Clustering Algorithm for Generalized Recurrences using Complete Lyapunov Functions

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Keywords: Complete Lyapunov Functions, Chain-recurrent Set, Clustering Algorithm, Mathematics, Dynamical Systems.

Abstract: Many advances and algorithms have been proposed to obtain complete Lyapunov functions for dynamical systems and to properly describe the chain-recurrent set, e.g. periodic orbits. Recently, a heuristic algorithm was proposed to classify and reduce the over-estimation of different periodic orbits in the chain-recurrent set, provided they are circular. This was done to investigate the effect on further iterations of the algorithm to compute approximations to a complete Lyapunov function. In this paper, we propose an algorithm that classifies the different connected components of the chain-recurrent set for general systems, not restricted to (circular) periodic orbits. The algorithm is based on identifying clustering of points and is independent of the particular algorithm to construct the complete Lyapunov functions.

1 INTRODUCTION

Dynamical systems describe the evolution of time-changing phenomena. In recent years, by the increasing implementation of numerical analysis methods in powerful programming languages, analysing dynamical systems has become more accessible. This contrasts with past years in which studying dynamical systems was a complex task that required to involve difficult mathematical techniques. In fact, the study of the chain-recurrent set and trajectories was only possible for a small collection of problems.

For that reason, several techniques to analyse stability have been inherited up to present days. Such techniques can vary in approach, difficulty and efficiency: from direct simulations of solutions with many different initial conditions, to computation of invariant manifolds which form the boundaries of attractors' basin of attraction (Krauskopf et al., 2005). Set oriented methods (Dellnitz and Junge, 2002) or the cell mapping approach (Hsu, 1987) are also techniques to analyse dynamical systems. Unluckily, all these methods require large computational effort.

Another approach to study dynamical systems is given by the Lyapunov theory approach and it allows to study their qualitative behaviour. In particular, it turns to be useful to find attractors and repellers.

In general, dynamical systems often arise from differential equations. Let us consider a general autonomous ordinary differential equation (ODE),

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $n \in \mathbb{N}$.

Aleksandr Lyapunov (Lyapunov, 1907) proposed in 1893 a method to describe the stability of an attractor without computing the explicit solution of the differential equation. His method consists of constructing an auxiliary scalar-valued function whose domain is a subset of the state-space. Along all solution trajectories, this function is strictly decreasing in a neighbourhood of an attractor, such as an equilibrium point or a periodic orbit. It attains its minimum on the attractor, hence, all solutions starting close to the latter will converge to it. In modern theory, the original function is known as a strict *Lyapunov function* in his honor. This is the classical definition (Lyapunov, 1992).

However, this definition is limited to the neighbourhood of one attractor. A generalization to this function, which is defined on the whole phase space, is called a *complete Lyapunov function* and was introduced in (Conley, 1978; Conley, 1988; Hurley, 1995; Hurley, 1998).

Unlike the classical case, a complete Lyapunov

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Argáez, C., Giesl, P. and Hafstein, S.
Clustering Algorithm for Generalized Recurrences using Complete Lyapunov Functions.
DOI: 10.5220/0007934101380146
In *Proceedings of the 16th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2019)*, pages 138-146
ISBN: 978-989-758-380-3

function describes the complete qualitative behaviour of the dynamical system on the whole phase space and divides it into two disjoint areas: The gradient-like flow, where the systems flows through, and the chain-recurrent set, where infinitesimal perturbations can make the flow recurrent.

The first mathematical proof of existence of complete Lyapunov functions was given by Conley (Conley, 1978). The proof is given for a dynamical system defined on a compact metric space. Hurley (Hurley, 1998) extended these results to separable metric spaces.

In this paper we use, and continue to expand, an algorithm to compute a complete Lyapunov function previously used in (Argáez et al., 2017a; Argáez et al., 2018b; Argáez et al., 2018c; P. Giesl C. Argáez and Wendland, 2018; Argáez et al., 2018a). This algorithm has proven to be computationally efficient. It is a modification of a general method to compute classical Lyapunov functions for one stable equilibrium using Radial Basis Functions (Argáez et al., 2019a).

The general idea is to approximate a "solution" to the ill-posed problem $V'(\mathbf{x}) = -1$, where $V'(\mathbf{x}) = \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})$ is the derivative along solutions of the ODE, i.e. the orbital derivative.

A function v is computed using Radial Basis Functions, a mesh-free collocation technique, such that $v'(\mathbf{x}) = -1$ is fulfilled at all points $\mathbf{x}$ in a finite set of collocation points X.

The discretized problem of computing v is well-posed and has a unique solution. However, the computed function v will fail to solve the PDE at some points of the chain-recurrent set, such as an equilibrium or a periodic orbit. For some $\mathbf{x}$ in the chain-recurrent set we must have $v'(\mathbf{x}) \geq 0$. This is the key component of our general algorithm to locate the chain-recurrent set; we determine the chain-recurrent set by localizing the area where $v'(\mathbf{x}) \not\approx -1$.

There are, however, two main issues that require extra attention after obtaining an approximation of the chain-recurrent set using this method, namely

- Classification of the chain-recurrent set into connected components
- Reducing the over-estimation of the chain-recurrent set

In this paper we will address the first problem and propose an algorithm which is able to classify the different connected components of the chain-recurrent set. This can then later be used to address the second problem.

1.0.1 Clustering Algorithms

Our first attempt to classify and then to reduce the over-estimation of the chain-recurrent set was rather an exercise in exploring its impact on previous results (Argáez et al., 2019b). Such an attempt was carried out under the application of a heuristic algorithm, only capable of working over circular orbits. The idea behind the algorithm was simple:

- Obtain an approximation to the chain-recurrent set
- Count the orbits in the approximation
- For each circular obit with radius r, define two new radii, r_{max} and r_{min}, enclosing the circular orbit, and define
$$r_1 = r_{min} + 0.52 * (r_{max} - r_{min})$$
$$r_2 = r_{max} - 0.52 * (r_{max} - r_{min})$$
- Remove from the chain-recurrent set all points with Euclidean norm $r \notin [r_2, r_1]$
- Use these results as a starting point for a new iteration obtaining a better approximation of the chain-recurrent set

As it can be seen, this algorithm was designed to work only for circular orbits. However, it showed the importance of constructing an independent algorithm capable of obtaining general-shaped orbits and of reducing the over-estimation of their elements. In this paper, we will address the problem of determining the connected components of general chain-recurrent sets.

2 ALGORITHM

To compute complete Lyapunov functions, we use our previous algorithms described in (Argáez et al., 2017a; Argáez et al., 2017b; Argáez et al., 2018b; Argáez et al., 2018c; P. Giesl C. Argáez and Wendland, 2018; Argáez et al., 2018a). We firstly transform the dynamical system with the quasi-normalization method introduced in (Argáez et al., 2018b) to the right-hand side of the ODE. That allows to homogenize the solutions' speed of the dynamical systems while maintaining the same trajectories. Therefore, the original dynamical system (1) gets substituted by

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x}), \quad \text{where} \quad \hat{\mathbf{f}}(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\sqrt{\delta^2 + \|\mathbf{f}(\mathbf{x})\|^2}}, \qquad (2)$$

with a small parameter $\delta > 0$ and where $\|\cdot\|$ denotes the Euclidean norm. More details can be found in (Argáez et al., 2018b).

2.1 Mesh-free Collocation

The construction of complete Lyapunov functions can be posed as a generalized interpolation problem. To solve it, mesh-free collocation methods, based on Radial Basis Functions (RBF), have proven to be a powerful methodology (Argáez et al., 2019a).

RBFs are real-valued functions, whose evaluation depends only on the distance from the origin. Examples of RBFs are Gaussians, multiquadrics and Wendland functions. Although, one could use any type of radial basis function, in our work we use Wendland functions, which are compactly supported and positive definite functions (Wendland, 1998), constructed as polynomials on their compact support. The corresponding Reproducing Kernel Hilbert Space is norm-equivalent to a Sobolev space.

Note that in the context of RBF, positive definite function ψ refers to the matrix $(\psi(\|\mathbf{x}_i-\mathbf{x}_j\|))_{i,j}$ being positive definite for $X=\{\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N\}$, where $\mathbf{x}_i\neq\mathbf{x}_j$ if $i\neq j$.

2.1.1 Wendland Functions

Their general form is $\psi(\mathbf{x}):=\psi_{l,k}(c\|\mathbf{x}\|)$, where $c>0$ and $k\in\mathbb{N}$ is a smoothness parameter. For our application the parameter l is fixed as $l=\lfloor\frac{n}{2}\rfloor+k+1$.

The Reproducing Kernel Hilbert Space corresponding to $\psi_{l,k}$ contains the same functions as the Sobolev space $W_2^{k+(n+1)/2}(\mathbb{R}^n)$ and the spaces are norm equivalent. The functions $\psi_{l,k}$ are defined by the recursion:
For $l\in\mathbb{N}$ and $k\in\mathbb{N}_0$, we define

$$\begin{aligned}\psi_{l,0}(r)&=(1-r)_+^l,\\ \psi_{l,k+1}(r)&=\textstyle\int_r^1 t\psi_{l,k}(t)\mathrm{d}t\end{aligned}\tag{3}$$

for $r\in\mathbb{R}_0^+$, where $x_+=x$ for $x\geq 0$ and $x_+=0$ for $x<0$.

2.1.2 Collocation Points

In all our computations we use $X=\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}\subset\mathbb{R}^n$ as collocation points, which is a subset of a hexagonal grid with fineness-parameter $\alpha_{\text{Hexa-basis}}\in\mathbb{R}^+$ constructed according to the next equation:

$$\begin{gathered}\left\{\alpha_{\text{Hexa-basis}}\sum_{k=1}^{n} i_k\omega_k : i_k\in\mathbb{Z}\right\},\\ \omega_k=\sum_{j=1}^{k-1}\varepsilon_j\mathbf{e}_j+(k+1)\varepsilon_k\mathbf{e}_k \text{ and } \varepsilon_k=\sqrt{\frac{1}{2k(k+1)}}.\end{gathered}\tag{4}$$

Here $\mathbf{e}_j$ is the usual jth unit vector. The hexagonal grid has been shown to minimize the condition numbers of the collocation matrices for a fixed fill distance (Iske, 1998).

Since $\mathbf{f}(\mathbf{x})=\mathbf{0}$ for all equilibria $\mathbf{x}$, we remove all equilibria from the set of collocation points X; not doing so would cause the collocation matrix to be singular.

The approximation v is then given by the function that satisfies the PDE $v'(\mathbf{x})=-1$ at all collocation points and it is norm minimal in the corresponding Reproducing Kernel Hilbert space. Practically, we compute v by solving a system of N linear equations, where N is the number of collocation points.

2.1.3 Evaluation Grid

Once we have solved the PDE on the collocation points, we use a different evaluation grid $Y_{\mathbf{x}_j}$, around each collocation point $\mathbf{x}_j$.

Such an evaluation grid can be constructed in many different ways. Important is, however, to always correlate each evaluation point to the original collocation point used to construct it.

In this paper, we use two different grids to evaluate the complete Lyapunov functions to guarantee that our chain-recurrent set classification method is independent of the evaluation points.

The first one is a directional grid introduced in (Argáez et al., 2018c; Argáez et al., 2018a), which places all evaluation points aligned to the flow of the ODE system.

$$\begin{aligned}&Y_{\mathbf{x}_j}\\ =\;&\left\{\mathbf{x}_j\pm\frac{r\cdot k\cdot\alpha_{\text{Hexa-basis}}\cdot\hat{\mathbf{f}}(\mathbf{x}_j)}{m\|\hat{\mathbf{f}}(x_j)\|} : k\in\{1,\ldots,m\}\right\}\end{aligned}$$

$\alpha_{\text{Hexa-basis}}$ is the parameter used to build the hexagonal grid defined above, $r\in(0,1)$ is the ratio up to which the evaluation points will be placed and $m\in\mathbb{N}$ denotes the number of points in the evaluation grid that will be placed on both sides of the collocation points aligned to the flow.

This means that there will not be any evaluated points to provide information about the dynamical system other than in the direction of the flow. On the other hand, this evaluation grid avoids exponential growth of evaluation points as the system's dimension becomes higher.

The second evaluation grid is a set originally proposed in (Argáez et al., 2019b) and allows to obtain information from all directions. It is built using a hexagonal grid of smaller size around each collocation point and it is, therefore, aligned to the basis in (4).

We define

$$\begin{aligned} Y_{\mathbf{x}_j} & \\ = & \left\{ \mathbf{x}_j + \frac{\alpha_{\text{Hexa-basis}}}{2i_{max}+1} \sum_{k=1}^{n} i_k \omega_k : (i_1, \ldots, i_n) \right. \\ & \left. \in [-i_{max}, i_{max}]^n \setminus \{(0, \ldots, 0)\} \right\} \end{aligned} \quad (5)$$

with $i_{max} \in \mathbb{N}$. $\alpha_{\text{Hexa-basis}}$, again, is the parameter used to build the collocation grid and ω_k is also defined in (4).

2.1.4 Construction of Complete Lyapunov Function; Classification of the Chain-recurrent Set

The solution of $V'(\mathbf{x}) = -1$ is approximated by v at the collocation points X. A tolerance parameter $-1 < \gamma \leq 0$ is defined and every collocation point $\mathbf{x}_j$ such that there exists a $\mathbf{y} \in Y_{\mathbf{x}_j}$ with $v'(\mathbf{y}) > \gamma$ is marked to be in the chain-recurrent set ($\mathbf{x}_j \in X^0$). The well-approximated points, i.e., for which the condition $v'(\mathbf{y}) \leq \gamma$ holds for all $\mathbf{y} \in Y_{\mathbf{x}_j}$, belong to our approximation of the area of the gradient-like flow ($\mathbf{x}_j \in X^-$).

After that, the Lyapunov function can be reconstructed with further iterations, in which now v is approximated by solving $V'(\mathbf{x}_j)$ equal to the average of the orbital derivative over all points in $Y_{\mathbf{x}_j}$ (or zero if the average is positive). The whole procedure is explained in Algorithm 2.1.

2.2 Clustering Algorithm to Classify Orbits

Our new clustering algorithm is based on the fact that the distance between two adjacent points in the collocation grid is $\alpha_{\text{Hexa-basis}}$, which is easy to see from (4) because

$$\begin{aligned} \|\omega_k\|^2/\alpha^2_{\text{Hexa-basis}} &= \\ &= \sum_{j=1}^{k-1} \varepsilon_j^2 + (k+1)^2 \varepsilon_k^2 \\ &= \frac{1}{2}\left(1 - \frac{1}{k}\right) + \frac{k+1}{2k} \\ &= 1. \end{aligned} \quad (6)$$

Based on this fact, we designed an algorithm that is capable to identifying different connected components by measuring gaps larger than $\alpha_{\text{Hexa-basis}}$.

Our algorithm is:

Algorithm 2.1. *1. Compute the approximated Lyapunov function v_i and the orbital derivative v_i' for $i = 0$ by solving $v_i'(\mathbf{x}_j) = -1$ at the collocation points*

2. Approximate the chain-recurrent set by X^0 by computing $v_i'(\mathbf{y})$ for all $\mathbf{y} \in Y_{\mathbf{x}_j}$ for each collocation point $\mathbf{x}_j$. If $v_i'(\mathbf{y}) > \gamma$ for any $\mathbf{y} \in Y_{\mathbf{x}_j}$, then $\mathbf{x}_j \in X^0$, else $\mathbf{x}_j \in X^-$, where $\gamma \leq 0$ is a predefined critical value

3. Measure all distances from the origin to the different failing points; this gives all radii of the chain-recurrent sets

4. Sort all points in an increasing order according to their distance from the origin

5. Measure the difference in radii-length between every two consecutive points. Gaps are considered to happen when difference in distance is greater than $\alpha_{\text{Hexa-basis}}$ for two consecutive points

6. All points before the first gap are considered to be a part of the first set of connected components. Between the first and the second gap, all points are considered to be part of the second set of connected component, etc. After we have found all gaps, the last one of them and the longest radius length define the last set of connected components

7. Classify the sets of connected components accordingly in different subsets: each element of a set of component needs to be checked to have neighbours. To do that, once the set of components is chosen, the distance between their elements are measured. If the distance between two points is bigger than $\alpha_{\text{Hexa-basis}}$ such points are not considered to be neighbours. If such distance is equal (or lower) than $\alpha_{\text{Hexa-basis}}$, they are classified to belong to the same component. All points in the boundary of the domain are removed from the subclassifications since it is observed that at the boundary of the domain the approximation tends to fail

8. Define $\tilde{r}_j = \left(\frac{1}{M}\sum_{\mathbf{y} \in Y_{\mathbf{x}_j}} v_i'(\mathbf{y})\right)_-$, where M is the total amount of evaluation points $Y_{\mathbf{x}_j}$ per collocation point and $x_- = x$ for $x < 0$ and $x_- = 0$ otherwise

9. Define $r_j = \frac{N}{\sum_{l=1}^{N} |\tilde{r}_l|}\tilde{r}_j$,

10. Compute the approximate solution v_{i+1} of $v_{i+1}'(\mathbf{x}_j) = r_j$ for $j = 1, \ldots, N$

11. Set $i \to i+1$ and repeat steps 2) to 10) until a predefined criterion is satisfied. The criteria could be: completion of a predefined defined number of iterations or until no more points get added to the chain-recurrent set

Note that in step 9 we normalize the right-hand side of the equation in step 10 in the l^1 norm. This is done to avoid that the right-hand side converges to zero.

3 RESULTS

We present how our algorithm works for five different systems.

3.1 Two Circular Periodic Orbits

We consider system (1) with right-hand side

$$\mathbf{f}(x,y) = \begin{pmatrix} -x(x^2+y^2-1/4)(x^2+y^2-1)-y \\ -y(x^2+y^2-1/4)(x^2+y^2-1)+x \end{pmatrix}. \quad (7)$$

This system has an asymptotically stable equilibrium at the origin. Moreover, the system has two periodic circular orbits: an asymptotically stable periodic orbit at $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ and a repelling periodic orbit at $\Omega_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1/4\}$.

To compute the complete Lyapunov function with our method we used the Wendland function $\psi_{5,3}$. The collocation points were set in a region $[-1.5,1.5] \times [-1.5,1.5] \subset \mathbb{R}^2$ and we used a hexagonal grid (4) with $\alpha_{Hexa-basis} = 0.0163$. The evaluation grid was computed with the hexagonal sub-grid (5) with parameter $i_{max} = 3$.

We computed this example with the almost-normalized method $\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x})$ with $\delta^2 = 10^{-8}$ and $\gamma = -0.25$. The complete Lyapunov function and its orbital derivative over the failing points are shown in Figure 1.

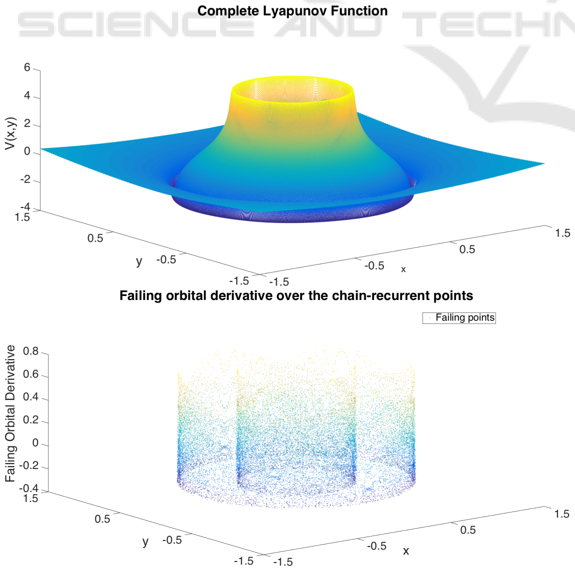

Figure 1: Upper figure: Complete Lyapunov function at the initial iteration for system (7). Lower figure: Values of the orbital derivative over the chain-recurrent set.

The distances that allowed to classify the orbits are shown in Fig. 2, note that these examples are computed with the original iteration ite=0.

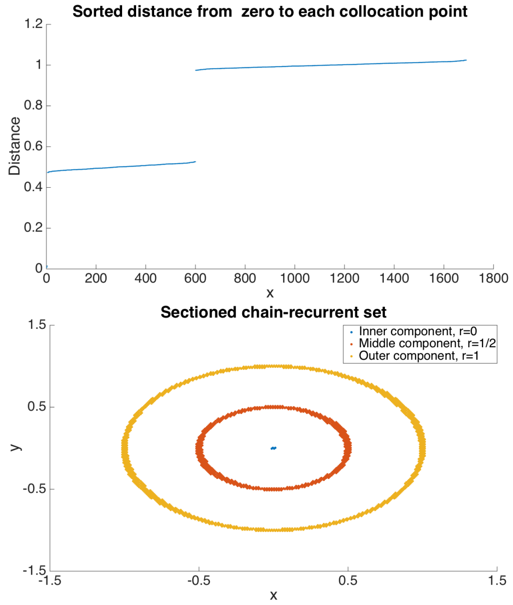

Figure 2: Upper: Sorted distance of each failing collocation point from zero for system (7). Lower: Three identified sets within the chain-recurrent set for (7). These sets are formed with two orbits with radii $r = 1$ and $r = 1/2$ and the failing points around the origin.

In Fig. 2 there are two main gaps, giving three connected components: The points near zero and the two remaining orbits. This allows us to classify the origin and the two orbits as subsets of the chain-recurrent set for system (7), see Fig. 2.

3.2 Van der Pol Oscillator

System (8) is a two-dimensional form of the Van der Pol oscillator. The system has an asymptotically stable periodic orbit and an unstable equilibrium at the origin.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{f}(x,y) = \begin{pmatrix} y \\ (1-x^2)y-x \end{pmatrix} \quad (8)$$

For computing the complete Lyapunov function associated to system (8), we set $\alpha_{\text{Hexa-basis}} = 0.05$ over the area defined by $[-4.0,4.0]^2 \subset \mathbb{R}^2$. The Wendland function parameters used are $(l,k,c) = (4,2,1)$, the critical value $\gamma = -0.5$, and $\delta^2 = 10^{-8}$. The evaluation grid was computed with the hexagonal subgrid with parameter $i_{max} = 3$.

The complete Lyapunov function at the initial iteration and the orbital derivative over the chain-recurrent set is shown in Fig. 3

Unlike system (7), the current system under analysis has only one periodic orbit which is not circular. Fig. 4 allows to understand how Euclidean norm of each element of the chain-recurrent set looks when sorted in an increasing order.

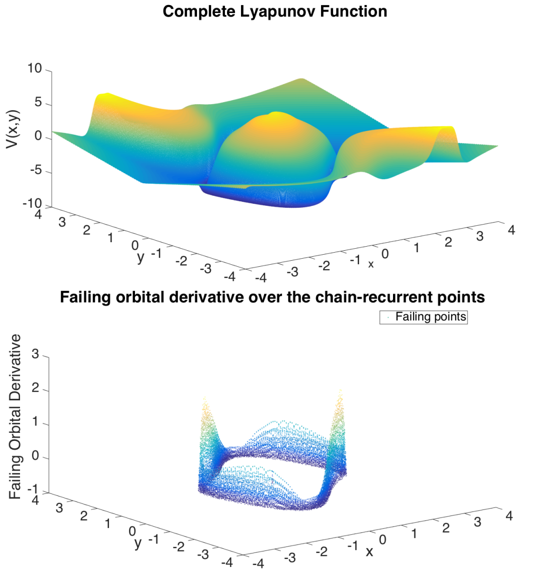

Figure 3: Upper figure: Complete Lyapunov function at the initial iteration for system (8). Lower figure: Values of the orbital derivative over the chain-recurrent set.

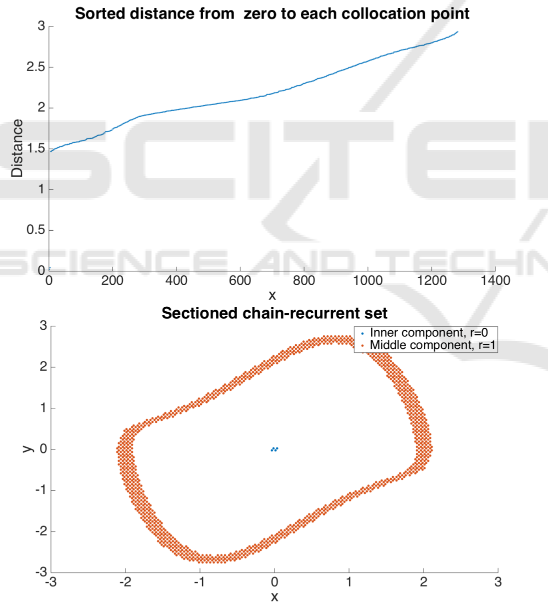

Figure 4: Upper: Sorted distance of each failing collocation point from zero for system (8). Lower: Two identified sets within the chain-recurrent set for (8).

3.3 Homoclinic Orbit

As in (Argáez et al., 2017a), we also consider here the following example

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{f}(x,y) = \begin{pmatrix} x(1-x^2-y^2)-y((x-1)^2+(x^2+y^2-1)^2)\\ y(1-x^2-y^2)+x((x-1)^2+(x^2+y^2-1)^2)\end{pmatrix}. \tag{9}$$

The origin is an unstable focus and the system has an asymptotically stable homoclinic orbit at a circle centred at the origin and with radius 1, connecting the equilibrium $(1,0)$ with itself.

We used the Wendland function $\psi_{4,2}$ for our computations. Our collocation points were defined in the region $[-1.5,1.5]\times[-1.5,1.5]\subset\mathbb{R}^2$ with a hexagonal grid (4) with $\alpha_{Hexa-basis}=0.0125$. In this example, we have used the normalized method, i.e. we replaced $\mathbf{f}$ by $\hat{\mathbf{f}}$ as in (2) with $\delta^2=10^{-8}$, and we used $\gamma=-0.75$. As before, the evaluation grid was computed with the hexagonal subgrid with parameter $i_{max}=3$.

The Lyapunov function at the initial iteration is shown in Fig. 5 along with the orbital derivative over the chain-recurrent set. The distances that allow to classify the components are shown in Fig. 6 along with the classified orbits. As with the Van der Pol oscillator, we have two connected components: the equilibrium at the origin and the unit circle. The equilibrium at the origin is clearly over-estimated.

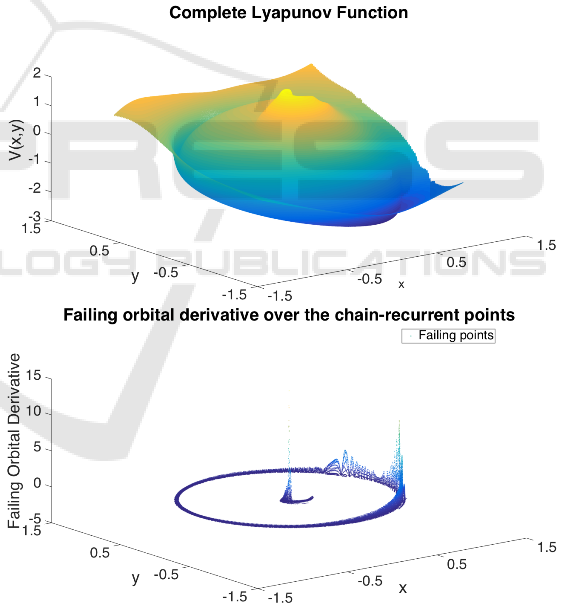

Figure 5: Upper figure: Complete Lyapunov function at the initial iteration for system (9). Lower figure: Values of the orbital derivative over the chain-recurrent set.

It can be seen in the examples given for systems (7), (8) and (9) that the connected components of the chain-recurrent set are subsets of lower dimension than the state space. This was the main tool used in our heuristic algorithm in (Argáez et al., 2019b). However, there are obviously examples where this assumption does not hold; we will see that our new algorithm can easily cope with such examples.

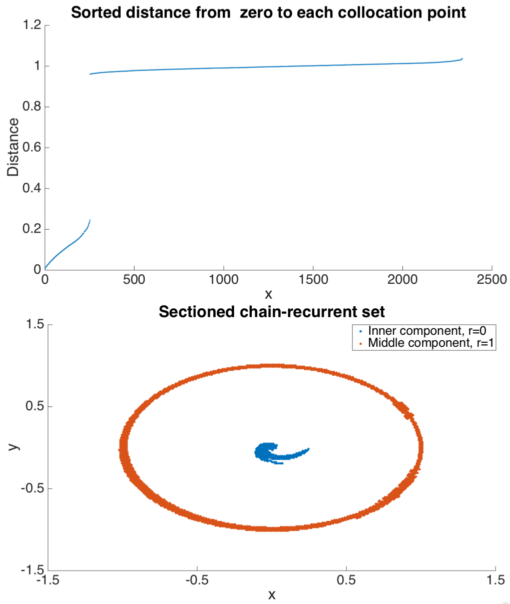

Figure 6: Sorted distance of each failing collocation point from zero for system (9). Lower: These sets form an orbit with radius $r = 1$ and the failing points around the origin for system (9).

3.4 Two-dimensional Chain-recurrent Set

We consider the system with right-hand side

$$\mathbf{f}(x,y) = \begin{pmatrix} x\Gamma(x,y) - y \\ y\Gamma(x,y) + x \end{pmatrix} \text{ with}$$

$$\Gamma(x,y) = \begin{cases} (1-x^2-y^2)^3 & \text{if } x^2+y^2 < 1 \\ 0 & \text{if } 1 \leq x^2+y^2 \leq 4 \\ (4-x^2-y^2)^3 & \text{if } x^2+y^2 > 4 \end{cases} \quad (10)$$

The origin is an unstable equilibrium, and there is a family of periodic orbits of radius $[1,2]$. Hence, the chain-recurrent set has two connected components: the origin and the annulus between the circles of radius 1 and 2.

We used the Wendland function $\psi_{4,2}$ for our computations and set our collocation points in the region $[-1.5,1.5] \times [-1.5,1.5] \subset \mathbb{R}^2$ with a hexagonal grid (4) with $\alpha_{Hexa-basis} = 0.0163$. In this example, we have used the normalized method, i.e. we replaced $\mathbf{f}$ by $\hat{\mathbf{f}}$ as in (2) with $\delta^2 = 10^{-8}$, and we used $\gamma = -0.97$.

The evaluation grid was computed with the directional grid with parameter $m = 11$. The Lyapunov function at the initial iteration is shown in Fig. **??** along with the orbital derivative over the chain-recurrent set. The distances that allowed to classify the orbits are shown in Fig. 8 along with the classified connected components. Our previous heuristic algorithm (Argáez et al., 2019b) would not have been able to classify this two-dimensional component of the chain-recurrent set.

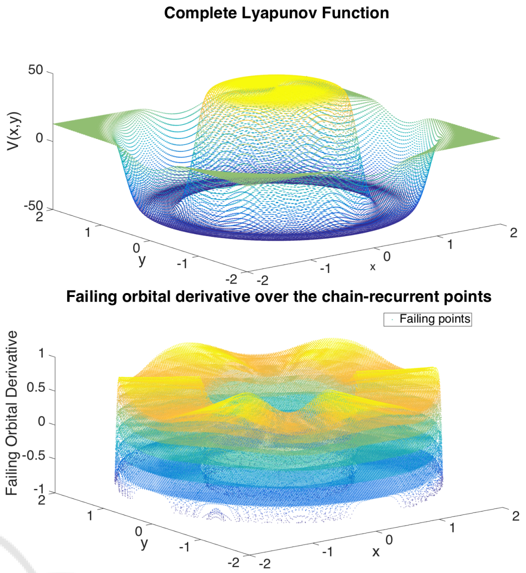

Figure 7: Upper figure: Complete Lyapunov function at the initial iteration for system (10). Lower figure: Values of the orbital derivative over the chain-recurrent set.

3.5 Three-dimensional Systems

Giving the nature of this algorithm, one could consider to extend its application to three dimensional systems. In fact, the algorithm is based on distances from the origin regardless of the system's dimension. Therefore, the algorithm explained in Sec. 2 is also valid for three dimensional systems, under the observation that now the distances from zero are measured for points with three coordinates instead of two.

Let us consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathbf{f}(x,y,z) = \begin{pmatrix} \mu x - y - xz, \\ x + \mu y \\ -z + x^2 z + y^2 \end{pmatrix} \quad (11)$$

which has been introduced in (Chen and J. Shen, 2014); we use $\mu = 0.1$. The area used to build the collocation points is: $[-0.7,0.7] \times [-0.7,0.7] \times [-0.4,0.4]$ with $\alpha = 0.039$, $\gamma = -0.25$. This problem was iterated 55 times.

Figure 9 shows the set of failing evaluation points, characterizing the chain-recurrent set. that failed together with the set of collocation points that were identify for our algorithm. It can be seen in the extreme superior part of the figure, a collection of points that were identify in the chain-recurrent set. However, no collocation points are associated to them in

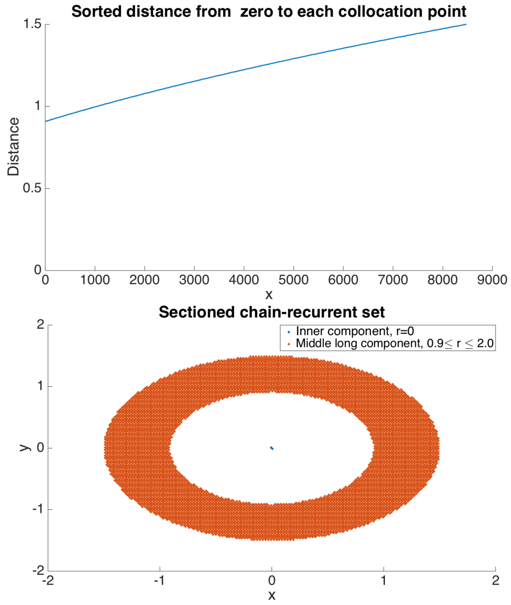

Figure 8: Upper: Sorted distance of each failing collocation point from zero for system (10). Lower: Two identified sets within the chain-recurrent set for (10). These sets are formed with one large orbit with radius $r \leq 1.5$ and $r \geq 0.9$ and the failing points around the origin.

the classification because they belong to the boundary of our domain. So, the subclassified collocation points are only those that fail by the approximation. A first step to clean the chain-recurrent set is to discharge all points in the boundary of the domain.

4 CONCLUSIONS

In this contribution we have introduced an algorithm capable of finding and classifying the different connected components of the chain-recurrent set. The algorithm works for arbitrary chain-recurrent sets and is thus a major improvement from our previous heuristic algorithm (Argáez et al., 2019b). Based on the algorithm in this paper, we can now address the problem of over-estimation of the chain-recurrent set in future work, following the idea of (Argáez et al., 2019b).

ACKNOWLEDGEMENTS

The first author in this paper is supported by the Icelandic Research Fund (Rannís) grant number 163074-052, Complete Lyapunov functions: Efficient numerical computation.

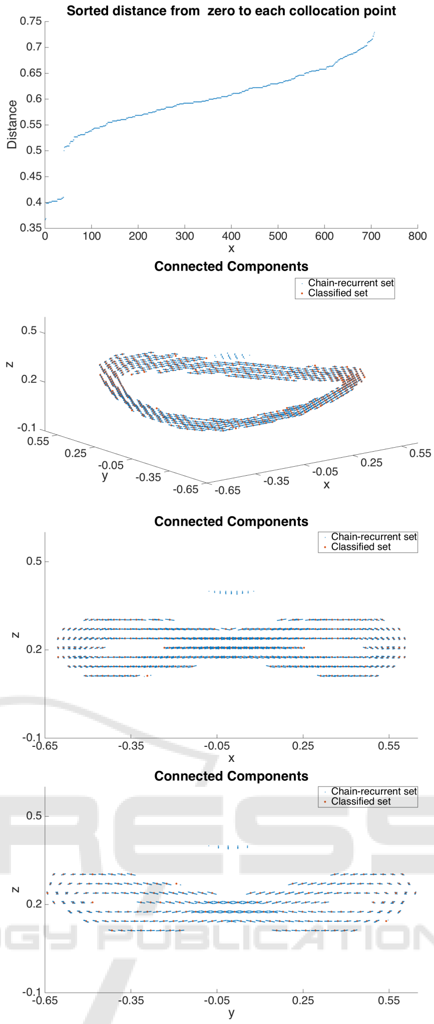

Figure 9: Upper first: Sorted distance of each failing collocation point from zero for system (11). Upper second: Identified set within the chain-recurrent set for system (11). Lower first: Identified set within the chain-recurrent set for system (11), *xz* plane. Lower second: Identified set within the chain-recurrent set for system (11), *yz* plane. All figures were obtained with 55 iterations.

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