Approximation of the Distance from a Point to an Algebraic Manifold

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Abstract: The problem of geometric distance d evaluation from a point X_0 to an algebraic curve in $\mathbb{R}^2$ or manifold $G(X) = 0$ in $\mathbb{R}^3$ is treated in the form of comparison of exact value with two its successive approximations $d_{(1)}$ and $d_{(2)}$. The geometric distance is evaluated from the univariate *distance equation* possessing the zero set coinciding with that of critical values of the function $d^2(X_0)$, while $d_{(1)}(X_0)$ and $d_{(2)}(X_0)$ are obtained via expansion of $d^2(X_0)$ into the power series of the algebraic distance $G(X_0)$. We estimate the quality of approximation comparing the relative positions of the level sets of $d(X)$, $d_{(1)}(X)$ and $d_{(2)}(X)$.

1 INTRODUCTION

We treat the problem of Euclidean distance evaluation from a point X_0 to manifold defined implicitly by the equation

$$G(X) = 0 \qquad (1)$$

in $\mathbb{R}^n, n \in \{2,3\}$. Here $G(X)$ is twice differentiable real valued function and it is assumed that the equation (1) defines a nonempty set in $\mathbb{R}^n$. This problem arises in image processing, multi-object movement simulation, and in the scattered data approximation problems. Being the problem of nonlinear constrained optimization, it can be solved with the aid of traditional Newton-like iteration methods. However for the applications connected with the parameter synthesis such as, for instance, the manifold selection best fitting to the given data set (Ahn et al., 2002; Aigner and Jutler, 2009; Cheng and Chiu, 2014), an analytical representation is needed for the distance as a function of parameters of the problem (point coordinates and coefficients of $G(X)$ if the latter is a polynomial).

We concern here with the following approximations for the distance d

$$d_{(1)} = |G|/\|\nabla G\|, \qquad (2)$$

$$d_{(2)} = d_{(1)}\left(1 + \frac{\nabla G^{\top} \cdot \mathcal{H}(G) \cdot \nabla G}{2\|\nabla G\|^4} G\right) \qquad (3)$$

Here ∇G stands for the gradient column vector, $^{\top}$ — for transposition, $\mathcal{H}(G)$ is the Hessian of $G(X)$, $\|\cdot\|$ is the Euclidean norm, and the right-hand sides in (2) and (3) are calculated at $X = X_0$.

Approximation (2) is known as *Sampson's distance* (Sampson, 1982). We aim at comparison of the qualities of approximations (2) and (3). We deal mostly with the case of algebraic manifolds (1), i.e. $G(X) \in \mathbb{R}[X]$. For this case, the tolerances of the approximations can be evaluated via comparison with the true (*geometric*) distance value determined from the so-called **distance equation** (Uteshev and Goncharova, 2017; Uteshev and Goncharova, 2018). In Section 2, we outline the background of this approach for the case of a quadric manifold, while in Section 3 we extend it to the case of manifold of an arbitrary order. In Section 4 we discuss an applicability of the proposed approximations to the case of non algebraic curve (1).

2 QUADRICS

For the particular case of quadric polynomials $G(X)$:

$$G(X) := X^{\top}\mathbf{A}X + 2B^{\top}X - 1 = 0, \qquad (4)$$

where $\mathbf{A} = \mathbf{A}^{\top} \in \mathbb{R}^{n\times n}$ and $\{X, B\} \subset \mathbb{R}^n$ are the column vectors, formula (2) is represented as

$$d_{(1)} = \frac{1}{2} \cdot \frac{|G(X_0)|}{\sqrt{(\mathbf{A}X_0 + B)^{\top}(\mathbf{A}X_0 + B)}}. \qquad (5)$$

The counterpart for the formula (3) can be originated from the following approximation

$$\widetilde{d}_{(2)} = d_{(1)}\sqrt{1 + \frac{1}{2}\frac{(\mathbf{A}X_0 + B)^{\top}\mathbf{A}(\mathbf{A}X_0 + B)}{[(\mathbf{A}X_0 + B)^{\top}(\mathbf{A}X_0 + B)]^2} G(X_0)}; \qquad (6)$$

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suggested in (Uteshev and Goncharova, 2018). For this particular case, both approximations $d_{(1)}$ and $\widetilde{d}_{(2)}$ can be deduced via the following consideration. First compute the **distance equation**, i.e. an algebraic equation $\mathcal{F}(z) = 0, \mathcal{F}(z) \in \mathbb{R}[z]$ whose roots coincide with the critical values of the squared distance function from X_0 to (4), and, generically, the smallest positive root of this equation equals d^2. For the case of an ellipse $G(x,y) := x^2/a^2 + y^2/b^2 - 1 = 0$ and $X_0 = (x_0, y_0)$, this equation takes the form

$$\mathcal{F}(z, x_0, y_0) := L^2 z^4 \tag{7}$$

$$-2L\left\{L(a^2+b^2+x_0^2+y_0^2)+a^2y_0^2-b^2x_0^2\right\}z^3$$

$$+\left\{6L[a^4y_0^2+a^2y_0^4-b^4x_0^2-b^2x_0^4+L(a^2b^2+x_0^2y_0^2)]\right.$$

$$\left.+[L^2-(a^2x_0^2+b^2y_0^2)]^2\right\}z^2-2a^2b^2\left\{a^2b^2MG_0^2\right.$$

$$-\left[(a^2+b^2)M^2+3a^2b^2M-6a^4b^4S_4\right]G_0$$

$$\left.+2a^2b^2M^2S_4\right\}z+a^4b^4G_0^2\left(M^2+4a^2b^2G_0\right)=0\,.$$

Here $L := a^2 - b^2$, $G_0 := G(x_0, y_0)$,

$$M := x_0^2 + y_0^2 - a^2 - b^2,\ S_4 := x_0^2/a^4 + y_0^2/b^4\,.$$

Represent the squared distance value as the formal series

$$d^2(x_0, y_0) = \ell_2 G_0^2 + \ell_3 G_0^3 + \dots$$

in powers of the *algebraic distance* G_0 (which can be treated as a small parameter in a vicinity of the considered ellipse) and substitute it into the distance equation. Equate to zero the coefficients of G_0^2 and G_0^3. This results in the approximations (5) and (6). Further expansion of the radical in (6) in power series of G_0, yields an analogue of approximation (3). Similar distance equation of the degree 6 in z can be written down for the point-to-quadric problem in $\mathbb{R}^3$. The general expression (valid for a quadric in $\mathbb{R}^n$ for arbitrary n) can be represented via a special function of the coefficients of a polynomial known as the *discriminant*.

We first remind a more general notion. For the univariate polynomials $\{f(x), g(x)\} \subset \mathbb{R}[x]$ their **resultant** $\mathcal{R}_x(f,g)$ can be defined in the form of Sylvester's determinant (Uspensky, 1948). For instance, if

$$f(x) := a_0x^4 + \dots + a_4,\ g(x) := b_0x^3 + \dots + b_3,$$

$a_0 \neq 0$, $b_0 \neq 0$, then this determinant equals

$$\left|\begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \end{array}\right|$$

while for the general case it is composed similarly from $\deg g$ rows of coefficients of $f(x)$ and $\deg f$ rows of coefficients of $g(x)$. The resultant is a polynomial function of the coefficients of $f(x)$ and $g(x)$, and its vanishment yields the necessary and sufficient condition for the existence of a common zero for these polynomials.

For the particular case $g(x) \equiv f'(x)$, the expression

$$\mathcal{D}_x(f(x)) := \mathcal{R}_x(f, f')/a_0$$

defines (up to a sign) the **discriminant** of the polynomial $f(x)$. Its vanishment yields the necessary and sufficient condition for the existence of a multiple zero for $f(x)$.

Theorem 1. *Let $G(X_0) \neq 0$. Distance from X_0 to the quadric (4) equals the square root from the minimal positive zero of the distance equation*

$$\mathcal{F}(z) := \mathcal{D}_\mu(\Phi(\mu, z)) = 0\,, \tag{8}$$

where

$$\Phi(\mu, z) := \det\left(\begin{bmatrix} \mathbf{A} & B \\ B^\top & -1 \end{bmatrix} + \mu \begin{bmatrix} -\mathbf{I} & X_0 \\ X_0^\top & z - X_0^\top X_0 \end{bmatrix}\right)$$

provided that this zero is not a multiple one. Here $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity matrix.

In (Uteshev and Goncharova, 2018) error estimations for the approximations (5) and (6) are deduced in terms of maximal deviations of the manifolds $d_{(1)} = const$ and $\widetilde{d}_{(2)} = const$ from the quadric (4).

Example 1. *For the ellipse $x^2/18^2 + y^2/5^2 = 1$, approximate equidistant curves $d_{(1)}(x,y) = 3$ and $d_{(2)}(x,y) = 3$ in comparison with the true equidistant $d(x,y) = 3$ are presented in Fig. 1 and 2 respectively.*

3 GENERAL ALGEBRAIC MANIFOLDS

For the general case of algebraic manifold, the construction of distance equation is also possible, at least, in principle. For the planar case, this construction is based on the following results (Uteshev and Goncharova, 2017):

Theorem 2. *Let* $G(0,0) \neq 0$ *and* $G(x,y)$ *be an even polynomial in* y. *Expand* G *in powers of* y^2 *and denote* $\widetilde{G}(x,y^2) \equiv G(x,y)$. *Equation* $G(x,y) = 0$ *does not define a real curve if*

(a) *equation* $G(x,0) = 0$ *does not have real zeros and*

(b) *equation*

$$\mathcal{F}(z) := \mathcal{D}_x(\widetilde{G}(x, z - x^2)) = 0 \qquad (9)$$

does not possess positive zeros. If any of these conditions fails then the distance from $X_0 = (0,0)$ *to the curve* $G(x,y) = 0$ *equals either the minimal absolute value of real zeros of the equation* $G(x,0) = 0$ *or the square root from the minimal positive zero of the equation (9) provided that this zero is not a multiple one.*

The generalization of this result to the case of an arbitrary polynomial $G(x,y)$, not necessarily even in any of its variables, can be performed by reduction to the just treated one via artificial *evenization* of the problem. Unfortunately this causes the appearance of an extraneous factor in the distance equation.

Theorem 3. *Let* $G(0,0) \neq 0$ *and* $G(x,y)$ *be not an even polynomial in* y. *Split* G *into the sum of even and odd terms in this variable:*

$$G(x,y) \equiv G_1(x,y^2) + yG_2(x,y^2),\ \{G_1, G_2\} \subset \mathbb{R}[x,y^2].$$

Denote

$$\widetilde{G}(x,y^2) := G(x,y)G(x,-y) \equiv G_1^2(x,y^2) - y^2 G_2^2(x,y^2)$$

and compute the polynomial $\mathcal{F}(z)$ *via (9). The latter is reducible over* $\mathbb{R}$:

$$\mathcal{F}(z) \equiv \mathcal{F}_1(z)\mathcal{F}_2^2(z)$$

with $\mathcal{F}_2(z) := \mathcal{R}_x(G_1(x, z - x^2), G_2(x, z - x^2))$.
Equation $G(x,y) = 0$ *does not define a real curve if*

(a) *equation* $G(x,0) = 0$ *does not possess real zeros and*

(b) *equation* $\mathcal{F}_1(z) = 0$ *does not possess positive zeros.*

If any of these conditions fails then the distance from $X_0 = (0,0)$ *to the curve* $G(x,y) = 0$ *equals either the minimal absolute value of real zeros of the equation* $G(x,0) = 0$ *or the square root from the minimal positive zero of the equation* $\mathcal{F}_1(z) = 0$ *provided that this zero is not a multiple one.*

Remark 1. The conditions **(a)** and **(b)** of Theorems 2 and 3 (as well as their counterparts from the undermentioned Theorem 5) can be verified without utilization of any numerical method. Indeed, there exist symbolic algebraic algorithms (Sturm's theorem (Uspensky, 1948) or Joachimsthal's theorem (Kalinina and Uteshev, 1993)) permitting one to find the exact number of real zeros of a univariate polynomial lying within any given interval.

Example 2. *For the cubic*

$$-2x^3 + 6xy^2 + y^3 - 13x^2 - 24xy - 7y^2$$
$$+3x + 9y - 6 = 0,$$

the polynomial $\mathcal{F}_1(z)$ *from Theorem 3 takes the form*

$$234000\,z^9 - 16231720\,z^8 + 424939357\,z^7$$
$$-5350750701\,z^6 + 34854257973\,z^5$$
$$-113424352224\,z^4 + 148842276936\,z^3$$
$$-13100614064\,z^2 - 25191108960\,z - 7233825600\,.$$

Its minimal positive zero equals $z_* \approx 0.737416$. *Distance from the origin to the cubic equals* $\sqrt{z_*} \approx$ 0.858729. □

Application of Theorems 2 and 3 for the polynomial $\widetilde{G}(x,y) \equiv G(x + x_0, y + y_0)$ results in the distance equation

$$\mathcal{F}_1(z, x_0, y_0) = 0$$

for an arbitrary $X_0 = (x_0, y_0)$. Unfortunately, for the polynomial $G(x,y)$ of a degree higher than 2, we are not able to provide an explicit representation for the coefficients of this equation in terms of algebraic distance $G(x_0, y_0)$ similar to that from (7) for the ellipse represented in canonical form. The only details on the structure of this equation are contained in the following result:

Theorem 4. *If* $\mathcal{F}_1(z,x,y)$ *is treated as a polynomial in* z *then, generically,* $\deg_z \mathcal{F}_1 = (\deg G)^2$ *and its free term is divisible by* $G^2(x,y)$.

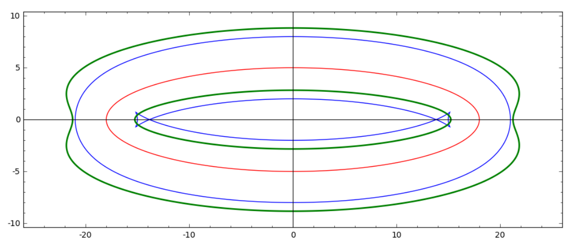

Figure 1: Example 1. Curve $d_{(1)} = 3$ (green boldface) vs. equidistant $d = 3$ (thin blue) to the ellipse (thin red).

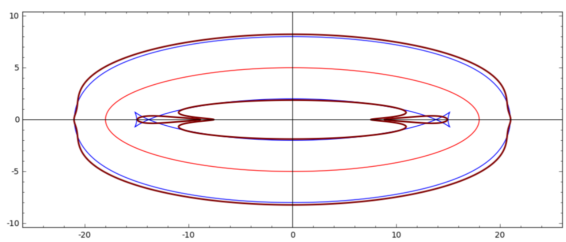

Figure 2: Example 1. Curve $d_{(2)} = 3$ (maroon boldface) vs. equidistant $d = 3$ (thin blue) to the ellipse (thin red).

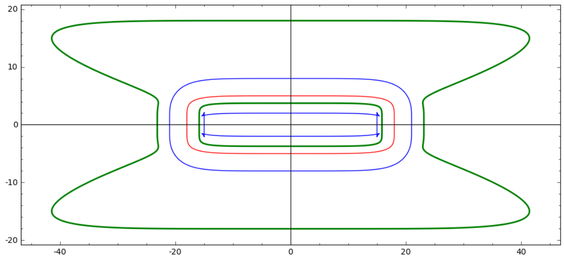

Figure 3: Example 3. Curve $d_{(1)} = 3$ (green boldface) vs. equidistant $d = 3$ (thin blue) to the superellipse (thin red).

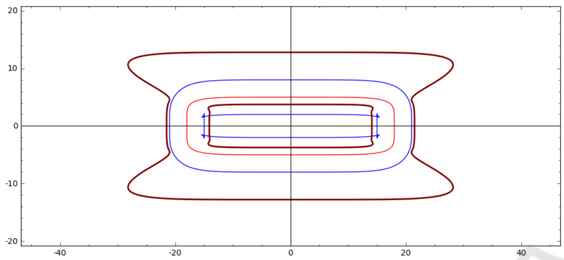

Figure 4: Example 3. Curve $d_{(2)} = 3$ (maroon boldface) vs. equidistant $d = 3$ (thin blue) to the superellipse (thin red).

Therefore, the trick exploited in Section 2 for justification of origination of the sequence of approximations (2) and (3) can not be repeated. However, we are still able to compare the loci of the level sets $d_{(1)}(x,y) = h$ and $d_{(2)}(x,y) = h$ of the proposed approximations with that of the equidistant manifold $\mathcal{F}_1(h^2,x,y) = 0$ for (1).

Example 3. *For the superellipse* $x^6/18^6 + y^6/5^6 = 1$, *one has* $\deg_z \mathcal{F}_1(z,x_0,y_0) = 36$. *The leading terms of* $\mathcal{F}_1(z,x,y)$ *treated as a polynomial in* x,y *are as follows:*

$$(x^2+y^2)^{30}(x^6/18^6+y^6/5^6)^2 .$$

The equidistant curve $\mathcal{F}_1(9,x,y) = 0$ *as well as the curves* $d_{(1)}(x,y) = 3$ *and* $d_{(2)}(x,y) = 3$ *are drawn in Fig. 3 and Fig. 4 correspondingly. Some sample values for the distance and its approximations are presented in the following table:*

(x_0,y_0)	$(8,4)$	$(19,1)$	$(10,6)$	$(14,6)$	$(20,2)$
$d_{(1)}$	1.856	0.877	0.675	0.739	1.568
$d_{(2)}$	0.297	0.978	0.865	0.966	1.876
d	0.993	1.000	1.025	1.199	2.011

Remark 2. If the curve $G(x,y) = 0$ contains a closed branch (oval) then the quality of distance approximations (2) and (3) for a point lying inside this branch becomes the farther the worse than for that lying outside at the same distance. One can observe this in the above example with the point $(8,4)$ inside the superellipse and the others outside.

The treatment of the 3D case is carried out similarly using the notion of the discriminant of a bivariate polynomial. We do not give here a formal definition but restrict ourselves by presenting formula for its iterative computation via the univariate discriminants

$$\mathcal{D}_{x,y}(f(x,y)) = \gcd(\mathcal{D}_x(\mathcal{D}_y(f(x,y)),\mathcal{D}_y(\mathcal{D}_x(f(x,y))) .$$

On computing every *internal* discriminant, one should get rid of an extraneous square factor.

Theorem 5. *Let* $G(0,0,0) \neq 0$ *and* $G(x_1,x_2,x_3)$ *be an even polynomial in* x_3. *Denote* $\widetilde{G}(x_1,x_2,x_3^2) \equiv G(x_1,x_2,x_3)$. *Equation* $G(x_1,x_2,x_3) = 0$ *does not define a real manifold if*

(a) *equation* $G(x_1,x_2,0) = 0$ *does not define a real curve and*

(b) *equation*

$$\mathcal{F}(z) := \mathcal{D}_{x_1,x_2}(\widetilde{G}(x_1,x_2,z-x_1^2-x_2^2) = 0 \qquad (10)$$

does not possess positive zeros.

If any of these conditions fails then the distance from $X_0 = (0,0,0)$ *to the manifold* $G(x_1,x_2,x_3) = 0$ *equals either the distance from* $(0,0)$ *to the curve* $G(x_1,x_2,0) = 0$ *or the square root from the minimal positive zero of the equation (10) provided that this zero is not a multiple one.*

Extension of the result of Theorem 5 to the case of arbitrary polynomial G is carried out in a manner similar to that utilized in Theorem 3.

Example 4. *For the cubic*

$$-2x_1^3+2x_1^2x_2+4x_1x_2^2+x_2^3-3x_2x_3^2$$

$$-3x_1x_2-x_2^2+x_3^2+x_1+2x_2-x_3-3=0$$

the distance equation is, generically, of the degree 21. *Some sample values for the distance and its approximations are presented in the following table:*

(x_{10},x_{20},x_{30})	$(4,4,-4)$	$(0,0,0)$	$(-1,5,-4)$	$(7,2,5)$
$d_{(1)}$	0.554	1.225	1.470	2.041
$d_{(2)}$	0.611	2.143	1.666	2.592
d	0.618	1.0585	1.447	2.243

Example 5. *The cross section of the torus*

$$(x_1^2+x_2^2+x_3^2+299)^2-1296x_1^2-1296x_2^2=0$$

and the corresponding surfaces $d_{(1)} = 3$ *and* $d_{(2)} = 3$ *with the half plane* $x_2 = 0, x_3 \geq 0$ *are displayed in Fig. 5 and Fig. 6.*

4 NON-ALGEBRAIC CURVES

For the case of implicitly defined non-algebraic curves, distance equation cannot be obtained in the closed form since the procedure for its construction

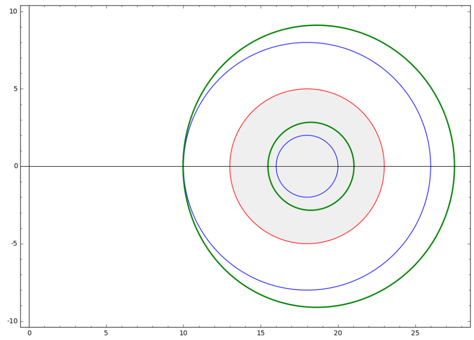

Figure 5: Example 5. Surface $d_{(1)} = 3$ (green boldface) vs. equidistant $d = 3$ (thin blue) to the torus (thin red).

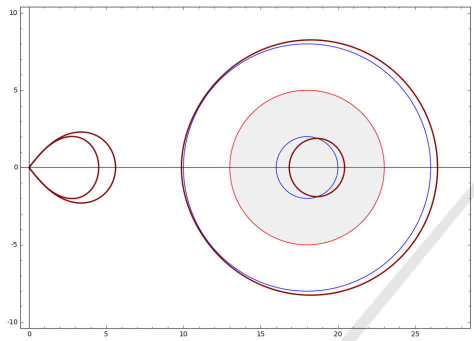

Figure 6: Example 5. Surface $d_{(2)} = 3$ (maroon boldface) vs. equidistant $d = 3$ (thin blue) to the torus (thin red).

suggested in the previous section is essentially algebraic, i.e. it is applicable only for polynomial functions. However, their equidistant curves can be found even for this case if one is able to establish the parametric representation for the interested curve.

Theorem 6. *If the curve is defined parametrically as*

$$x = \zeta(t),\ y = \eta(t) \text{ for } t \in [a,b],$$

then its equidistant curves lying at the distance d are defined as

$$x = \zeta \pm \frac{d\eta'}{\sqrt{(\zeta')^2 + (\eta')^2}},\ y = \eta \mp \frac{d\zeta'}{\sqrt{(\zeta')^2 + (\eta')^2}}.$$

for $t \in [a,b]$.

Example 6. *For the tractrix*

$$x^2 - 25\left(\ln\frac{5+\sqrt{25-y^2}}{y} - \sqrt{25-y^2}\right)^2 = 0,$$
$$y \in (0;5),$$

parametric representation is as follows

$$x = 5\left(\ln\tan\frac{t}{2} + \cos t\right),\ y = 5\sin t.$$

The true equidistant curves $d = 1$ *and their comparison with the approximations* $d_{(1)}(x,y) = 1$ *and* $d_{(2)}(x,y) = 1$ *are presented in Fig. 7, 8 and 9 respectively.*

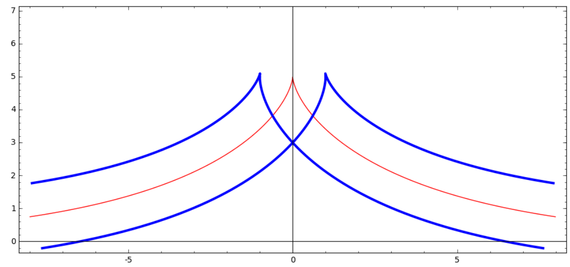

Figure 7: Example 6. Equidistant $d = 1$ (boldface blue) to the tractrix (thin red).

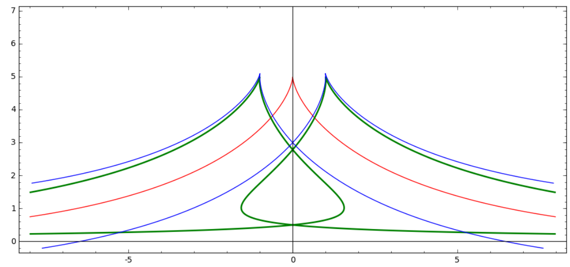

Figure 8: Example 6. Curve $d_{(1)} = 1$ (green boldface) vs. equidistant $d = 1$ (thin blue) to the tractrix (thin red).

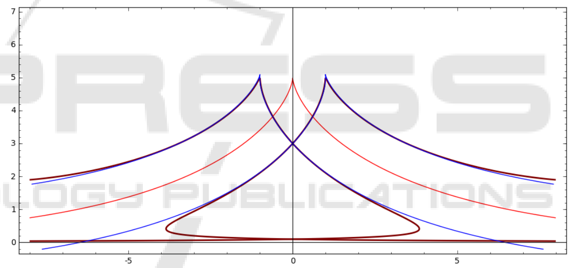

Figure 9: Example 6. Curve $d_{(2)} = 1$ (maroon boldface) vs. equidistant $d = 1$ (thin blue) to the tractrix (thin red).

The result of Theorem 6 can evidently be extended to the case of parametric surface in $\mathbb{R}^3$.

5 CONCLUSION

We have investigated the quality of two approximations for the distance from a point to implicitly defined curve in $\mathbb{R}^2$ and manifold in $\mathbb{R}^3$. Using an analytical representation of the true distance value via the point coordinates and parameters of the manifold, it is possible to compare the relative position of the level sets of approximation with respect to the equidistant manifolds. Adequacy of the suggested approximations for arbitrary manifold needs further quantitative validation, similar to that performed in (Uteshev and Goncharova, 2018) for the case of a quadric. However, even empirical considerations demonstrated in the present report give grounds for hope to constructively resolve the best fitting manifold problem mentioned in Introduction.

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