_{\mathcal{K}\in\mathfrak{M}}, \\ div^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}^*}(\xi_{\mathfrak{D}}))_{\mathcal{K}^*\in\mathfrak{M}^*}, \\ div^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}) = (div_{\mathcal{K}^*}(\xi_{\mathfrak{D}}))_{\mathcal{K}^*\in\partial\mathfrak{M}^*}, \end{cases}$$

with

$$div_{\mathcal{K}}\xi = \frac{1}{m_{\mathcal{K}}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} m_\sigma \xi_{\mathcal{D}}.\nu_{\sigma,\mathcal{K}},\ \text{for all } \mathcal{K}\in\mathfrak{M},$$

$$div_{\mathcal{K}^*}\xi = \frac{1}{m_{\mathcal{K}^*}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*}\xi_{\mathcal{D}}.\nu_{\sigma^*,\mathcal{K}^*},\ \text{for all } \mathcal{K}^*\in\mathfrak{M}^*,$$

$$div_{\mathcal{K}^*}\xi = \frac{1}{m_{\mathcal{K}^*}}\Big[\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*}\xi_{\mathcal{D}}.\nu_{\sigma^*,\mathcal{K}^*} + \sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext}} \frac{m_\sigma}{2}\xi_{\mathcal{D}}.\nu_{\sigma,\mathcal{K}}\Big],\ \text{for all } \mathcal{K}^*\in\partial\mathfrak{M}^*.$$

Let us now define the scalar products $<.,.>_{\mathcal{T}}$ on $\mathbb{R}^{\mathcal{T}}$ and $<.,.>_{\mathcal{D}}$ on $(\mathbb{R}^2)^{\mathcal{D}}$ by

$$< v_{\mathcal{T}}, u_{\mathcal{T}} >_{\mathcal{T}} = \frac{1}{2}\left(\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} v_{\mathcal{K}} + \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} v_{\mathcal{K}^*}\right),$$

for all $u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$.

$$< \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} >_{\mathcal{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\xi_{\mathcal{D}}.\varphi_{\mathcal{D}},\ \text{for all } \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathcal{D}}. \quad (6)$$

The corresponding norms are denoted by $\|.\|_{p,\mathcal{T}}$ and $\|.\|_{p,\mathfrak{D}}$ for all $1 \le p \le +\infty$.

- For all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and for all $1 \le p < +\infty$

$$\|u_{\mathcal{T}}\|_{p,\mathcal{T}} = \left(\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|u_{\mathcal{K}}|^p + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|u_{\mathcal{K}^*}|^p\right)^{1/p}. \quad (7)$$

- For all $\xi_{\mathfrak{D}} \in \left(\mathbb{R}^2\right)^{\mathfrak{D}}$ and for all $1 \le p < +\infty$.

$$\|\xi_{\mathfrak{D}}\|_{p,\mathfrak{D}} = \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|\xi_{\mathcal{D}}|^p\right)^{1/p}. \quad (8)$$

- For all $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$

$$\|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} = \max\left(\max_{\mathcal{K}\in\mathfrak{M}}|u_{\mathcal{K}}|, \max_{\mathcal{K}^*\in\mathfrak{M}^*}|u_{\mathcal{K}^*}|\right). \quad (9)$$

- For all $\xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathcal{D}}$

$$\|\xi_{\mathfrak{D}}\|_{\infty,\mathcal{D}} = \max_{\mathcal{D}\in\mathfrak{D}}|\xi_{\mathcal{D}}|, \quad (10)$$

Definition 2.3 (Convection term)**.** Let $divc^{\mathcal{T}} : (\mathbb{R}^2)^{\mathfrak{D}} \times \mathbb{R}^{\mathcal{T}} \to \mathbb{R}^{\mathcal{T}}$ the convection operator defined for all $\xi_{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ and $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ by

$$divc^{\mathcal{T}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = [divc^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), 0, divc^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}), divc^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}})],$$

such that

$$\begin{cases} divc^{\mathfrak{M}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}\in\mathfrak{M}}, \\ divc^{\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\mathfrak{M}^*}, \\ divc^{\partial\mathfrak{M}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = (divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}))_{\mathcal{K}^*\in\partial\mathfrak{M}^*}, \end{cases}$$

with

- For all $\mathcal{K}\in\mathfrak{M}$,

$$divc_{\mathcal{K}}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}}}\sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}} \\ \sigma=\mathcal{K}/\mathcal{L}}} m_\sigma[(\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^- v_{\mathcal{L}}],$$

- For all $\mathcal{K}^*\in\mathfrak{M}^*$

$$divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}^*}}\sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*} \\ \sigma^*=\mathcal{K}^*/\mathcal{L}^*}} m_{\sigma^*}[(\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^- v_{\mathcal{L}^*}],$$

- For all $\mathcal{K}^*\in\partial\mathfrak{M}^*$

$$divc_{\mathcal{K}^*}(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}^*}}\Big(\sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*} \\ \sigma^*=\mathcal{K}^*/\mathcal{L}^*}} m_{\sigma^*}[(\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^+ v_{\mathcal{K}^*} - (\xi_{\mathfrak{D}}.\nu_{\sigma^*,\mathcal{K}^*})^- v_{\mathcal{L}^*}] + \sum_{\substack{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext} \\ \sigma=\mathcal{K}/\mathcal{L}}} \frac{m_\sigma}{2}[(\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^+ v_{\mathcal{K}} - (\xi_{\mathfrak{D}}.\nu_{\sigma,\mathcal{K}})^- v_{\mathcal{L}}].$$

where $x^+ = max(x,0)$ and $x^- = max(0,-x)$.

2.3 The Numerical Scheme

A DDFV scheme for the the discretisation of the problem (1) is given by the following set of equations:

For all $\mathcal{K} \in \mathfrak{M}$ and $\mathcal{K}^* \in \mathfrak{M}^*$, let

$$n^0_{\mathcal{K}} = \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} u^0(x)dx \text{ and } n^0_{\mathcal{K}^*} = \frac{1}{m_{\mathcal{K}^*}} \int_{\mathcal{K}^*} u^0(x)dx. \quad (11)$$

At each time step k, the numerical solution will be given by $(n^{k+1}_{\mathcal{T}}, S^{k+1}_{\mathcal{T}})$. Then, the scheme for (1) writes for all $0 < k < N_T - 1$

$$\begin{cases} \frac{n^{k+1}_{\mathcal{T}} - n^k_{\mathcal{T}}}{\Delta t} - div^{\mathcal{T}}(\nabla^{\mathfrak{D}} n^{k+1}_{\mathcal{T}}) + divc^{\mathcal{T}}(n^k_{\mathcal{T}} \nabla^{\mathfrak{D}} S^{n+1}_{\mathcal{T}}) = 0, \\ -div^{\mathcal{T}}(\nabla^{\mathfrak{D}} S^{k+1}_{\mathcal{T}}) + \mu S^{k+1}_{\mathcal{T}} = n^k_{\mathcal{T}}. \\ \nabla^{\mathcal{D}} n^k_{\mathcal{T}}.\nu = \nabla^{\mathcal{D}} S^k_{\mathcal{T}}.\nu = 0, \forall \mathcal{D} \in \mathfrak{D}_{ext}. \end{cases} \quad (12)$$

Whith $div^{\mathcal{T}}$ and $\nabla^{\mathfrak{D}}$ are defined respectively by definition 2.2 and definition 2.1.

3 THE MAIN RESULTS

3.1 Existence of DDFV Solutions

Theorem 3.1. *Let Ω be an open, bonded, connected, polygonal domain of $\mathbb{R}^2$ and let $\mathcal{T}$ be a discretization of $\Omega \times (0,T)$ such that*

$$\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \leq \frac{mes(D_{\mathcal{K},\sigma})}{3}. \quad (13)$$

Let $n_0 \in L^2(\Omega), n_0 \geq 0$ in Ω. Then there exists a solution $\{(n^{k+1}_{\mathcal{T}}, S^{k+1}_{\mathcal{T}}), 0 \leq k \leq N_T - 1\}$ to (11) and (12) satisfying:

$$\begin{cases} \text{for all } \mathcal{K} \in \mathfrak{M} \text{ and } \mathcal{K}^* \in \mathfrak{M}^*, \text{ for all } 0 \leq k \leq N_T. \\ n^k_{\mathcal{K}} \geq 0 \text{ and } n^k_{\mathcal{K}^*} \geq 0 \\ \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} n^k_{\mathcal{K}} + \frac{1}{2} \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} n^k_{\mathcal{K}^*} = \\ \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} m_{\mathcal{K}} n^0_{\mathcal{K}} + \frac{1}{2} \sum_{\mathcal{K}^* \in \mathfrak{M}^*} m_{\mathcal{K}^*} n^0_{\mathcal{K}^*} = \|n_0\|_{L^1(\Omega)}, \\ \text{for all } 0 \leq k \leq N_T. \end{cases}$$

Proof. Let $k \in \{0,1,2,3,...,N_T\}$ and let $(n^k_{\mathcal{T}}, S^k_{\mathcal{T}})$ be a solution to (1), we introduce the set:

$$X^{\mathcal{T}} = \{v \in \mathbb{R}^{\mathcal{T}}; v \geq 0 \text{ in } \Omega, \|v\|_{L^1(\Omega)} \leq \|n_0\|_{L^1(\Omega)}\}.$$

Firstly we constructed $\overline{n}$ and $\overline{S}$, then we demonstrate in the first step the unicity of the solution, after in the second step we using the Browr's fixed point to proof the existance of the solution.

We construct $\overline{S} \in X_{\mathcal{T}}$ using the following schemes

$$\begin{cases} -\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} m_\sigma \quad \nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}} + m_{\mathcal{K}} \overline{S}_{\mathcal{K}} = \\ \qquad m_{\mathcal{K}} \mu n^k_{\mathcal{K}}, \text{ for all } \mathcal{K} \in \mathfrak{M}, \\ -\sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} \quad \nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*} + m_{\mathcal{K}^*} \overline{S}_{\mathcal{K}^*} = \\ \qquad m_{\mathcal{K}^*} \mu n^k_{\mathcal{K}^*} \text{ for all } \mathcal{K}^* \in \mathfrak{M}^*, \end{cases} \quad (14)$$

and we comput $\overline{n} \in X_{\mathcal{T}}$ using the schemes

$$\begin{cases} m_{\mathcal{K}} \frac{\overline{n}_{\mathcal{K}} - n^k_{\mathcal{K}}}{\Delta t} - \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} m_\sigma \nabla_{\mathcal{D}} \overline{n}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}} \\ + \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} m_\sigma [\overline{n}_{\mathcal{K}} (\nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}})^+ \\ -\overline{n}_{\mathcal{L}} (\nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma,\mathcal{K}})^-] = 0, \text{ for all } \mathcal{K} \in \mathfrak{M}, \\ m_{\mathcal{K}^*} \frac{\overline{n}_{\mathcal{K}^*} - n^k_{\mathcal{K}^*}}{\Delta t} - \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} \nabla_{\mathcal{D}} \overline{n}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*} + \\ \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*} [\overline{n}_{\mathcal{K}^*} (\nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*})^+ - \\ \overline{n}_{\mathcal{L}^*} (\nabla_{\mathcal{D}} \overline{S}_{\mathcal{T}}.\nu_{\sigma^*,\mathcal{K}^*})^-] = 0, \text{ for all } \mathcal{K}^* \in \mathfrak{M}^*. \end{cases} \quad (15)$$

Step 1: The system (14) can be written as $A\overline{S} = b$, where for all $\mathcal{K}, \mathcal{L} \in \mathfrak{M}$ such that $\sigma = \mathcal{K}|\mathcal{L}$ and for all for all $\mathcal{K}^*, \mathcal{L}^* \in \mathfrak{M}^*$ such that $\sigma^* = \mathcal{K}^*`|\mathcal{L}^*$, A is defined by:

$$A_{\mathcal{K},\mathcal{K}} = \sum_{\sigma \in \mathcal{K}} \frac{m_\sigma^2}{2m_{\mathcal{D}}} + m_{\mathcal{K}},$$

$$A_{\mathcal{K}^*,\mathcal{K}^*} = \sum_{\sigma^* \in \mathcal{K}^*} \frac{m^{*2}_\sigma}{2m_{\mathcal{D}}} + m_{\mathcal{K}^*}.$$

$$\begin{cases} A_{\mathcal{K},\mathcal{L}} = -\frac{m_\sigma^2}{2m_{\mathcal{D}}}, \\ A_{\mathcal{K},\mathcal{K}^*} = -\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \\ A_{\mathcal{K},\mathcal{L}^*} = \frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \end{cases}$$

and

$$\begin{cases} A_{\mathcal{K}^*,\mathcal{L}^*} = -\frac{m_{\sigma^*}^2}{2m_{\mathcal{D}}}, \\ A_{\mathcal{K}^*,\mathcal{K}} = -\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \\ A_{\mathcal{K}^*,\mathcal{L}} = \frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} \nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \end{cases}$$

and $b_{\mathcal{K}} = \mu m_{\mathcal{K}} n^k_{\mathcal{K}}$., $b_{\mathcal{K}^*} = \mu m_{\mathcal{K}^*} n^k_{\mathcal{K}^*}$. Since for all $\mathcal{K} \in \mathfrak{M}$ and $\mathcal{K}^* \in \mathfrak{M}^*$:

$$\begin{cases} |A_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} [|A_{\mathcal{K},\mathcal{L}}| + |A_{\mathcal{K},\mathcal{L}^*}| + \\ |A_{\mathcal{K},\mathcal{K}^*}|] = m_{\mathcal{K}} - 2 \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}} |cos(\alpha_{\mathcal{D}}|), \\ \text{and} \\ |A_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} [|A_{\mathcal{K}^*,\mathcal{L}^*}| + |A_{\mathcal{K}^*,\mathcal{L}}| + \\ |A_{\mathcal{K}^*,\mathcal{K}}|] = m_{\mathcal{K}^*} - 2 \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*} m_\sigma}{2m_{\mathcal{D}}} |cos(\alpha_{\mathcal{D}}|). \end{cases}$$

Using the hypothesis (13) we have

$$\begin{cases} |A_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \left[|A_{\mathcal{K},\mathcal{L}}| + |A_{\mathcal{K},\mathcal{L}^*}| + |A_{\mathcal{K},\mathcal{K}^*}|\right] \geq 0 \\ |A_{\mathcal{K}^*,\mathcal{K}^*}| - \\ \sum_{\sigma^* \in \mathcal{E}_{\mathcal{K}^*}} \left[|A_{\mathcal{K}^*,\mathcal{L}^*}| + |A_{\mathcal{K}^*,\mathcal{L}}| + |A_{\mathcal{K}^*,\mathcal{K}}|\right] \geq 0. \end{cases}$$

Then the matrix A is strictly diagonally dominant with respect to the columns and hence, A is invertible. This shows the unique solvability of (14).

Now, the system (15) equivalent to the system $B\overline{n} = C$, with:

$$B_{\mathcal{K},\mathcal{K}} = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \frac{m_\sigma^2}{2m_{\mathcal{D}}} + \frac{m_{\mathcal{K}}}{\Delta t} + \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} m_\sigma (\nabla_{\mathcal{D}} \overline{S}.\nu_{\sigma,\mathcal{K}})^+,$$

and

$$B_{\mathcal{K}^*,\mathcal{K}^*} = \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} \frac{m_{\sigma^*}^2}{2m_{\mathcal{D}}} + \frac{m_{\mathcal{K}^*}}{\Delta t} + \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^+.$$

$$\begin{cases} B_{\mathcal{K},\mathcal{L}} = -\frac{m_\sigma^2}{2m_{\mathcal{D}}} - m_\sigma(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{K}})^-, \\ B_{\mathcal{K},\mathcal{K}^*} = \frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}\nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \\ B_{\mathcal{K},\mathcal{L}^*} = -\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}\nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \end{cases}$$

and

$$\begin{cases} B_{\mathcal{K}^*,\mathcal{L}^*} = -\frac{m_{\sigma^*}^2}{2m_{\mathcal{D}}} - m_{\sigma^*}(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^-, \\ B_{\mathcal{K}^*,\mathcal{K}} = \frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}\nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \\ B_{\mathcal{K}^*,\mathcal{L}} = -\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}\nu_{\mathcal{K},\sigma}.\nu_{\mathcal{K}^*,\sigma^*}, \end{cases}$$

and $c_{\mathcal{K}} = \frac{m_{\mathcal{K}} n_{\mathcal{K}}^k}{\Delta t}$, $c_{\mathcal{K}^*} = \frac{m_{\mathcal{K}^*} n_{\mathcal{K}^*}^k}{\Delta t}$. Since for all $\mathcal{K}\in\mathfrak{M}$ and $\mathcal{K}^*\in\mathfrak{M}^*$:

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\left[|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|\right] = \\ \frac{m_{\mathcal{K}}}{\Delta t} + \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma|(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{K}})^+| \\ - \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma|(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{K}})^-| - \\ 2\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}|cos(\alpha_{\mathcal{D}}|), \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}}\left[|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|\right] = \\ \frac{m_{\mathcal{K}^*}}{\Delta t} + \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}|(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^+| \\ - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}|(\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^-| - \\ 2\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}}\frac{m_{\sigma^*}m_\sigma}{2m_{\mathcal{D}}}|cos(\alpha_{\mathcal{D}}|), \end{cases}$$

for all $\sigma\in\mathcal{E}_{\mathcal{K}}$ and $\sigma^*\in\mathcal{E}_{\mathcal{K}^*}$ we have

$$\begin{cases} \nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{K}} = -\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{L}} \\ \nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*} = -\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{L}^*} \end{cases} \tag{16}$$

which yields

$$\begin{cases} (\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{K}})^- = (\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma,\mathcal{L}})^+, \\ (\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{K}^*})^- = (\nabla_{\mathcal{D}}\overline{S}.\nu_{\sigma^*,\mathcal{L}^*})^+, \end{cases} \tag{17}$$

that's give

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\left[|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|\right] = \\ \frac{m_{\mathcal{K}}}{\Delta t} - 2\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\frac{m_\sigma m_{\sigma^*}}{2m_{\mathcal{D}}}|cos(\alpha_{\mathcal{D}}|) \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}}\left[|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|\right] = \\ \frac{m_{\mathcal{K}^*}}{\Delta t} - 2\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}}\frac{m_{\sigma^*}m_\sigma}{2m_{\mathcal{D}}}|cos(\alpha_{\mathcal{D}}|) \end{cases}$$

using the hypothesis (13) we have

$$\begin{cases} |B_{\mathcal{K},\mathcal{K}}| - \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\left[|B_{\mathcal{K},\mathcal{L}}| + |B_{\mathcal{K},\mathcal{L}^*}| + |B_{\mathcal{K},\mathcal{K}^*}|\right] \geq 0 \\ |B_{\mathcal{K}^*,\mathcal{K}^*}| - \sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}}\left[|B_{\mathcal{K}^*,\mathcal{L}^*}| + |B_{\mathcal{K}^*,\mathcal{L}}| + |B_{\mathcal{K}^*,\mathcal{K}}|\right] \geq 0. \end{cases}$$

Then the matrix B is strictly diagonally dominant with respect to the columns and hence, B is invertible. This shows the unique solvability of (15). Then $\overline{n}$ is nonnegative, implies that $\overline{n}$ satisfies (14).

In (15), summing the first equation over $\mathcal{K}\in\mathfrak{M}$ and the second equation over $\mathcal{K}^*\in\mathfrak{M}^*$, we obtain

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\overline{n}_{\mathcal{K}} = \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n_{\mathcal{K}}^k. \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}\overline{n}_{\mathcal{K}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n_{\mathcal{K}^*}^k. \end{cases} \tag{18}$$

That's give

$$\begin{cases} \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}\overline{n}_{\mathcal{K}} + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}\overline{n}_{\mathcal{K}^*} = \\ \frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} n_{\mathcal{K}}^k + \frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} n_{\mathcal{K}^*}^k = \|n^0\|_{L^1(\Omega)}. \end{cases} \tag{19}$$

step 2: Let $\mathfrak{H}: X^{\mathcal{T}} \to X^{\mathcal{T}}$ the operator define by the solution to (14) and (15) such that $\mathfrak{H}(n) = \overline{n}$, it must be shown that the operator $\mathfrak{H}$ is continuous to apply Brouwer fixed point thm (i.e) we have to prove that $\overline{n}^\beta \to \overline{n}$ as $\beta\to\infty$ such that:

$$\begin{cases} (n^\beta)_{\beta\in\mathbb{N}} \subset X^{\mathcal{T}} \text{ be a sequence verified} \\ n^\beta \to n \text{ as } \beta\to\infty \text{ in } X^{\mathcal{T}}, \\ \mathfrak{H}(n^\beta) = \overline{n}^\beta, \\ \mathfrak{H}(n) = \overline{n}. \end{cases} \tag{20}$$

It easy to show that $\overline{S}^\beta - \overline{S} \to 0$ in $X^{\mathcal{T}}$ as $\beta\to\infty$, since the map $n \to \overline{S}$ is linear on the finite dimensional space $X^{\mathcal{T}}$ and continuous. Later, using (15) and an adaptation of the proof of theorem 2.1 in (G.Chamoun and R.TalhoukF.Filbet, 2006) leads to:

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|\overline{n}_{\mathcal{K}}^\beta - \overline{n}_{\mathcal{K}}| \leq 2\Delta t\left(\sum_{\mathcal{K}\in\mathfrak{M}}|\overline{n}_{\mathcal{K}}|^2\right)^{1/2} \\ \left(\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma|\nabla_{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma,\mathcal{K}}|^2\right)^{1/2}, \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|\overline{n}_{\mathcal{K}^*}^\beta - \overline{n}_{\mathcal{K}^*}| \leq 2\Delta t\left(\sum_{\mathcal{K}^*\in\mathfrak{M}^*}|\overline{n}_{\mathcal{K}^*}|^2\right)^{1/2} \\ \left(\sum_{\mathcal{K}^*\in\mathfrak{M}^*}\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}|\nabla_{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2\right)^{1/2}, \end{cases}$$

Let $c_1 > 0$ such that $2\Delta t\sum_{\mathcal{K}\in\mathfrak{M}}|\overline{n}_{\mathcal{K}}|^2 \leq c_1\|n_{\mathcal{T}}^0\|_{L^2(\Omega)}$ and $2\Delta t\sum_{\mathcal{K}^*\in\mathfrak{M}^*}|\overline{n}_{\mathcal{K}^*}|^2 \leq c_1\|n_{\mathcal{T}}^0\|_{L^2(\Omega)}$, then

$$\begin{cases} \sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|\overline{n}_{\mathcal{K}}^\beta - \overline{n}_{\mathcal{K}}| \leq \\ c_1\|n_{\mathcal{T}}^0\|_{L^2(\Omega)}\left(\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma|\nabla_{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma,\mathcal{K}}|^2\right)^{1/2}, \\ \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|\overline{n}_{\mathcal{K}^*}^\beta - \overline{n}_{\mathcal{K}^*}| \leq \\ c_1\|n_{\mathcal{T}}^0\|_{L^2(\Omega)}\left(\sum_{\mathcal{K}^*\in\mathfrak{M}^*}\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}|\nabla_{\mathcal{D}}(\overline{S}^\beta - \overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2\right)^{1/2}, \end{cases}$$

then

$$\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|\overline{n}^{\beta}_{\mathcal{K}}-\overline{n}_{\mathcal{K}}|^2+\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|\overline{n}^{\beta}_{\mathcal{K}^*}-\overline{n}_{\mathcal{K}^*}|^2 \leq$$
$$c_1^2\|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)}(\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\sigma}|\nabla_{\mathcal{D}}(\overline{S}^{\beta}-\overline{S}).\nu_{\sigma,\mathcal{K}}|^2$$
$$+\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*}\sum_{\sigma^*\in\mathcal{E}_{\mathcal{K}^*}} m_{\sigma^*}|\nabla_{\mathcal{D}}(\overline{S}^{\beta}-\overline{S}).\nu_{\sigma^*,\mathcal{K}^*}|^2),$$

using the poincare inequality, we have

$$\frac{1}{2}\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}}|\overline{n}^{\beta}_{\mathcal{K}}-\overline{n}_{\mathcal{K}}|^2+\frac{1}{2}\sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*}|\overline{n}^{\beta}_{\mathcal{K}^*}-\overline{n}_{\mathcal{K}^*}|^2 \leq$$
$$c_1^2\|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)}\|\nabla_{\mathcal{D}}(\overline{S}^{\beta}-\overline{S})\|^2_{\mathcal{D}} \leq c_1)^2\|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)}\|\overline{S}^{\beta}-\overline{S}\|^2_{2,\mathcal{T}},$$

then

$$\begin{cases} \|\overline{n}^{\beta}_{\mathcal{K}}-\overline{n}_{\mathcal{K}}\|^2_{2,\mathcal{T}} \leq \\ c_1^2\|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)}\|\nabla_{\mathcal{D}}(\overline{S}^{\beta}-\overline{S})\|^2_{\mathcal{D}} \leq c_1^2\|n^0_{\mathcal{T}}\|^2_{L^2(\Omega)}\|\overline{S}^{\beta}-\overline{S}\|^2_{2,\mathcal{T}}. \end{cases}$$

$\|n^0_{\mathcal{T}}\|_{L^2(\Omega)}$ is bounded and $\overline{S}^{\beta}-\overline{S}\rightarrow 0$ as $\beta\rightarrow\infty$, then $\overline{n}^{\beta}\rightarrow\overline{n}$ in $H^{\mathcal{T}}$ implies that $\mathfrak{H}$ is a continuous operator.

Therefore using the Brouwer fixed point thm the operator $\mathfrak{H}$ has a fixed point, hence the prove of thm. □

3.2 Numerical Experiments

In this section, we show three numerical simulations of model (1) in a two dimensional space to show the efficiency of the DDFV scheme. the system (1) is describes the evolution over time of the cell density $n(x,t)$ and the chemical signal concentration variable $S(x,t)$, Some of the tests cases come from the paper (Bessemoulin-Chatard and Jungel, 2014) where a finite volume scheme is used, and our results compare very well to the ones in this reference. We simulate the model in a two dimensional domain $\Omega=(0;5)\times(0;5)$ for which we consider a nonuniform and non-admissible grid figure.1,

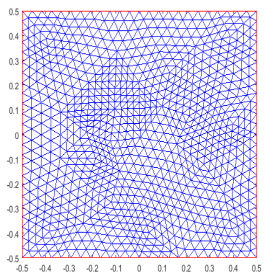

Figure 1: The mesh supported in the numerical tests with $h=0.0471$

3.2.1 Test 1

Firstly, we chose the nonsymmetric initial data on a square $(0;5)\times(0;5)$ and we present the numerical solution of (1) for different values of t. in this subsection, $\mu=1$, $\xi=1$, the time step is $\Delta t=10^{-3}$, the number of triangles is 1296 and the nonsymmetric initial functions is given by

$$n_{0,1}(x,y)=\frac{M}{2\pi\theta}exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2\theta}\right), \qquad (21)$$

with the total mass is $M=6\pi$, $\theta=10^{-2}$ and $x_0=y_0=0.1$.

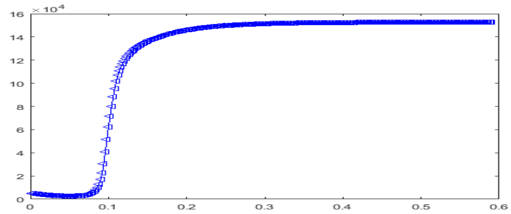

Figure 2: Test 1: Time evolution of $\|n\|_{L^{\infty}(\Omega)}$ computed from the radially non-symmetric initial datum $n_{0,1}$ with $M=6\pi$

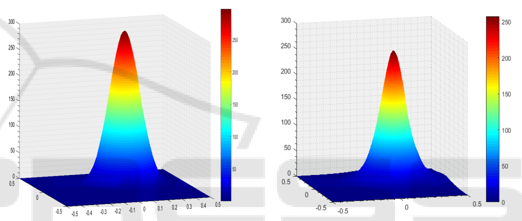

Figure 3: Initial datum $n_{0,1}$, $t=0$ and $t=0.005$

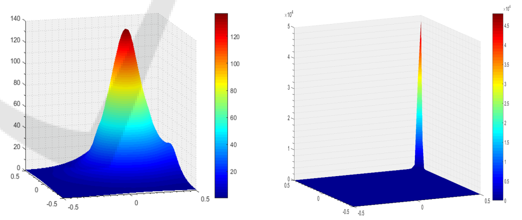

Figure 4: Initial datum $n_{0,1}$, $t=0.015$ and $t=1$

3.2.2 Test 2

Next, we present the numerical solution of (1) for different values of t. in this subsection, $\mu=1$, $\xi=1$, the time step is $\Delta t=10^{-3}$, the number of triangles is 1296 and the nonsymmetric initial functions is given by:

$$\begin{cases} n_{0,2}(x,y)=\frac{4\pi}{2\pi\theta}exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2\theta}\right)+ \\ \frac{2\pi}{2\pi\theta}exp\left(-\frac{(x-x_1)^2+(y-y_1)^2}{2\theta}\right), \end{cases} \qquad (22)$$

with $\theta=10^{-2}$, $x_0=y_0=0.1$, and $x_1=y_1=-0.2$.

Test 2: Cell density computed from nonsymmetric initial data with $M=6\pi$ for different values of t.

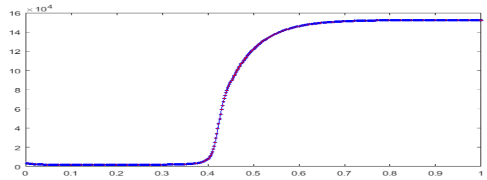

Figure 5: Test 2: Time evolution of $\|n\|_{L^\infty(\Omega)}$ computed from the radially non-symmetric initial datum $n_{0,2}$ with the total mass is $M = 6\pi$

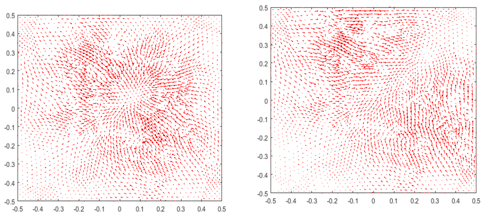

Figure 6: Initial datum $n_{0,2}$, $t = 0.01$ and $t = 0.045$

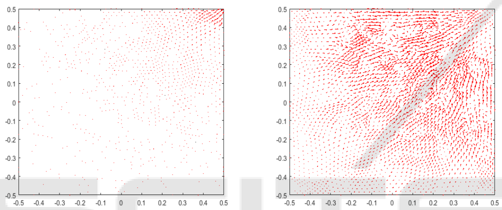

Figure 7: Initial datum $n_{0,2}$, $t = 0.55$ and $t = 0.15$

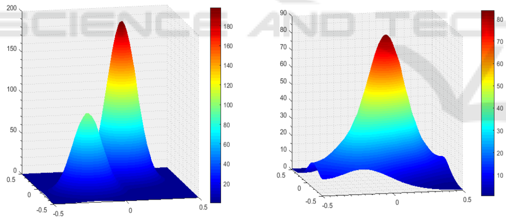

Figure 8: Initial datum $n_{0,1}$, $t = 0$ and $t = 0.01$.

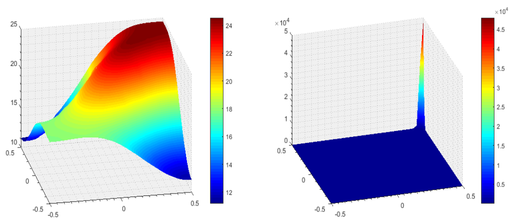

Figure 9: Initial datum $n_{0,1}$, $t = 0.045$ and $t = 1$

3.2.3 Test 3

We now consider the case of radially symmetric initial functions, we present the numerical solution of (1) for different values of δ. in this subsection, $\mu = 1$, $\Delta t = 2\times 10^{-2}$ and the radially symmetric initial function is given by:

$$n_{0,3}(x,y) = \frac{M}{2\pi\theta} exp\left(-\frac{x^2+y^2}{2\theta}\right), \qquad (23)$$

with the mass $M = 20\pi$, $\xi = 1$ and $\theta = 10^{-2}$.

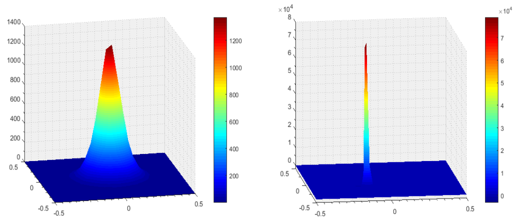

Figure 10: Initial datum $n_{0,3}$, $t = 0$ and $t = 1.5$

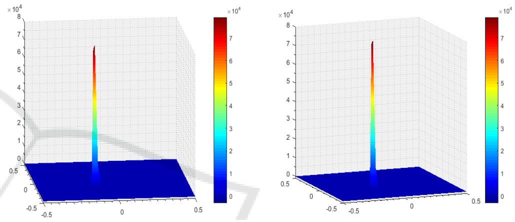

Figure 11: Initial datum $n_{0,3}$, $t = 3$ and $t = 5$

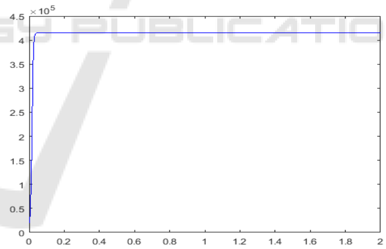

Figure 12: Test 3 : Time evolution of $\|n\|_{L^\infty(\Omega)}$ computed from the radially symmetric initial datum $n_{0,3}$ with $M = 20\pi$.

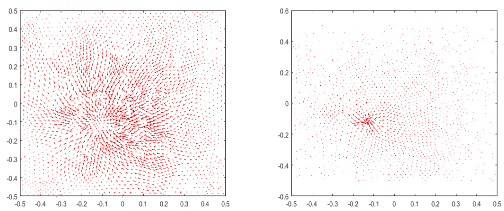

Figure 13: Test 3 : Chemical signal concentration computed from symmetric initial data with $M = 20\pi$ for $t = 0$ and $t = 5.5$.

4 CONCLUSION

Our numerical results, in the case of radially non-symmetrical initial data (test 1 and 2), show that the blow-up occurs at the nearest corner of the point of inoculation from the 0.1 instant for $n_{0;1}$ and 0.4 for $n_{0;2}$, which is compatible with cellular dynamics. In the case where the initial datum is radially symmetrical($test3$, the figures show that the explosion of the solution of Keller-Segel classical models occurs at the center of the domain.

REFERENCES

A.Chertock and A.Kurganov (2008). A second-order positivity preserving central-upwind scheme for chemotaxis and haptotaxis models. *Numer. Math.*, 111:169–205.

A.J.Carrillo, S.Hittmeir, A. u. (2012). Cross diffusion and nonlinear diffusion preventing blow up in the keller-segel model. *Math. Mod. Meth. Appl. Sci.*

AK.Domelevo, P. O. (2005). A finite volume method for the laplace equation on almost arbitrary two-dimensional grids. *M2AN Math. Model. Numer. Anal.*, 39(6):1203–1249.

A.Marrocco (2003). Numerical simulation of chemotactic bacteria aggregation via mixed finite elements. *Math. Mod. Numer. Anal.*, 4:617–630.

B.Andreianov, F. and F.Hubert (2007). Discrete duality finite volume schemes for leray-lions type elliptic problems on general 2d-meshes. *Num. Meth. for PDEs*, 23:145–195.

Bessemoulin-Chatard, M. and Jungel, A. (2014). A finite volume scheme for a keller-segel model with additional cross-diffusion. *IMA Journal of Numerical Analysis*, 34:96–122.

Boyer, F. and Hubert, F. (2008). Finite volume method for 2d linear and nonlinear elliptic problems with discontinuities. *SIAM J Numer Anal*, 46:3032—-3070.

C. Chainais-Hillairet, S. K. and Mouton, A. (2013). Study of discrete duality finite volume schemes for the peaceman model. *SIAM J. Sci. Comput.*, 35(6):A2928–A2952.

C.Chainais-Hillairet (2009). Discrete duality finite volume schemes for two-dimensional driftdiffusion and energy-transport models. *Internat J Numer Methods Fluids*, 59:239–57.

C.Chainais-Hillairet, S. Krell, A. and Mouton (2015). Convergence analysis of a ddfv scheme for a system describing miscible fluid flows in porous media. *Numerical Methods for Partial Differential Equations, Wiley*, 315(3):1098–2426.

Delcourte, S. (2007). *Développement de méthodes de volumes finis pour la mécanique des fluides*. Ph.d. thesis (in french), University of Toulouse III, France.

F. Hermeline, S. L. and Omnes, P. (2008). A finite volume method for the approximation of maxwell's equations in two space dimensions on arbitrary meshes. *J Comput Phys*, 227:9365—9388.

F.Hermeline (2000). A finite volume method for the approximation of diffusion operators on distorted meshes. *J. Comput. Phys.*, 160(2):481–499.

G.Chamoun, M. and R.TalhoukF.Filbet (2006). A finite volume scheme for the patlak-keller-segel chemotaxis model. *Numer. Math.*, 104:457–488.

Ha$\widehat{s}$kovec, J. and Schmeiser, C. (2009). Stochastic particle approximation for measure valued solutions of the 2d keller-segel system. *J. Stat. Phys.*, 135:133–151.

Ha$\widehat{s}$kovec, J. and Schmeiser, C. (2011). Convergence of a stochastic particle approximation for measure solutions of the 2d keller-segel system. *Commun. Part. Diff. Eqs.*, 36:940–960.

Keller, E. and Segel, L. (1970). Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:399–415.

Krell, S. (2011). Stabilized ddfv schemes for stokes problem with variable viscosity on general 2d meshes. *Numer Methods Partial Differential Eq*, pages 1666––1706.

Krell, S. (2012). Finite volume method for general multifluid flows governed by the interface stokes problem. *Math Models Methods Appl Sci*, 22(5):1150–025.

Krell, S. and Manzini, G. (2012). The discrete duality finite volume method for stokes equations on three-dimensional polyhedral meshes. *SIAM J Numer Anal*, 50:808—-837.

N.Saito (2007). Conservative upwind finite-element method for a simplified keller-segel system modelling chemotaxis. *IMA J. Numer. Anal.*, 27:332–365.

N.Saito (2012). Error analysis of a conservative finite-element approximation for the keller-segel model of chemotaxis. comm. *Pure Appl. Anal.*, 11:339–364.

N.Saito and T.Suzuki (2005). Notes on finite difference schemes to a parabolic-elliptic system modelling chemotaxis. *Appl. Math. Comput.*, 171:72–90.

Patlak, C. (2953). Random walk with persistence and external bias. *Bull. Math. Biophys*, 15:311–338.

R. Strehl, A. Sokolov, D. K. and Turek, S. (2010). A flux-corrected finite element method for chemotaxis problems. *Comput. Meth. Appl. Math.*, 10:219–232.

S.Delcourte, K. D. and P.Omnes (2007). A discrete duality finite volume approach to hodge decomposition and div-curl problems on almost arbitrary two-dimensional meshes. *SIAM J Numer Anal*, 45:1142—-1174.

Y. Coudiere, J.-P. V. and Villedieu, P. (1999). Convergence rate of a finite volume scheme for a two-dimensional convection-diffusion problem. *M2AN Math. Model. Numer. Anal.*, 33:493–516.

Y.Coudière and F.Hubert (2011). A 3d discrete duality finite volume method for nonlinear elliptic equations. *SIAM J Sci Comput*, 33:1739—-1764.

Y.Coudière and G.Manzini (2010). The discrete duality finite volume method for convectiondiffusion problems. *SIAM J Sci Comput*, 47:4163—-4192.