# Band-limited Orthogonal Functional Systems for Optical Fresnel Transform

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Keywords: Fresnel Transform, Dual Orthogonal Property, Eigenvalue Problem, Jacobi Method, Hermitian Matrix.

Abstract: The fundamental formula in an optical system is Rayleigh diffraction integral. In practice, we deal with Fresnel diffraction integral as approximate diffraction formula. By optical instruments, an optical wave is subject to a band limited. To reveal the band-limited effect in Fresnel transform plane, we seek the function that its total power in finite Fresnel transform plane is maximized, on condition that an input signal is zero outside the bounded region. This problem is a variational one with an accessory condition. This leads to the eigenvalue problems of Fredholm integral equation of the first kind. The kernel of the integral equation is Hermitian conjugate and positive definite. Therefore, eigenvalues are real non-negative numbers. Moreover, we also prove that the eigenfunctions corresponding to distinct eigenvalues have dual orthogonal property. By discretizing the kernel and integral calculus range, the eigenvalue problems of the integral equation depend on a one of the Hermitian matrix in finite dimensional vector space. We use the Jacobi method to compute all eigenvalues and eigenvectors of the matrix. We consider the application of the eigenvectors to the problem of approximating a function and showed the validity of the eigenvectors in computer simulation.

## 1 INTRODUCTION

In scalar diffraction theory, the Huygens-Fresnel principle is used to explain diffraction phenomenon. The integral theorem of Helmholtz and Kirchhoff plays an important role in the development of the scalar theory of diffraction. Although scalar wave propagation is fully described by a single scalar wave equation, fundamental formula in an optical system is Rayleigh diffraction integral. In practice, we deal with Fresnel diffraction integral as approximate diffraction formula. The Fresnel transform has been studied mathematically and shown to be a one-parameter group of unitary and factor-type operators from its algebraic and topological properties in Hilbert space ( $H(E_2)$) (Aoyagi, 1973, and Aoyagi et al., 1973a). In recently, it is also used in image processing, optical information processing, optical waveguides, computer-generated holograms, iterative phase retrieval techniques, speckle pattern interferometry and so on. In optical applications, an orthogonal functional system plays an important role. Up to now, many orthogonal functional systems have been derived in connection with the Fourier transform and applied to many applications. The extension of optical fields through an optical instrument is practically limited to some finite area. By band-limited effect in Fourier transform plane, sampling functional systems have been derived and have orthogonal property. From sampling theorem about the Fourier transform, orthogonal functional systems are formulated from the point of view of functional analysis (Ogawa, 2009). In the literature, there are many sampling theorems and examples about the Fourier transform. Its applications and references therein (Jerri, 1977). However, the property of the orthogonal function about Fresnel transform is not revealed sufficiently.

In this paper, the band-limited effect in Fresnel transform plane is investigated. For that, we seek the function that its total power in finite Fresnel transform plane is maximized, on condition that an input signal is zero outside the bounded region. This problem is a variational one with an accessory condition. This leads to the eigenvalue problems of Fredholm integral equation of the first kind (Kondo, 1954). The kernel of the integral equation is Hermitian conjugate and positive definite. Therefore, eigenvalues are real non-negative numbers. Moreover, we prove that the eigenfunctions corresponding to distinct eigenvalues have dual orthogonal property. By discretizing the kernel and



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integral range, the eigenvalue problems of the integral equation depend on a one of the Hermitian matrix in finite dimensional vector space ($\mathbb{C}^n$). We use the Jacobi method to compute all eigenvalues and eigenvectors of the matrix. We consider the application of the eigenvectors to the problem of approximating a function. We show the validity and limitations of the eigenvectors in computer simulation.

## 2 FRESNEL TRANSFORM

From the physical and mathematical standpoint, the fundamental formula in scalar diffraction theory is the Rayleigh diffraction integral guided by Helmholtz equation. The Rayleigh diffraction integral is defined as the Rayleigh diffraction operator on $H(E_2)$ which indicates the Hilbert space of all complex-valued square-integrable function defined on 2 dimensional Euclidean space. The Rayleigh diffraction operator is a bounded additive operator. The derivations of the transform formula of Fresnel diffraction are straightforward and reflect the traditional view that wave fields can be thought of as being generated by a distribution of point sources. Since wave field is expressed as a superposition of plane waves traveling in different directions, we can derive the Fresnel diffraction formula by restricting attention to plane wave components which are diffracted through small angles.

Assume that we place a diffracting screen on the $z = 0$ plane. The parameter $z$ represents the normal distance from the input plane. Let ξ , η be the coordinates of any point in that plane. Parallel to the screen at $z$ is a plane of observation. Let $x, y$ be the coordinates of any point in this latter plane. If $f(\xi,\eta)$ represents the amplitude transmittance in $H(E_2)$ , then the Fresnel transform is defined by

$$g(x,y;z) = \frac{k\exp(ikz)}{i2\pi z} \iint_{-\infty}^{\infty} f(\xi,\eta) \times \exp\left[\frac{ik}{2z}\{(x-\xi)^2+(y-\eta)^2\}\right] d\xi d\eta, \quad (1)$$

where $k$ is the wave number and $i = \sqrt{-1}$ . The inverse Fresnel transform is defined by

$$f(\xi,\eta) = -\frac{k\exp(-ikz)}{i2\pi z} \iint_{-\infty}^{\infty} g(x,y;z) \times \exp\left[-\frac{ik}{2z}\{(x-\xi)^2+(y-\eta)^2\}\right] dxdy. \quad (2)$$

Figure 1 shows a general optical system and its coordinate system. Fresnel transform and inverse Fresnel transform, which give a basis for Fresnel diffraction, are formulated systematically and mathematically in terms of Fresnel diffraction operator on $H(E_2)$. The Fresnel transform has been studied mathematically and shown to be a one-parameter group of unitary and factor-type operators from its algebraic and topological properties. In addition, a generalized Fresnel transform have been formulated by considering the transformation of the scalar-wave propagating between two quadratic surfaces within a paraxial approximation (Aoyagi et al., 1973b).

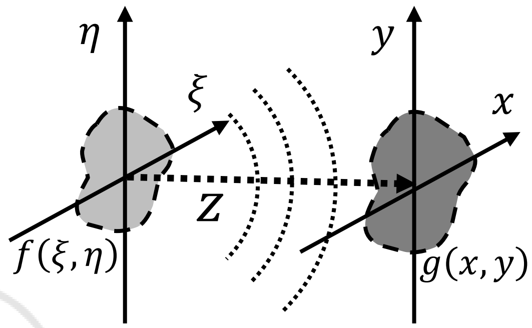


Figure 1: Sketch of a general optical system and its coordinate system.

## 3 EIGENVALUE PROBLEM

To simplify the discussion, we consider only one-dimensional Fresnel transform. The one-dimensional Fresnel transform is defined by

$$g(x,z) = \frac{1}{\sqrt{i2\pi z}} \int_{-\infty}^{\infty} f(\xi) \times\exp\left\{\frac{i}{2z}(x-\xi)^2\right\} d\xi, \quad (3)$$

where we set the wave number unit. The inverse Fresnel transform is defined by

$$f(\xi) = \sqrt{\frac{i}{2\pi z}} \int_{-\infty}^{\infty} g(x,z) \times \exp\left\{-\frac{i}{2z}(x-\xi)^2\right\} dx. \quad (4)$$

Assume that $f(\xi)$ is limited within the finite region $R$ on the ξ-plane and its total power $P_R$, namely the inner product of the function, is constant.

$$P_R = \int_R |f(\xi)|^2 \, d\xi = \text{const.} \quad (5)$$

Assume that $g(x)$ is the Fresnel transform of the function $f(\xi)$ which is bounded by a finite region R. Then, the total power $P_S$ of $g(x)$ in the bounded region $S$ is

$$P_S = \int_S |g(x)|^2\,dx = \int_S g^*(x)g(x)\,dx, \tag{6}$$

where $g^*(x)$ denotes the complex conjugate function of $g(x)$. We seek the function $f(\xi)$ that maximizes $P_S$ provided that the total power $P_R$ is fixed. This problem is a variational one with an accessory condition. We use the method of Lagrange multiplier to solve this problem (Aoyagi et al., 2018). Then, we can derive the integral equation, such that,

$$\int_R K_S(\xi,\xi')f(\xi')\,d\xi' = \lambda f(\xi), \tag{7}$$

where the kernel function $K_S(\cdot,\cdot)$ is defined by

$$\begin{aligned} K_S(\xi,\xi') &= \frac{1}{2\pi z}\exp\left\{-\frac{i}{2z}(\xi^2-\xi'^2)\right\} \\ &\times \int_S \exp\left\{\frac{i}{z}(\xi-\xi')x\right\}dx. \end{aligned} \tag{8}$$

This leads to the eigenvalue problems of Fredholm integral equation of the first kind. The kernel of the integral equation is Hermitian conjugate and positive definite. Therefore, eigenvalues are real non-negative numbers. This equation corresponds to some modification of the integral equation for the prolate spheroidal wave functions (Slepian et al., 1961, Landau et al., 1961, Landau et al., 1962). The integral equation and differential equation for the prolate spheroidal wave function have been generalized and revealed its properties (Slepian, 1964). Moreover, discrete prolate spheroidal functions and their mathematical properties have been investigated in great detail (Slepian, 1978). The prolate spheroidal wave functions have been applied to some optical problems (Itoh, 1970).

In our previous paper (Aoyagi et al., 2018), it was shown that the kernel of the integral equation is Hermitian conjugate and positive definite. It was also shown that by setting the finite region S in Fresnel transform plane to the fixed real number, the kernel is of Hermitian symmetry.

If $\lambda_m$ and $\lambda_n$ are distinct eigenvalues of the above integral equation, i.e. $m \neq n$, and $\varphi_m$, $\varphi_n$ are corresponding eigenfunctions, we can express them as the following integral formulas.

$$\int_R K_S(\xi,\xi')\varphi_m(\xi')\,d\xi' = \lambda_m\varphi_m(\xi). \tag{9}$$

$$\int_R K_S(\xi,\xi')\varphi_n(\xi')\,d\xi' = \lambda_n\varphi_n(\xi). \tag{10}$$

Let us consider the complex conjugate of the kernel of the integral equation.

$$\begin{aligned} K_S^*(\xi,\xi') &= \frac{1}{2\pi z}\exp\left\{\frac{i}{2z}(\xi^2-\xi'^2)\right\} \\ &\times \int_S \exp\left\{-\frac{i}{z}(\xi-\xi')x\right\}dx. \end{aligned} \tag{11}$$

From eq. (8), we have

$$\begin{aligned} K_S(\xi',\xi) &= \frac{1}{2\pi z}\exp\left\{-\frac{i}{2z}(\xi'^2-\xi^2)\right\} \\ &\times \int_S \exp\left\{\frac{i}{z}(\xi'-\xi)x\right\}dx \\ &= \frac{1}{2\pi z}\exp\left\{\frac{i}{2z}(\xi^2-\xi'^2)\right\} \\ &\times \int_S \exp\left\{-\frac{i}{z}(\xi-\xi')x\right\}dx. \end{aligned} \tag{12}$$

Therefore, we obtain

$$K_S^*(\xi,\xi') = K_S(\xi',\xi), \tag{13}$$

and the integral kernel $K_S(\xi,\xi')$ is of Hermitian symmetry. If we multiply the both sides of eq. (9) by $\varphi_n^*(\xi)$ and integrate with respect to over $R$, we obtain

$$\begin{aligned} &\int_R\int_R K_S(\xi,\xi')\varphi_m(\xi')\varphi_n^*(\xi)d\xi'd\xi \\ &= \lambda_m \int_R \varphi_m(\xi)\varphi_n^*(\xi)d\xi. \end{aligned} \tag{14}$$

After taking the complex conjugate of eq. (10), we multiply the both sides by $\varphi_m(\xi)$ and integrate with respect to over $R$, we obtain

$$\begin{aligned} &\int_R\int_R K_S^*(\xi,\xi')\varphi_m(\xi)\varphi_n^*(\xi')d\xi'd\xi \\ &= \lambda_n^* \int_R \varphi_m(\xi)\varphi_n^*(\xi)d\xi. \end{aligned} \tag{15}$$

From eq. (15) and eq. (13), we obtain

$$\begin{aligned} &\int_R\int_R K_S(\xi',\xi)\varphi_m(\xi)\varphi_n^*(\xi')d\xi'd\xi \\ &= \lambda_n^* \int_R \varphi_m(\xi)\varphi_n^*(\xi)d\xi. \end{aligned} \tag{16}$$

Because the left side of eq. (14) and the right side of eq. (16) are equal and $\lambda_n$ is real number, we have

$$(\lambda_m-\lambda_n)\int_R \varphi_m(\xi)\,\varphi_n^*(\xi)d\xi = 0. \tag{17}$$

For $\lambda_m \neq \lambda_n$, we conclude

$$\int_R \varphi_m(\xi)\,\varphi_n^*(\xi)d\xi = 0. \tag{18}$$

That is to say, $\varphi_m(\xi)$ and $\varphi_n(\xi)$ are orthogonal on $R$.

Let us consider the extension of the domain of ξ into one-dimensional Euclidean space $E$. Now we can redefine the following integral equation.

$$\int_R K_S(\xi,\xi')\varphi(\xi')d\xi' = \lambda\varphi(\xi) \;\; \xi \in E, \tag{19}$$

where $E$ denotes one-dimensional Euclidean space. Then, for the eigenfunctions $\varphi_m$, and $\varphi_n$, $m \neq n$ , we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_m(\xi)\varphi_n^*(\xi)d\xi \\ &= \frac{1}{\lambda_m\lambda_n}\int_R\int_R \varphi_m(\xi')\varphi_n^*(\xi'') \\ &\times \int_{-\infty}^{\infty} K_S(\xi,\xi')K_S^*(\xi,\xi'')d\xi d\xi' d\xi''. \end{aligned} \tag{20}$$

We need to consider the integral part about the kernel.

$$\begin{aligned} &\int_{-\infty}^{\infty} K_S(\xi,\xi')K_S^*(\xi,\xi'')d\xi \\ &= \frac{1}{(2\pi z)^2}\exp\left\{\frac{i}{2z}(\xi'^2 - \xi''^2)\right\} \\ &\times \int_{-\infty}^{\infty}\int_S\int_S \exp\left\{\frac{i}{z}(\xi-\xi')x \right. \\ &\left. - \frac{i}{z}(\xi-\xi'')x'\right\} dx dx' d\xi \\ &= \frac{1}{(2\pi z)^2}\exp\left\{\frac{i}{2z}(\xi'^2 - \xi''^2)\right\} \\ &\times \int_S\int_S \exp\left\{-\frac{i}{z}(\xi' x + \xi'' x')\right\} \\ &\times \int_{-\infty}^{\infty} \exp\left\{\frac{i}{z}(x-x')\xi\right\} d\xi dx dx'. \end{aligned} \tag{21}$$

By using the delta function $\delta(\cdot)$, as shown in the Appendix, such that,

$$\delta(x-x') = \frac{1}{2\pi z}\int_{-\infty}^{\infty} \exp\left\{\frac{i}{z}(x-x')\xi\right\} d\xi, \tag{22}$$

the above equation can be expressed by the following form.

$$\begin{aligned} &\frac{1}{2\pi z}\exp\left\{\frac{i}{2z}(\xi'^2 - \xi''^2)\right\} \\ &\times \int_S\int_S \exp\left\{-\frac{i}{z}(\xi' x + \xi'' x')\right\}\delta(x \\ &- x')dx dx' \\ &= \frac{1}{2\pi z}\exp\left\{\frac{i}{2z}(\xi'^2 - \xi''^2)\right\} \\ &\times \int_S \exp\left\{-\frac{i}{z}(\xi' - \xi'')x\right\} dx \\ &= K_S^*(\xi',\xi''). \end{aligned} \tag{23}$$

Substituting eq. (23) into eq. (21), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \varphi_m(\xi)\varphi_n^*(\xi)d\xi \\ &= \frac{1}{\lambda_m\lambda_n}\int_R\int_R \varphi_m(\xi')\varphi_n^*(\xi'')K_S^*(\xi',\xi'')d\xi' d\xi' \\ &= \frac{1}{\lambda_m}\int_R \varphi_m(\xi')\,\varphi_n^*(\xi')d\xi'. \end{aligned} \tag{24}$$

If the functional systems $\{\varphi_m(\xi)\}$ are orthogonal on $E$, these also are orthogonal on $R$. Therefore, the orthogonal functional systems have dual orthogonal property.

Dual orthogonal property means that the functional systems have the orthogonality of the functions over two different intervals. It can expand any function in two different intervals. Orthogonal functional systems have important role in expanding the objective functions by using basis functions. In numerical computation, it is necessary to discretize the objective function. We derived dual orthogonal functional systems and revealed its property. These lead to reveal the relation between functions and their Fresnel transforms.

## 4 NUMERICAL COMPUTATION

It is difficult in general to seek the strict solution of the integral equation. So we desire to seek the approximate solution in practical exact accuracy. By discretizing the kernel function and integral calculus range at equal distance, and using the value of the discrete sampling points, we can write

$$\sum_{j=1}^{N} K_{ij}x_j = \lambda x_i, \tag{25}$$

where $i$, $j$ are the natural number, $1 \leq i \leq M$. The matrix $K_{ij}$ is the Hermitian matrix if the kernel is discretized evenly-spaced and $M = N$.

Therefore, the eigenvalue problems of the integral equation depend on one of the Hermitian matrix in finite dimensional vector space. In general finite dimensional vector spaces ($\mathbb{C}^n$), the eigenvalues of Hermitian matrix are real numbers and then eigenvectors from different eigenspaces are orthogonal (Anton et al., 2003). We use the Jacobi method (Press et al., 1992) to compute all eigenvalues and eigenvectors of the matrix. The Jacobi method is a procedure for the diagonalization of complex symmetric matrices, using a sequence of plane rotations through complex angles. All eigenvectors

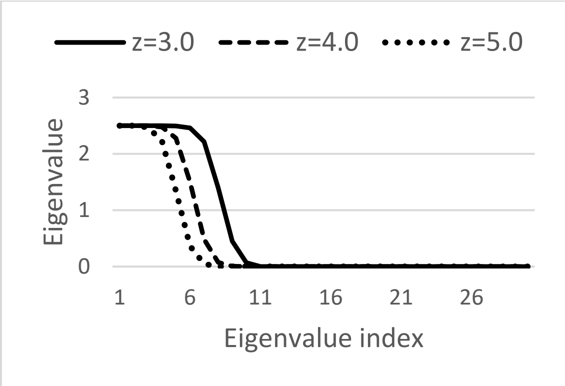


Figure 2: Plots of the eigenvalues in descending order. $S = [-6{,}6], R = [-6{,}6]$.

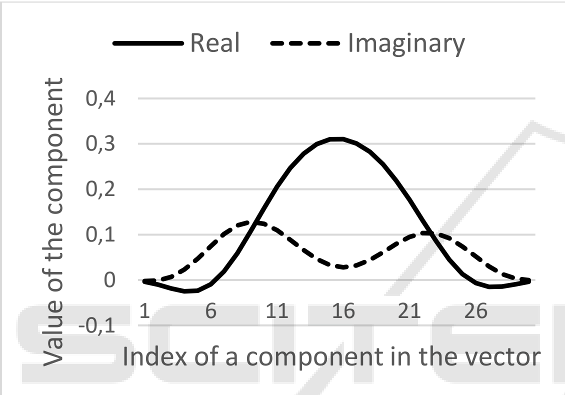


Figure 3:Plots of the eigenvectors for the largest eigenvalue. $S = [-6{,}6], R = [-6{,}6]$. z=5.0.

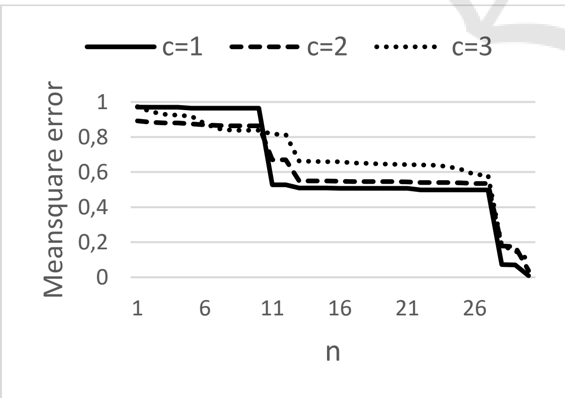


Figure 4: Plots of the normalized mean square error versus the number of eigenvectors.

computed by the Jacobi method is of orthonormal vectors automatically. Now, we set $M = N = 30$.
Figure 2 shows the eigenvalues in descending order, if z is 3.0, 4.0 and 5.0. They are nonnegative and real number. Figure 3 shows the real part and imaginary part of the eigenvectors for the largest eigenvalue at

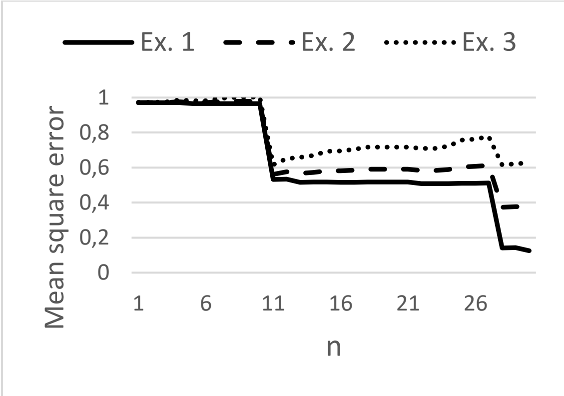


Figure 5: Plots of the normalized mean square error versus the number of eigenvectors. Original function is added by noise with white Gaussian. (Ex. 1) 18.0 dB SNR;(Ex. 2) 8.4dB SNR;(Ex. 3) 4.0dB SNR.

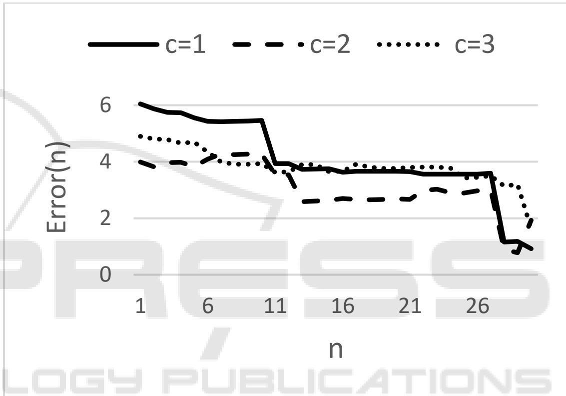


Figure 6: Plots of the mean square error versus the number of eigenvectors for the phase without noise.

$z = 5.0$. Because of 30 dimensional vector space, except for this, there are 29 eigenvectors.

We consider the application of the above eigenvectors to the problem of approximating a function. Theoretically, we deal with a problem of expressing an arbitrary element on a finite $N$-dimensional Hilbert space $H_N$ with an orthonormal basis. For any element $\boldsymbol{v}$ in $H_N$, by using orthonormal basis $\{\psi_n\}_{n=1}^{N}$, we can write

$$\boldsymbol{v} = \sum_{n=1}^{N} \langle \boldsymbol{v}, \psi_n \rangle \psi_n, \tag{26}$$

where $\langle \cdot,\cdot \rangle$ is an inner product (Reed et al., 1972) . Now, we set $N = 30$. Let us consider the set $\mathbb{C}^{30}$ of all 30-tuples

$$\boldsymbol{v} = (v_1, v_2, \cdots, v_{30}), \tag{27}$$

where $v_1, v_2, \cdots, v_{30}$ are complex numbers.
Now, let us consider a following test function.

$$f(x) = \sin(cx), x \in [0{,}2\pi] \tag{28}$$

where c is natural number. We evenly discretize the test function at 30 points to reconstruct by using the eigenvectors. Figure 4 illustrates the mean square error versus the number of eigenvectors. The normalized mean square error is defined by

$$\mathrm{Error}(n) = \frac{\|\boldsymbol{v}_n - \boldsymbol{v}\|_2}{\|\boldsymbol{v}\|_2}, \tag{29}$$

where $\boldsymbol{v}_n$ is the sum in Eq. (26) up to $n$, $\boldsymbol{v}$ is the original vector and $\|\cdot\|_2$ is the $\ell^2$-norm. From Fig. 4, we can see that the error decreases with increasing number of eigenvectors used in the expansion.

Next, let us consider another following test function.

$$f(x) = \sin(x) + \mathrm{q},\ x \in [0, 2\pi], \tag{30}$$

where q indicates noise and is a normally distributed deviate with zero mean and unit variance. To measure the effect of noise on the function, we use the signal-to-noise ratio (SNR) (Trussel, 2008). This is usually defined as the ratio of signal power $\sigma_f^2$, to noise power $\sigma_q^2$,

$$\mathrm{SNR} = \frac{\sigma_f^2}{\sigma_q^2}, \tag{31}$$

and in decibels

$$\mathrm{SNR_{dB}} = 10\log_{10}\left(\frac{\sigma_f^2}{\sigma_q^2}\right). \tag{32}$$

In $\mathbb{C}^{30}$, the function power is usually estimated by the simple summation

$$\sigma_f^2 = \frac{1}{30}\sum_{i=1}^{30}\left\{f(x_i) - \mu_f\right\}^2, \tag{33}$$

where $\mu_f$ is the mean of the function. Figure 5 illustrates the mean square error versus the number of eigenvectors with noise. The SNR in example 1 (Ex.1) is 18.019354, example 2 (Ex. 2) is 8.476929 and example 3 (Ex. 3) is 4.039954. From Fig. 5, we can see that the original test function is reconstructed in the state that is almost perfection if SNR increases. In general, it is difficult for the small value of SNR to reconstruct original test function completely. Figure 6 illustrates the mean square error versus the number of eigenvectors for the phase without noise. The mean square error is defined by

$$\mathrm{Error}(n) = \|\boldsymbol{v}_n - \boldsymbol{v}\|_2. \tag{34}$$

From Fig. 6, we can see also that the error decreases with increasing number of eigenvectors used in the expansion for the phase.

## 5 CONCLUSIONS

Band-limited effects with respect to Fourier transform have already been investigated and well known. However, those with respect to Fresnel transform have not been studied and revealed sufficiently. We have investigated the band-limited effect in Fresnel transform plane. For that, we have sought the function that its total power in finite Fresnel transform plane is maximized, on condition that an input signal is zero outside the bounded region. We have shown that this leads to the eigenvalue problems of Fredholm integral equation of the first kind. It is important to reveal the mathematical properties of the integral equation for finite Fresnel transform. Orthogonal eigenfunctions are derived from its properties. Orthogonal functional systems are significant tools in analysing a diffraction image. We have also shown that the eigenfunctions corresponding to distinct eigenvalues have dual orthogonal property. These functional systems and its properties show clearly the relation between functions and their Fresnel transforms. It is difficult in general to seek the strict solution of the integral equation. So we desired to seek the approximate solution in practical exact accuracy. Furthermore, we applied it to the problem of approximating a function and evaluated the error. We confirmed the validity of the eigenvectors for finite Fresnel transform by computer simulations.

In this study, there are many parameters, especially, the band-limited areas $S, R$, the wave number $k$ and the normal distance $z$. It is necessary to consist of orthogonal functional systems with the optimal parameters for finite Fresnel transform in application of an optical system. Moreover, in general, the matrix given by discretizing the kernel of the integral equation is not the Hermitian matrix. If so, it is difficult to compute accurately all eigenvalues and eigenvectors. It is also necessary to consider other computational methods for this. Although the kernel function was discretizing at 30 point, it is necessary to increase the number of sampling points. Although we considered only one dimensional Fresnel transform, it is necessary to derive the integral equation for the two dimensional Fresnel transform. These become the future problems. Theoretically, it is important to search for a spectral representation of finite Fresnel transform which are defined as a bounded linear operator in Hilbert space.

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# APPENDIX

The delta function can be defined as follows (Goodman, 2005);

$$\delta(x) = \lim_{N\to\infty} N\mathrm{sinc}(Nx), \tag{35}$$

where

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{36}$$

Noting that

$$\begin{aligned}\int_{-N}^{N} \exp\left\{i\frac{1}{z}(x-x')\xi\right\}d\xi &= \left[\exp\left\{i\frac{1}{z}(x-x')N\right\} - \exp\left\{-i\frac{1}{z}(x-x')N\right\}\right] \Big/ \left\{i\frac{1}{z}(x-x')\right\} \\ &= \frac{2z}{x-x'}\sin\frac{1}{z}(x-x')N,\end{aligned} \tag{37}$$

we can define $S_N(\cdot)$ as following.

$$\begin{aligned} S_N(x-x') &= \sin\frac{\pi}{z}(x-x')N \Big/ \left\{\frac{\pi}{z}(x-x')N\right\} \\ &= \frac{x-x'}{2z}\int_{-\pi N}^{\pi N} \exp\left\{i\frac{1}{z}(x-x')\xi\right\}d\xi \Big/ \left\{\frac{\pi}{z}(x-x')N\right\} \\ &= \frac{1}{2\pi N}\int_{-\pi N}^{\pi N} \exp\left\{i\frac{1}{z}(x-x')\xi\right\}d\xi. \end{aligned} \tag{38}$$

We conclude that

$$\begin{aligned}\delta(x-x') &= \lim_{N\to\infty} NS_N(x-x') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{i\frac{1}{z}(x-x')\xi\right\}d\xi.\end{aligned} \tag{39}$$