The Time Operator of Reals

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Abstract: The purpose of the paper is to establish the continuum in terms of the complex systems physics. It is based upon the intuitionistic mathematics of Brouwer, implying a processual definition of real numbers that concerns the measurement problem. The Brouwerian continuum is proved to be a categorical skeleton of complex systems whose determining feature is the existence of time operator. Acting on continuous signals, the time operator represents multiresolution hierarchy of the measurement process. The wavelet domain hidden Markov model, that recapitulates statistical properties of the hierarchy, is applicable to a wide range of signals having experimental verification. It indicates a novel method that has already been proved tremendously useful in applied mathematics.

1 INTRODUCTION

Announcing decline and fall of reductionism in mathematics, Gregory Chaitin (1995, pp.156-159) considers randomness in arithmetic elucidated by some results of the computation theory. In the conclusion concerning experimental mathematics, he highlights an impact of the computer that has so vast mathematical experience people are forced to proceed in a more pragmatic fashion. In that manner, mathematicians go ahead proofs postulating some hypothesis based on the results of computer experiments. Chaitin particularly points out a relation to contemporary physics wherein the randomness is regarded to be a crucial agent, which is the core of an emergent paradigm in science. He ends by a remark that the question of how one should actually do mathematics requires at least another generation of work.

The occurrence of randomness in a formal theory, set upon deterministic assumptions, corresponds to the complex systems physics that considers systems for which the best method of their description is not clear a priori (Sambrook et al., 1997, p.203). The statistical complexity suggested by Grassberger (1986), that concerns stochastic computing in terms of the Bernoulli-Turing machine, is an analog of the deterministic one designed upon the Turing machine (Crutchfield et al., 1990). Within stochastic computation theory, deterministic and random behaviors are considered to be elemental extremes deprived of a vital component since they share common failure to support emergent properties. Being an amalgam of both, the complex patterns have an inherent tendency towards hierarchical organization (Sambrook et al., 1997, p.200).

The hierarchy has substantial implications concerning cognition, since observation and comprehension are related to the neural architecture whose structure is a reflection of the cognitive complexity (Sambrook et al., 1997, pp.204-206). It corresponds to an evolution in the hierarchical manner (Simon, 1962, pp.470-477) indicating a concept of time operated not only physically or biologically, but in terms of organization theory. In that respect, time represents a primordial intuition being the very base of conscious life – as stated by Brouwer in his attempt to found the continuum upon such an *intuitionism* (Tasić, 2001, pp.36-45). He considers mathematics to be an intellection of increasingly complex features, meaning actually self-organization of complex systems due to the definition originated by Shalizi et al. (2004) that concerns the increase of complexity over time. The issue requires

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a formulation of complex systems by the time operator acting on the space of continuous signals (Prigogine, 1980). It is regarded a straightforward generalization of multiresolution that relates the measurement process (Antoniou et al., 2003, p.107).

The paper is aimed to establish the Brouwerian continuum in terms of the complex systems physics. In the preliminary section, the intuitionistic mathematics is briefly outlined, as well as the concept of the time operator. The next section elaborates the continuum structure that concerns the measurement problem, defining the time operator of multiresolution hierarchy in the space of continuous signals. The last section contains some concluding remarks.

2 PRELIMINARIES

2.1 The Intuitionistic Mathematics

Brouwer`s contribution to the foundations of mathematics should be regarded in a context of the XIX and XX centuries whose significant persons were Hilbert, Russell and Whitehead. Russell`s and Whitehead`s view were based upon logic,[1] stating that it represents the fundament of mathematical thought and Hilbert`s considered a formal language to be the design of mathematics. All of them were labelled by Brouwer to be the *platonic* ones, since implying timeless conceptions. Elucidated in physical terms, they have been the deterministic theories.

Brouwer claims irreducibility of mathematics to a language, aiming to separate it both from formalism and logicism.[2] According to his opinion, a basis of consciousness is the time continuum transcending any language in order to provide an original creation. It represents a continual activity of the creative subject, that is not formally determined. Consequently, for mathematics there is no certain language (Brouwer, 1929). To elucidate such a conception, he uses the term *choice sequence*, diverging intuitionism also from constructive mathematics based upon deterministic decisions (Tasić, 2001, pp.38-40).

Considering the intuitionistic logic, one should mention its deviation from the laws of excluded middle and double negation. Brouwer regards a structure to be discrete if the law of excluded middle holds in the form $x = y \vee x \neq y$, which is not generally applicable.[3] Intuitionism is therefore a logic of the continuum, unlike formal logic that is actually the discrete one. Hereupon, the double negation law $\neg x \neq y \Rightarrow x = y$ is also violated allowing the existence of infinitesimals since a negation of diversity from zero $\neg \varepsilon \neq 0$ is not equivalent to the identity $\varepsilon = 0$ (Bell, 1998). Consequently, the continuum does not reduce to a pointwise structure, considering that the primordial intuition is not dependent on a formal spatiality. However, the sense of time has been seriously damaged by treating it as an additional dimension of formal space that the modern science peddles for an ultimate reality (Tasić, 2001, pp.36-37).

Brouwer actually refers to the elimination of time stated by Meyerson (1908) through his definition of the modern science history in terms of progressive realizing the fundamental bias in human reason, which is about reducing of difference and change to identity and constancy. As early as the XVIII century, d`Alambert (1754) noted that one could regard the duration to be a fourth dimension supplementing the common three-dimensional design and Lagrange (1796, p.223), more than hundred years before Einstein and Minkowski, went so far to term it the *four-dimensional geometry*. The climax of such a historical trail was Albert Einstein in categorical rejecting the existence of change that was considered by him to be a mere illusion (Prigogine, 1980, pp.201-203).[4] In that respect, contemporary restoration of time indicates a postmodernism in science, whose forerunner Brouwer has been (Tasić, 2001, p.49).

[1] The logicism had actually come from Gottlob Frege whose heirs Russell and Whitehead were.

[2] *Interpreting mathematics as being based on logic is like considering the human body to be an application of the science of anatomy* (Brouwer, 1907).

[3] *The Principle of Excluded Middle has only a scholastic and heuristic value and therefore the theorems that in their proof cannot avoid the use of this principle lack any mathematical content* (Brouwer, 1928).

[4] Einstein`s intension to reach the timeless world of supreme rationality was definitely manifested in the celebrated Einstein-Bohr debate on the foundations of quantum theory. A core of the debate was about the fundamental role of randomness in specifying a system`s state, that was emphatically denied by Einstein who was supporting the objective epistemology of science. Although he overturned the Newtonian mechanics, Einstein was firmly holding the Cartesian intent of reducing physics to geometry in terms of the formal spatiality that was but a deterministic assumption.

2.2 The Time Operator

To be a conception of the postmodern science, one considers the quantum physics whose origins date back to the beginning of the XX century (Toulmin, 1985). The statistical formulation, concerning an evolution of probabilities, has given rise to the operator mechanics by Koopman and von Neumann. In terms of the theory, a system corresponds to group of transformations V^t acting on the Hilbert space $L^2_\mu(\Omega)$ through $V^t f = f \circ Z^t$, whereat a dynamics of the phase space Ω is represented by one-parameter family $Z^t : \Omega \to \Omega$ that preserves probability measure μ. Due to the measure preservation, V^t is unitary and, according to Stone`s theorem, the group has an infinitesimal generator L termed the *Liouvillian* which is a self-adjoint operator defined on the dense subspace (Koopman, 1931).

The uncertainty principle concerns a pair of conjugate observables, whose main example is manifested by the position $Q = q\,\cdot$ and the momentum operator

$$P = \frac{\partial}{i\partial q}.[5]$$

Formulated in terms of the commutator $[Q, P] = QP - PQ$, the uncertainty principle implies relation $[Q, P] = iI$. In the same manner, Prigogine (1980) defined the time operator T to satisfy the uncertainty relation

$$[T, L] = iI, \tag{1}$$

considering the Liouvillian L.

The variable f whose domain is Ω evolves by action of the group $V^t = e^{-iLt}$, whilst the distribution density ρ is acted by adjoint operators $V^{t\dagger} = e^{iLt}$ implying the Liouville equation

$$\frac{\partial \rho}{\partial t} = iL\rho$$

Consequently, the Liouvillian

$$L = \frac{\partial}{i\partial t}$$

and the time operator $T = t\,\cdot$ correspond to conjugate observables similarly to the position and the momentum in the quantum theory[6]. In terms of the group action, the relation (1) is equivalent to

$$[T, V^t] = tV^t \tag{2}$$

which comes down to $[T, V] = V$, supposing a cyclic group generated by $V \equiv V^1$.

The existence of the time operator in the system induces a change of representation $\Lambda = \lambda(T)$,[7] transforming with no loss of information the group to a semigroup action (Misra et al., 1979). The semigroup

$$W^{t\dagger} = \Lambda V^{t\dagger} \Lambda^{-1}, t \geq 0 \tag{3}$$

corresponds to an irreversible evolution of the complex system,[8] conjugated to the reversible one of the group. It addresses a stochastic process irreducible to the deterministic description, whose existence is analog of Gödel's incompleteness theorem (Misra et al., 1983). In the modern science based upon an elimination of time, the irreversibility is cognized through the measurement problem that demands a departure from determinism in favor of the statistical causality (Prigogine, 1980, pp.65-67).[9]

3 THE TIME CONTINUUM

3.1 The Continuum of Reals

The concept of continuum corresponds to real numbers and the measurement process, originating from the geometrical algebra of Euclid. In the V book of the *Elements*, he elaborates the doctrine of proportion that concerns commensuration of magnitudes. According to the Euclidean algorithm, magnitudes a and b measure each other in the form

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[5] Operators act on functions defined in the phase space having a coordinate q. In the quantum theory, the position $Qf(q \ldots) = q \cdot f(q \ldots)$ is related to multiplication and the momentum $Pf(q \ldots) = \partial f(q \ldots)/i\partial q$ to the Hermitian derivation by a coordinate.

[6] The value t, however, is not a coordinate of the phase space and there is no guarantee of the time operator existence. If it does exist, the system is termed to be *complex*.

[7] λ is an operator function in the sense of operational calculus.

[8] Although there are also the operators $W^{t\dagger} = \Lambda V^{t\dagger} \Lambda^{-1}, t < 0$, they are not positivity preserving and therefore not corresponded to system`s evolution that maps one distribution to another (Misra et al., 1979, p.23).

[9] In the quantum theory, the measurement corresponds to reduction of the wave packet, which is a nonunitary transformation. Von Neumann (1955) has expressed its difference from a unitary evolution by the Schrödinger equation in terms of the entropy increase, invoking a substantial role of the observer. Therefore, the problem of irreversibility appears at the very core of physics.

$$\frac{a}{b}=\cfrac{1}{n_1+\cfrac{1}{\ddots+\cfrac{1}{n_i+\cfrac{1}{\ddots}}}} \tag{4}$$

that is termed the *continued fraction* having the *spectrum* $n_1,\dots,n_i,\dots$[10] The proportion

$$\frac{a}{b}=\frac{c}{d},$$

indicated by matching of the respective terms in both spectrums, induces identity on the continuum of reals.[11] Consequently, the real number corresponds to the fraction expansion (4) implying the measurement process taken place step by step in a manner of time.

The time evolution is represented by the Ford (1938) diagram of circles, whose intersections with a vertical line correspond to the sequence

$$\xi_i=\cfrac{1}{n_1+\cfrac{1}{\ddots+\cfrac{1}{n_i}}} \tag{5}$$

identifying a real number x. Its elements

$$\xi_i=\frac{h_i}{k_i}$$

are termed the *Diophantine approximations* of x, being the most approxiate to the real number in regard to the fractions

$$\frac{h}{k}, k\le k_i$$

with denominators not greater than that of ξ_i. Denominators and numerators of the sequence are given by recurrence equations of the form

$$h_{i+1}=n_{i+1}h_i+h_{i-1}, k_{i+1}=n_{i+1}k_i+k_{i-1} \tag{6}$$

considering the initial conditions $h_0=0, h_1=1$ and $k_0=1, k_1=n_1$. The difference of successive members is

$$\Delta\xi_i=\xi_{i+1}-\xi_i=\frac{h_{i+1}}{k_{i+1}}-\frac{h_i}{k_i}==\frac{h_{i+1}k_i-h_ik_{i+1}}{k_{i+1}k_i}==\frac{[\![h,k]\!]_i}{k_{i+1}k_i} \tag{7}$$

supposing $[\![h,k]\!]_i=h_{i+1}k_i-h_ik_{i+1}$ which implies

$$[\![h,k]\!]_i=(a_{i+1}h_i+h_{i-1})k_i-h_i(a_{i+1}k_i+k_{i-1})=h_{i-1}k_i-h_ik_{i-1}=-[\![h,k]\!]_{i-1} \tag{8}$$

and, having in mind $[\![h,k]\!]_0=h_1k_0-h_0k_1=1$, one gets $[\![h,k]\!]_i=(-1)^i$, i.e.,

$$\Delta\xi_i=\frac{(-1)^i}{k_ik_{i+1}} \tag{9}$$

Accordingly, the continued fraction concerning a real number takes the form of an alternating series

$$x=\Delta\xi_0+\dots+\Delta\xi_i+\dots==\frac{1}{k_0k_1}-\cdots\frac{(-1)^i}{k_ik_{i+1}}\cdots \tag{10}$$

that is a sparse representation (Mallat, 2009) composed of terms from the redundant dictionary

$$\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots$$

The equation (10) corresponds to a binary code wherein 0 is assigned to terms of the dictionary that do not participate in the series and 1 to those that do participate, having a sign $\pm$ which alters in front of them. Such a record of the real number is, however, highly redundant since the entire dictionary cannot be involved in a series. Therefore, one should eliminate excess zeros, which is achieved by coding the spectrum $n_1,\dots,n_i,\dots$ A binary code like that is composed of alternative values ±1 different from zero at the positions $n_1,\dots,n_1+\dots+n_i,\dots$, which gives rise to the Minkowski question mark function

$$?:\cfrac{1}{n_1+\cfrac{1}{\ddots+\cfrac{1}{n_i+\frac{1}{\ddots}}}}\mapsto\frac{1}{2^{n_1-1}}-\cdots\frac{(-1)^{i-1}}{2^{n_1+\dots+n_i-1}}\cdots \tag{11}$$

[10] One assumes that $a\le b$, i.e., $a/b\le1$. If not, the representation applies to b/a.

[11] Defining proportion in terms of the continued fraction spectrum is not done by Euclid, but by Omar Khayyam in *A Commentary on the Difficulties Concerning the Postulates of Euclid's Elements*. The conception presented in the *Elements*, that originates from Eudoxus of Cnidus, means that the proportion

$$\frac{a}{b}=\frac{c}{d}$$

holds if each of propositions $ma<nb, ma=nb, ma>nb$ is equivalent to the respective one of $mc<nd, mc=nd, mc>nd$. Displeased with such a philosophy, Khayyam redefined it by the use of continuous fractions, where the concept of real number was established. Eudoxus` definition was employed by Richard Dedekind, concerning his construction of the real line in terms of rational cuts.

transforming the continued fraction to the binary code. It is an automorphism of the continuum, mapping a real number

$$x = \frac{1}{n_1 + \frac{1}{\ddots + \frac{1}{n_i + \ddots}}}$$

to the analog value

$$?(x) = \sum_i \frac{(-1)^{i-1}}{2^{n_1+\cdots n_i - 1}}.$$

Under the continuum, one implies a skeletal category being specified up to an isomorphism. In that regard, the transformation ? is considered to be automorphism of the structure, since isomorphism (12) hereinafter is also an identity.

The Ford diagram is structured by a hierarchy of scales, each corresponding to the insertion of circles tangent to two of them at the previous scales, as well as to the number line. In such a hierarchy, each circle is attributed to an irreducible fraction that represents its contact to the line.

Assuming designators of the circles to be

$$\frac{r_1}{s_1} \text{ and } \frac{r_2}{s_2},$$

an inserted circle between them corresponds to the fraction

$$\frac{r_1+r_2}{s_1+s_2}.$$

One denotes it

$$\frac{r_1}{s_1} \oplus \frac{r_2}{s_2},$$

which defines an operation termed the *mediant* or *the Farey sum*.[12] The question mark function maps the mediant to the arithmetic mean (Minkowski, 1905)

$$?(x_1 \oplus x_2) = \frac{?(x_1) + ?(x_2)}{2} \tag{12}$$

being isomorphism of the topological quasigroups, whose action turns circles of the Ford diagram into squares. The diagram of squares has a binary tree structure whereby the nodes, coordinated by x and y, correspond to paracomplex number $x + \iota y$, $\iota^2 = 1$ forming an algebra of segments $[x - y, x + y]$ (Warmus, 1956). Branching of the segments

$$[\tfrac{k-1}{2^j}, \tfrac{k+1}{2^j}], 1 \le k \le 2^j - 1$$

designates bit by bit of a real number and, in that regard, hierarchy of continuum is related to the binary coding. The Renyi map

$$R(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2x - 1 & 1/2 < x \le 1 \end{cases} \tag{13}$$

that is a shift in terms of binary digits, represents self-similarity of the structure mapping both left and right subtree to the entire one.

3.2 Wavelets and Multiresolution

Aiming to construct the time operator corresponded to an evolution of continuum, the Hilbert space of continuous signals is considered. In that respect, one suggests the space $L^2(\mathbb{I})$ consisted of square integrable functions on the domain $\mathbb{I} = [0,1]$. Crucial for the issue are hierarchical bases consentient to the continuum structure, which are the wavelet bases on the interval. According to a wavelet base of the support $\mathbb{I} = [0,1]$, a signal $f \in L^2(\mathbb{I})$ is decomposed in the form

$$f = A_0 + \sum_{j \ge 0} \sum_{k=1}^{2^j} D_{j,k} \psi_{j,k} \tag{14}$$

wherein j indexes the dyadic scale and k the spatial position of an element $\psi_{j,k}$ (Daubechies, 1992, pp.304-307). The Haar base

$$\chi_{j,k}(x) = \begin{cases} -1 & \frac{k}{2^j} \le x \le \frac{k + 1/2}{2^j} \\ +1 & \frac{k + 1/2}{2^j} < x \le \frac{k+1}{2^j} \end{cases} \tag{15}$$

is obtained by translations and dilatations of the mother wavelet

$$\chi(x) = \begin{cases} -1 & 0 \le x < 1/2 \\ +1 & 1/2 \le x < 1 \end{cases}.$$

In the decomposition sum (14), A_0 is the average value which is a projection onto the subspace of constant signals $\mathbb{1}$. The hierarchical structure of a wavelet base is reflected to the detail coefficients $D_{j,k}$ forming the binary tree. Each node at a scale j of the tree has two successors at the next one $j+1$ sharing its position in the hierarchy. The succession is related to the measurement process whose steps correspond

[12] It is due to John Farey (1816) who noticed that the successive fractions $\zeta < \xi < \eta$, whose denominators in the reduced form are up to a given value, relate by $\xi = \zeta \oplus \eta$. For example, the fractions up to the denominator value 5 form the order

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

whereby each successive threesome is related by the Farey sum.

to the scales of hierarchy. In that respect, the time operator concerning a wavelet base is given by

$$T\psi_{j,k} = (j+1)\psi_{j,k} \quad (16)$$

wherein the eigenvalues correspond to the scales of eigenvectors $\psi_{j,k}$, increased by a unit in order it to be the invertible operator the dense subset of $L^2(\mathbb{I}) \ominus \mathbb{1}$. The operator domain includes finite sums

$$\sum_{j=0}^{J} \sum_{k=1}^{2^j} \blacksquare \psi_{j,k}$$

whose components

$$\sum_{k=1}^{2^j} \blacksquare \psi_{j,k}$$

constitute detail subspaces $\mathcal{D}_j$ wandering by the unilateral shift

$$Uf(x) = f \circ R(x) = \\ = \begin{cases} f(2x) & 0 \le x \le 1/2 \\ f(2x-1) & 1/2 < x \le 1 \end{cases} \quad (17)$$

induced by the Renyi map (13).[13] It generates the time succession of the space $L^2(\mathbb{I}) \ominus \mathbb{1}$, establishing a multiresolution analysis whose basic axiom is the shift property

$$\mathcal{D}_j = U^{-1}(\mathcal{D}_{j+1}) \quad (18)$$

Presented in the form

$$T = \sum_j (j+1) P_{\mathcal{D}_j} \quad (19)$$

whereby each projector

$$P_{\mathcal{D}_j} f = \sum_k D_{j,k} \psi_{j,k} \quad (20)$$

corresponds to the details at a resolution scale, the time operator appears to be a straightforward generalization of multiresolution (Antoniou et al., 2003, p.107). The basic axiom (18) is equivalent to the uncertainty relation

$$[T, U] = U \quad (21)$$

that concerns the definition of time in complex systems (2).

In order to establish a measure preserving group, the Renyi map R should be extended naturally to the baker transform

$$B(x,y) = \begin{cases} \left(2x, \frac{y}{2}\right) & 0 \le x \le 1/2 \\ \left(2x-1, \frac{y+1}{2}\right) & 1/2 < x \le 1 \end{cases} \quad (22)$$

inducing the bilateral shift $Vf = f \circ B$ (Antoniou et al., 2003, pp.35-38). According to that, the signal space $L^2(\mathbb{I})$ is embedded into $L^2(\mathbb{I} \times \mathbb{I}) = L^2(\mathbb{I}) \otimes L^2(\mathbb{I})$ by the rule $\sim: f \mapsto f \otimes 1$, i.e., $\tilde{f}(x,y) = f(x)$. The relation

$$P_{\mathcal{D}_{j+1}} U = U P_{\mathcal{D}_j} \quad (23)$$

that holds on the space $L^2(\mathbb{I})$, gives rise on $L^2(\mathbb{I} \times \mathbb{I})$ to

$$P_{\mathfrak{D}_{j+1}} = V P_{\mathfrak{D}_j} V^{-1} \quad (24)$$

meaning that the projectors onto detail subspaces interrelate by conjugation. The time operator T^{χ} of the baker map has been explicitly constructed (Antoniou et al., 2003, pp. 47-60), whose natural projection onto $L^2(\mathbb{I})$ corresponds to the Haar multiresolution (15). However, any wavelet mulitiresolution could be obtained likewise, since the change of base $C: \chi_{j,k} \otimes \chi_{l,m} \mapsto \psi_{j,k} \otimes \psi_{l,m}$ does not violate the shift property (18) and therefore the operator $T = CT^{\chi}C^{-1}$ also satisfies the uncertainty relation (21).[14]

The time operator T induces a change of representation $\Lambda = \lambda(T)$ transforming the group evolution, generated by $V^{\dagger}$, to the semigroup one (3) by

$$W^{\dagger} = \Lambda V^{\dagger} \Lambda^{-1} \quad (25)$$

whose adjoint operator corresponds to the Markov process (Misra et al., 1979, p.9)

$$Wf(\omega) = \int f(\varpi) p(\omega, d\varpi) \quad (26)$$

The semigroup action signifies a blurring of the signal, related to an extension of the spatial domain due to an action of the operator U on $L^2(\mathbb{I})$.

3.3 Complexity and Self-organization

In the poem *On the Nature of Reality*, Lucretius describes not only how things vanish at a distance – but also how they appear to change (Lucretius, 1948,

[13] Mutually orthogonal subspaces $\mathcal{D}_j$ are termed to be *wandering* if the shift property $\mathcal{D}_j = U^{-1}(\mathcal{D}_{j+1})$ holds. They are termed *generating* if the sequence is a cover of the space (Antoniou et al., 2000, p.446).

[14] A problem occurs concerining preservation of positivity, since the approximation operators $\sum_{j \le J} P_{\mathfrak{D}_j}$ does not preserve it any more (Gustafson, 2007, pp. 16-18). It is reflected to the positivity preserving of Λ and $W^{\dagger}$, that should not correspond to an evolution of probability. However, it is all about a mere change of base conjugating the approximation operators to those of the Haar multiresolution, that still preserve positivity. In order to resolve the concerns, one requires a base independent definition that presumably affects the concept of probability distribution. The authors do not discuss it in details.

p.156).[15] The effect concerns blurring of the signal, wherewith the details are successively suppressed. On the other hand, the emergence of details unfolding the time of a system is termed *self-organization* that is the increase of complexity (Shalizi et al., 2004). The concept originates from Grassberger (1986) who defined the statistical complexity to be minimal information required for an optimal prediction. Crutchfield et al. (1990) elaborated the conception by accurate definitions of the optimal predictor and its state. In that manner, the causal structure has been established corresponding to the intrinsic computability of a process in terms of the Bernoulli-Turing machine.

The detail coefficients of a signal $\mathbf{D} = \left(D_{j,k}\right)$ are considered to be distorted measurements of a causal variable $\mathbf{S}$ evolving in a stochastic manner. It is factorized into local variables $S_{j,k}$ related to each node of the tree. The Markovian tree $\mathbf{S} = (S_{j,k})$ contains all correlations in a signal, that occur only along branches linking the local states due to a structure of the continuum. The wavelet domain hidden Markov model established like that has been proved tremendously useful in a variety of applications, including speech recognition and artificial intelligence (Crouse et al., 1998, p.887).

Elaborating a statistical model of the wavelet transform, a signal and its coefficients are regarded to be random realizations. In that respect, one requires the space of distributed signals $L^2(\mathbb{I} \times \mathbb{I})$ whose constituent $f\colon \mathbb{I} \longmapsto L^2(\mathbb{I})$ is a variable of the domain $\mathbb{I}$ implying Lebegue`s probability measure, that varies over the codomain $L^2(\mathbb{I})$. The coefficient distributions are given by

$$D_j = \langle V^{-j} f, \psi \rangle \tag{27}$$

since from (24) it follows that details at a resolution scale j are

$$P_{\mathfrak{D}_j} f = V^j P_{\mathfrak{D}_0} V^{-j} f = V^j \left(D_j \psi\right) \tag{28}$$

considering

$$P_{\mathfrak{D}_0} f = \langle f, \psi \rangle \psi.$$

According to that, the detail coefficients at a common scale j are regarded to be equally distributed

$$D_{j,k} \doteq D_j \tag{29}$$

which reflects to the causal variable $\mathbf{S} = (S_{j,k})$ whose distribution is independent of the position index k.[16] The information contained in local variables

$$C_j = \mathrm{H}\left(S_{j,\blacksquare}\right) \tag{30}$$

dependent on the scale only, is termed the *local complexity* whose increase in the temporal domain represents self-organiozation. One implies the Shannon entropy $\mathrm{H}(\blacksquare) = -\sum_i p_i \log p_i$, being the extensive measure of a random variable.

The time is unfolding in a manner of the complexity increase, and so it is significant to find an optimal base of the signal wherein self-organization is the most prominent. The complexity

$$C = \mathrm{H}(\mathbf{S}) \tag{31}$$

termed the *global* one, is proven to be a measure of the optimal representation (Milovanović et al., 2013). The signal information is decomposed through the canonical equation

$$\mathrm{H}(\mathbf{D}) = \mathrm{H}(\mathbf{S}) + \mathrm{H}(\mathbf{D}|\mathbf{S}) \tag{32}$$

wherein $\mathrm{H}(\mathbf{S})$ corresponds to the complexity and the extensive term $\mathrm{H}(\mathbf{D}|\mathbf{S})$ represents an irreducible randomness that remains even after all correlations are given. Adding white noise to the signal,[17] only the randomness should increase while the complexity remains unchanged. Thereby the optimal base performs superior denoising, since it best respects self-organization of a system corresponded to the time operator. A multiresolution it provides temporally decomposes the signal, specifying its significance by a complexity insight. In that regard, the multiscale pyramids are proposed to be likely models of the visual perception (Koenderink, 1997, p.xx).

An incisive phenomenology of the fact has been presented by John Ruskin (1844, p.174):

> *Go to the top of Highgate Hill on a clear summer morning at five o'clock, and look at Westminster Abbey. You will receive*

[15] For instance, distant square towers look rounded. A pair of distant islands appear to merge into single one. When distance is increased, details generalize and distinctions merge or vanish (Koenderink, 1997, p.xv).

[16] Statistical stationarity of the system, that concerns translational invariance of a signal distribution, enables reduction of the model parameters. The practice is known as *tying* in the hidden Markov model literature (Rabiner, 1989), aiming to estimate the parameters robustly by a use of the Baum-Welch algorithm given an observation f from the signal space $L^2(\mathbb{I})$. It is about sharing statistical information between related variables at certain scales, whose distribution parameters are tied to a common value. The algorithm usually converges in as few as ten iterations supposing a locally two state causal structure (Crouse et al., 1998, p.893).

[17] The term *white noise* means uncorrelated Gaussian noise independent of the signal it is added to.

> *an impression of a building enriched with multitudinous vertical lines. Try to distinguish one of these lines all the way down from the one next to it: You cannot. Try to count them: You cannot. Try to make out the beginning or end of any of them: You cannot. Look at it generally, and it is all symmetry and arrangement. Look at it in its parts, and it is all inextricable confusion.*

Yet Ruskin adamantly insists that the draughtsman should render the confusion veridically, meaning that the complexity is optimally represented. Such a rendering is however done in a hierarchical manner, since it describes a complex object (Simon, 1962, p.477).

Koendreink (1997, pp.xvii-xx) indicates that one is faced with a fundamental and important, though unfortunately ill understood, aspect of perception. Having taken a first look at the subject, he admitted a shock by the fact that there existed essentially no science on the topic. The only discipline that carried about such phenomena turned out to be cartography (Greenwood, 1964). Although there is certainly a lot of science in cartography, its arguably the most important aspect has always remained an art conducted largely on intuition. It corresponds to an aesthetical criterion relating truth to the original creation (Milovanović et al., 2016), that has been termed by Gaston Bachelard (1961) the *poetics of space*.

Concerning physical reality, Koenderink concludes the same as Mandelbrot (1983) in terms of the fractal geometry that a complex description of nature is required. The conception is concisely exposed in the book *Powers of Ten*, giving to the number a significance corresponded to multiresolution (Morrison et al., 1982). A link between the number of ten and multiscaling is about the continuum structure designed by the measurement process. According to the Lochs (1964) theorem, the number of terms in continued fraction requisite for determining a decimal digit tends to be

$$\frac{6\log 2\log 10}{\pi^2} \approx 0.97,$$

meaning that each step of measurement roughly designates digit by digit almost certainly. A significance of the decimal system in coding numbers is based therefore on the structure of continuum. Since the continuous signals are equally distributed with no dependence on a horizontal position of the hierarchy (29), a real number in the tree representation corresponds to a choice sequence unfolding in time from the top downwards.

In that manner, the time continuum appears to be a model of intuitionistic logic wherein the excluded middle $x = y \lor x \neq y$ does not hold, considering a dynamical identity unfolded by choice (Milovanović, 2018). However, the law of excluded middle is valid concerning the diversity since it holds

$$x \neq y \lor \neg x \neq y \quad (33)$$

Respecting the intuitionism, a negation of identity $=$ is diversity $\neq$, but its negation is undiversity $\approx$ that is a discrete relation. A negation $\not\approx$ of undiversity is diversity $\neq$, and thus the law (33) takes the form

$$x \approx y \lor x \not\approx y \quad (34)$$

indicating a discrete structure corresponded to the formal logic. It is obtained through a negative translation of the intuitionistic one (Glivenko, 1929), meaning that formalism is a form of the intuitionistic continuum reduced to a discrete method.[18]

The discretization due to the double negation of identity makes a pointwise structure based upon the undiversity of elements.[19] In terms of the signal space, it gives rise to a point operator that is required to be sufficiently smooth in order to transfer the concepts of continuity and differentiation onto the discrete functions (Florack, 1997, pp.57-65).[20] Regarding the time continuum, however, all functions on the domain have been considered continuous since representing morphisms of the structure (Brouwer, 1924).

4 CONCLUSION

Elaborating relation between wavelets and stochastic processes, Antoniou et al. (1999, p.96) asserted that wavelets had not been motivated by any underlying dynamics of the phase space. He concludes that the ergodic theory is richer than the wavelet one, since the former involves *fundamentally an underlying dynamical system of point trajectories*. However, it has been demonstrated that wavelets involve the baker transform, acting in the domain $\mathbb{I} \times \mathbb{I}$ of distributed signals. Due to the existence of the time

[18] The extreme concerns deterministic computation based upon the Turing machine that is a reduction of the stochastic one using the Bernoulli-Turing machine. In that manner, the concept of statistical complexity is reduced to the algorithmic one (Crutchfield et al., 1990).

[19] It also emerges in JavaScript, being the legendary *cast-to-bool* operator written in a form of the double negation (!!).

[20] The point operator represents a measurement process corresponded to blurring of the signal (26).

operator, the group comes down to the semigroup action generated by the Renyi map. The complex systems physics is therefore implied by the very concept of real numbers, that addresses the measurement problem.

According to Brouwer`s view, time is a primordial intuition being the base of conscious life. Mathematics is regarded to be the paradigm of self-organization, i.e., an intellection of increasingly complex features. In that respect, the basic structure is the time continuum that is a categorical skeleton of complex systems. The dynamical identity it implies is unfolded by choice, similarly to one of Jungian psychology whereby the natural number emerges to be a timestamp (von Franz, 1974).

A complex description of nature following the evolution of continuum is designed by fractal geometry, wherewith time is established in terms of multiresolution. Considering the statistics of continuous signals, the wavelet domain hidden Markov model has been proved tremendously useful in a variety of applications (Crouse et al., 1998). It is obtained in a manner of experimental mathematics, elucidating the complex systems physics to be the paradigmatic framework for such an activity. Referring to Chaitin (1995, pp.156-159), one concludes that another generation of mathematics has come – aligned to the Brouwerian method.

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