Risk-sensitive Markov Decision Processes with Risk Constraints of Coherent Risk Measures in Fuzzy and Stochastic Environment

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Abstract: Risk-sensitive decision making with constraints of coherent risk measures is discussed in Markov decision processes. Risk-sensitive expected rewards under utility functions are approximated by weighted average value-at-risks, and risk constraints are described by coherent risk measures. In this paper, coherent risk measures are represented as weighted average value-at-risks with the best risk spectrum derived from decision maker's risk averse utility, and the risk spectrum can inherit the risk averse property of the decision maker's utility as weighting. By perception-based extension for fuzzy random variables, a dynamic portfolio model with coherent risk measures is introduced. To find feasible regions, firstly a dynamic risk-minimizing problem is discussed by mathematical programming. Next a risk-sensitive reward maximization problem under the feasible coherent risk constraints is demonstrated. A few numerical examples are given to understand the obtained results.

1 INTRODUCTION

Risk-sensitive decision making is one of most important themes in management sciences and so on. Risk-sensitive expected rewards and risk measures are reasonable and effective tools in risk-sensitive decision making. Risk-sensitive expectation, which was introduced by (Howard and Matheson, 1972), is given by

$$f^{-1}(E(f(\cdot))), \tag{1}$$

where f and f^{-1} are decision maker's utility function and its inverse function and $E(\cdot)$ is an expectation. Risk-sensitive expectation is a method to estimate random risks through utility functions, and it is studied by several authors. (Bäuerle and Rieder, 2014). However this criterion with non-linear utility functions f has computational complexity in general. For example, let $\{X_t\}$ is a sequence of random variables. Then $\sum_t E(f(X_t))$ implies a sum of decision maker's expected utility values and it is non-sense. While $f^{-1}(E(f(X_t)))$ belongs to a space of values where random variables X_t take, and their sum with respect to t has meaning. Therefore in dynamic optimization problems we need to compute a sum of values with criterion (1) with the inverse function f^{-1} by Bellman equations. When f are non-linear utility functions, it is difficult to compute the optimal values immediately (Bäuerle and Rieder, 2014).

In decision making, several risk measures have been proposed for economic theory, financial analysis, asset management and engineering. The variance was classically used as a risk measure in decision processes, and the risk measure has been improved from both practical and theoretical aspects. Nowadays drastic declines of asset prices are studied, and *value-at-risk (VaR)* is used widely to estimate the risk of asset price decline in practical management (Jorion, 2006). VaR is defined by percentiles at a specified probability, however it does not have coherency. *Coherent risk measures* have been studied to improve the criterion of risks with worst scenarios (Artzner et al., 1999). Several improved risk measures based on VaR are proposed: for example, conditional value-at-risks, expected shortfall, entropic value-at-risk (Rockafellar and Uryasev, 2000), (Tasche, 2002). Recently (Kusuoka, 2001) gave a spectral representation for coherent risk measures, and (Acerbi, 2002) and (Adam et al., 2008) discussed its applications to portfolio selection and so on. Further (Yoshida, 2018) has introduced a *spectral weighted average value-at-risk* as the best coherent risk measure derived from utility functions. Using this derived coherent risk measure, the risk measure can inherit the risk averse property of the decision maker's utility function as risk spectrum

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weighting. This paper adopts the spectral weighted average value-at-risks to estimate risk-sensitive rewards under constraints, which is also a kind of risk-sensitive extended model of (Yoshida, 2017).

Fuzzy random variables, which were introduced by (Kwakernaak, 1978), are applied to decision making under uncertainty with fuzziness such as linguistic data in engineering, economics et al.. To represent uncertainty, we use *fuzzy random variables* which have two kinds of uncertainties, i.e. randomness and fuzziness. In this paper, randomness is used to represent the uncertainty regarding the belief degree of frequency, and fuzziness is applied to linguistic imprecision of data because of a lack of information about the current stock market. In this paper, using fuzzy random variables, we deal with optimization of portfolio allocation in an environment with both randomness and fuzziness. We extend coherent risk measures and a risk-sensitive estimation for real-valued random variables to one regarding fuzzy random variables from the viewpoint of perception-based method in (Yoshida, 2007), and we apply the perception-based criteria to estimate the uncertainties. (Yoshida, 2006) introduced the mean, the variance and the covariances of fuzzy random variables, using *evaluation weights* and θ-mean functions. This paper estimates fuzzy numbers and fuzzy random variables by probabilistic expectation and these criteria, which are characterized by *possibility and necessity criteria* for subjective estimation and *pessimistic-optimistic indexes* for subjective decision.

In Section 2, we introduce coherent risk measures and their spectral representation for coherent risk measures based on (Kusuoka, 2001), and a coherent risk measure is given with the best risk spectrum derived from decision maker's utility. In Section 3, we introduce coherent risk measures and a risk-sensitive estimation for fuzzy random variables by perception-based extension, and we give estimation tools with evaluation weights and θ-mean functions in order to evaluate the randomness and fuzziness for fuzzy random variables. In Section 4, we discuss a risk-sensitive decision problem under risk constraints by use of coherent risk measures. Then risk-sensitive rewards are approximated by weighted average value-at-risks with the risk spectrum derived from the utility, and the risk constraints are described by coherent risk measures which are represented by weighted average value-at-risks. In Section 5 we investigate the lower bound of risk values to find feasible regions of the constraints. In Section 6 we discuss maximization of risk-sensitive rewards under risk conditions. In Section 7, we give a few numerical examples to understand the obtained results.

2 COHERENT RISK MEASURES DERIVED FROM RISK AVERSE UTILITY

Let $\mathbf{R} = (-\infty, \infty)$ and let P be a non-atomic probability on a sample space Ω. Let $\mathcal{X}$ be the family of all integrable real-valued random variables X on Ω with a continuous distribution $x \mapsto F_X(x) = P(X < x)$ for which there exists a non-empty open interval I such that $F_X(\cdot) : I \to (0,1)$ is strictly increasing and onto. Then there exists a strictly increasing and continuous inverse function $F_X^{-1} : (0,1) \to I$. For a probability $p \in (0,1)$, *value-at-risk (VaR)* is given by the percentile of the distribution F_X, i.e.

$$\mathrm{VaR}_p(X) = F_X^{-1}(p). \tag{2}$$

Then *average value-at-risk (AVaR)* at a probability $p \in (0,1]$ is given by

$$\mathrm{AVaR}_p(X) = \frac{1}{p} \int_0^p \mathrm{VaR}_q(X)\, dq. \tag{3}$$

The following fundamental concepts are well-known (Artzner et al., 1999, Kusuoka, 2001).

Definition 1. Let a map $\rho : \mathcal{X} \mapsto \mathbf{R}$.

(i) Random variables $X (\in \mathcal{X})$ and $Y (\in \mathcal{X})$ are called *comonotonic* if $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ holds for almost all $\omega, \omega' \in \Omega$.

(ii) ρ is called *comonotonically additive* if $\rho(X+Y) = \rho(X) + \rho(Y)$ holds for all comonotonic $X, Y \in \mathcal{X}$.

(iii) ρ is called *law invariant* if $\rho(X) = \rho(Y)$ holds for all $X, Y \in \mathcal{X}$ satisfying $P(X < \cdot) = P(Y < \cdot)$.

(iv) ρ is called *continuous* if $\lim_{n\to\infty} \rho(X_n) = \rho(X)$ holds for $\{X_n\} \subset \mathcal{X}$ and $X \in \mathcal{X}$ such that $\lim_{n\to\infty} X_n = X$ almost surely.

Hence the following definition characterizes coherent risk measures (Artzner at al., 1999).

Definition 2. A map $\rho : \mathcal{X} \mapsto \mathbf{R}$ is called a *coherent risk measure* if it satisfies the following (i) – (iv):

(i) $\rho(X) \geq \rho(Y)$ for $X, Y \in \mathcal{X}$ satisfying $X \leq Y$. (*monotonicity*)

(ii) $\rho(cX) = c\rho(X)$ for $X \in \mathcal{X}$ and $c \in \mathbf{R}$ satisfying $c \geq 0$. (*positive homogeneity*)

(iii) $\rho(X + c) = \rho(X) - c$ for $X \in \mathcal{X}$ and $c \in \mathbf{R}$. (*translation invariance*)

(iv) $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$. (*subadditivity*)

It is known in (Artzner et al., 1999) that $-\mathrm{AVaR}_p(\cdot)$ is a coherent risk measure however

$-\mathrm{VaR}_p(\cdot)$ is not coherent because sub-additivity (iv) does not hold, where $-$ means the minus singature. Conditional value-at-risks and expected shortfall are also famous coherent risk measures (Rockafellar and Uryasev, 2000, Tasche, 2002). Now, for a probability $p \in (0,1]$ and a non-increasing right-continuous function λ on $[0,1]$ satisfying $\int_0^1 \lambda(q)\,dq = 1$, we define a *weighted average value-at-risk* with weighting λ on $(0,p)$ by

$$\mathrm{AVaR}_p^\lambda(X) = \int_0^p \mathrm{VaR}_q(X)\,\lambda(q)\,dq \Big/ \int_0^p \lambda(q)\,dq. \tag{4}$$

Then λ is called a *risk spectrum*, and $-\mathrm{AVaR}_p^\lambda$ becomes a coherent risk measure. Further recently (Kusuoka, 2001) proved coherent risk measures are represented by weighted average value-at-risks in the following *spectral representation* (Yoshida, 2018).

Lemma 1. *Let $\rho : \mathcal{X} \mapsto \mathbf{R}$ be a law invariant, comonotonically additive, continuous coherent risk measure. Then there exists a risk spectrum λ such that*

$$\rho(X) = -\mathrm{AVaR}_1^\lambda(X) \tag{5}$$

for $X \in \mathcal{X}$. Further, $-\mathrm{AVaR}_p^\lambda$ is a coherent risk measure on $\mathcal{X}$ for $p \in (0,1)$.

In this paper we use a law invariant, comonotonically additive, continuous coherent risk measure ρ, and we also deal with a case when value-at-risks are represented as

$$\mathrm{VaR}_p(X) = E(X) + \kappa(p)\cdot\sigma(X) \tag{6}$$

with the mean $E(X)$ and the standard deviation $\sigma(X)$ of random variables $X \in \mathcal{X}$, where $\kappa : (0,1) \mapsto \mathbf{R}$ is an increasing function. From (4) and (6) we have

$$\mathrm{AVaR}_p^\lambda(X) = E(X) + \kappa^\lambda(p)\cdot\sigma(X), \tag{7}$$

where

$$\kappa^\lambda(p) = \int_0^p \kappa(q)\,\lambda(q)\,dq \Big/ \int_0^p \lambda(q)\,dq. \tag{8}$$

Let $f : I \mapsto \mathbf{R}$ be a C^2-class risk averse utility function satisfying $f' > 0$ and $f'' \le 0$ on I, where I is an open interval. For a probability $p \in (0,1]$ and a random variable $X \in \mathcal{X}$, a non-linear risk-sensitive form

$$f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,dq\right) \tag{9}$$

is an average value-at-risk of X on the downside $(0,p)$ under utility f. We note that (9) is reduced to (3) if f is risk-neutral, i.e. it is a linear increasing function. Hence we have the following lemma from (Yoshida, 2018).

Lemma 2. *A risk spectrum λ which minimizes the distance between (9) and (4):*

$$\sum_{X\in\mathcal{X}} \left(f^{-1}\left(\frac{1}{p}\int_0^p f(\mathrm{VaR}_q(X))\,dq\right) - \mathrm{AVaR}_p^\lambda(X)\right)^2 \tag{10}$$

for $p \in (0,1]$ is given by

$$\lambda(p) = e^{-\int_p^1 C(q)\,dq}\,C(p) \tag{11}$$

with a component function C in (Yoshida, 2018) if λ is non-increasing.

For exponential utility function f, the corresponding component function C is given concretely in Example 2. The component functions C for several utilities f are also investigated in (Yoshida, 2018). In Lemma 2 the coherent risk measure $-\mathrm{AVaR}_p^\lambda$ has a kind of semi-linear property such as Definition 2(ii)(iii) and it brings us effective computation, and the risk spectrum λ can also inherit the risk averse property of the non-linear utility function f as weighting on $(0,p)$. Regarding risk-sensitive rewards (1), in the sequel we use the risk spectrum λ in Lemma 2 because $-\mathrm{AVaR}_p^\lambda$ is the best coherent risk measure derived from risk averse utility f.

3 FUZZINESS AND EXTENDED CRITERIA

A *fuzzy number* is represented by its membership function $\tilde{n} : \mathbf{R} \to [0,1]$ which is normal, upper-semicontinuous, fuzzy convex and has a compact support (Zadeh, 1965). Let $\mathcal{N}$ be the set of all fuzzy numbers. For a fuzzy number $\tilde{n} \in \mathcal{N}$, its α-cuts are given by closed intervals $\tilde{n}_\alpha = \{x \in \mathbf{R} \mid \tilde{n}(x) \ge \alpha\} = [\tilde{n}_\alpha^-, \tilde{n}_\alpha^+]$ for $\alpha \in (0,1]$. An addition and a scalar multiplication for fuzzy numbers are defined by their α-cuts. For fuzzy numbers $\tilde{n}, \tilde{m} \in \mathcal{N}$, *fuzzy max order* $\tilde{n} \succeq \tilde{m}$ means that $\tilde{n}_\alpha^\pm \ge \tilde{m}_\alpha^\pm$ for all $\alpha \in (0,1]$. A fuzzy-number-valued map $\tilde{X} : \Omega \to \mathcal{N}$ is called a *fuzzy random variable* if $\tilde{X}_\alpha^\pm \in \mathcal{X}$ for all $\alpha \in (0,1]$, where $\tilde{X}_\alpha(\omega) = \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \ge \alpha\} = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)]$ for $\omega \in \Omega$. Let $\tilde{\mathcal{X}}$ be the family of all fuzzy random variables on Ω. (Kruse and Meyer, 1987) gave the expectation of fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}$ in the following *perception-based definition* based on Zadeh's extension principle:

$$\tilde{E}(\tilde{X})(x) = \sup_{X\in\mathcal{X}:E(X)=x}\ \inf_{\omega\in\Omega} \tilde{X}(\omega)(X(\omega)) \tag{12}$$

for $x \in \mathbf{R}$, where $E(\cdot)$ is the expectation for real-valued random variables. Then, the expectation

$\tilde{E}(\tilde{X})$ is a fuzzy number with α-cut $\tilde{E}(\tilde{X})_\alpha = [E(\tilde{X}^-_\alpha), E(\tilde{X}^+_\alpha)]$. Define criterion (1) by

$$\varphi(X) = f^{-1}(E(f(X))) \tag{13}$$

for $X \in \mathcal{X}$. For a weighted average value-at-risk AVaR^λ_p, the criterion φ and a coherent risk measure ρ, their extensions for a fuzzy random variable $\tilde{X} \in \tilde{\mathcal{X}}$ are also fuzzy numbers:

$$\widetilde{\mathrm{AVaR}}^\lambda_p(\tilde{X})(x) = \sup_{X \in \mathcal{X} : \mathrm{AVaR}^\lambda_p(X) = x} \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)), \tag{14}$$

$$\tilde{\varphi}(\tilde{X})(x) = \sup_{X \in \mathcal{X} : \varphi(X) = x} \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)), \tag{15}$$

$$\tilde{\rho}(\tilde{X})(x) = \sup_{X \in \mathcal{X} : \rho(X) = x} \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \tag{16}$$

for $x \in \mathbf{R}$. Then their α-cuts are given respectively by $\tilde{\varphi}(\tilde{X})_\alpha = [\varphi(\tilde{X}^-_\alpha), \varphi(\tilde{X}^+_\alpha)]$ and $\tilde{\rho}(\tilde{X})_\alpha = [\rho(\tilde{X}^+_\alpha), \rho(\tilde{X}^-_\alpha)]$, and the extended measure $\tilde{\rho}(\cdot)$ has the following properties similarly to Definition 2 (Yoshida, 2008).

Lemma 3. $\tilde{\rho}(\cdot)$ *is monotonically decreasing, positively homogeneous, translation invariant and subadditive.*

In the latter sections we use a coherent risk measure ρ in Lemma 1 and its extension $\tilde{\rho}$ in (16) to estimate risks in a financial model. We also need defuzzification methods. A defuzzification of a fuzzy number $\tilde{n} \in \mathcal{N}$ with a θ-*mean* and an *evaluation weight* $w(\alpha)$ is given by

$$E^\theta(\tilde{n}) = \frac{\int_0^1 (\theta \cdot \tilde{n}^-_\alpha + (1-\theta) \cdot \tilde{n}^+_\alpha)\, w(\alpha)\, d\alpha}{\int_0^1 w(\alpha)\, d\alpha}, \tag{17}$$

where $\tilde{n}_\alpha = [\tilde{n}^-_\alpha, \tilde{n}^+_\alpha]$. Here θ is called decision maker's *pessimistic index* if $\theta = 1$, and it is also called the *optimistic index* if $\theta = 0$. $w(\alpha)$ is called the *possibility evaluation* if $w(\alpha) = 1$ for $\alpha \in [0,1]$, and it is also called the *necessity evaluation* if $w(\alpha) = 1 - \alpha$ for $\alpha \in [0,1]$ (Yoshida, 2006, 2008). Then $E^\theta(\cdot)$ has the following properties.

Lemma 4. *For* $\theta \in [0,1]$, $E^\theta(\cdot)$ *is positively homogeneous, additive and monotonically increasing.*

The randomness of fuzzy random variables is evaluated by probabilistic expectation, and its fuzziness is estimated by the θ-mean and the weight $w(\alpha)$ as follows: For a fuzzy random variable $\tilde{X} \in \tilde{\mathcal{X}}$, the mean of the expectation $E(E^\theta(\tilde{X}))$ is a real number

$$E(E^\theta(\tilde{X})) = E\left(\frac{\int_0^1 (\theta \cdot \tilde{X}^-_\alpha + (1-\theta) \cdot \tilde{X}^+_\alpha)\, w(\alpha)\, d\alpha}{\int_0^1 w(\alpha)\, d\alpha} \right). \tag{18}$$

From Lemma 4, we obtain the following results (Yoshida, 2006, 2007).

Lemma 5. *For* $\theta \in [0,1]$, $E(E^\theta(\cdot))$ *is positively homogeneous, additive and monotonically increasing, and it has the following properties (i) and (ii):*

(i) $E(E^\theta(\cdot)) = E^\theta(\tilde{E}(\cdot))$.

(ii) $E(E^\theta(\tilde{n})) = E^\theta(\tilde{n})$ *and* $E(E^\theta(X)) = E(X)$ *for* $\tilde{n} \in \mathcal{N}$ *and* $X \in \mathcal{X}$.

Let $\tilde{\mathcal{X}}_a$ be a family of fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}$ for which there exist a random variable $X \in \mathcal{X}$ and a fuzzy number $\tilde{n} \in \mathcal{N}$ such that

$$\tilde{X}(\omega)(x) = 1_{\{X(\omega)\}}(x) + \tilde{n}(x) \tag{19}$$

for $\omega \in \Omega$ and $x \in \mathbf{R}$, where $1_{\{\cdot\}}$ denotes the characteristic function of a singleton. Then we can easily check the following proposition for the weighted average value-at-risks AVaR^λ_p, coherent risk measures ρ and their extensions $\widetilde{\mathrm{AVaR}}^\lambda_p$ and $\tilde{\rho}$ (Yoshida, 2008).

Proposition 1. *For* $\theta \in [0,1]$*, it holds that*

$$E^\theta(\widetilde{\mathrm{AVaR}}^\lambda_p(\tilde{X})) = \mathrm{AVaR}^\lambda_p(E^\theta(\tilde{X})), \tag{20}$$

$$E^{1-\theta}(\tilde{\rho}(\tilde{X})) = \rho(E^\theta(\tilde{X})) \tag{21}$$

for fuzzy random variables $\tilde{X} \in \tilde{\mathcal{X}}_a$.

4 RISK ALLOCATION WITH COHERENT RISK MEASURES

Let a *state space* by $S = \mathbf{R}$ and an *action space* by $\mathcal{A} = \{(x^1, x^2, \cdots, x^n) \in \mathbf{R}^n \mid \sum_{i=1}^n x^i = 1 \text{ and } x^i \geq 0\ (i = 1, 2, \cdots, n)\}$, where n is a positive integer. In this paper we focus on risk-sensitive expected rewards to choose alternatives consisting of n assets. Let a positive integer T be a terminal time, and let time $t = 1, 2, \cdots, T$. Let $\tilde{X}^i_t (\in \tilde{\mathcal{X}})$ be a *fuzzy reward* for *asset* i $(= 1, 2, \cdots, n)$. Hence we put their expectations and covariances respectively by $\mu^i_t = E^\theta(\tilde{E}(\tilde{X}^i_t)) = E(E^\theta(\tilde{X}^i_t))$ and $\sigma^{ij}_t = E((E^\theta(\tilde{X}^i_t) - \mu^i_t)(E^\theta(\tilde{X}^j_t) - \mu^j_t))$ for $i, j = 1, 2, \cdots, n$. We give *Markov policies* by $\pi = \{\pi_t\}_{t=1}^T$ where mappings $\pi_t = (\pi^1_t, \pi^2_t, \cdots, \pi^n_t) : \Omega \mapsto \mathcal{A}$ for $t = 1, 2, \cdots, T$, and then π_t is called a *strategy*. They are chosen depending only on the current state X^{π}_{t-1}. Put a collection of all Markov policies by Π. A reward with a strategy $\pi_t = (\pi_t^1, \pi_t^2, \cdots, \pi_t^n)$ is given by

$$\tilde{X}_t^{\pi} = \sum_{i=1}^{n} \pi_t^i \tilde{X}_t^i. \tag{22}$$

Let a probability $p \in (0,1)$ and let a positive constant δ. Let f be a C^2-class risk averse utility function which is given in Section 2. While let ρ be a coherent risk measure for risk constraints. Let β be a positive constant. Hence we focus on the following optimization problem with (13), (15) and (16).

Problem (P1). Maximize the risk-sensitive estimation

$$\sum_{t=1}^{T} \beta^{t-1} f^{-1}(E(f(E^{\theta}(\tilde{X}_t^{\pi})))) \tag{23}$$

with respect to strategies $\pi_t \in \Pi$ under risk constraint

$$E^{1-\theta}(\tilde{\rho}(\tilde{X}_t^{\pi})) \le \delta \tag{24}$$

for time $t = 1, 2, \cdots, T$.

From the results of Lemma 2, $f^{-1}(E(f(\cdot))) = f^{-1}(\int_0^1 \mathrm{VaR}_q(f(\cdot))\, dq)$ is approximated by $\mathrm{AVaR}_1^{\lambda}(\cdot)$ with a risk spectrum λ. While by Lemma 1 there exists a risk spectrum ν such that $\rho = -\mathrm{AVaR}_p^{\nu}$. Hence we estimate the downside risks on $(0,p)$. By Proposition 1 this paper discusses the following optimization instead of Problem (P1).

Problem (P2). Maximize the risk-sensitive estimation

$$\sum_{t=1}^{T} \beta^{t-1} \mathrm{AVaR}_1^{\lambda}(E^{\theta}(\tilde{X}_t^{\pi})) \tag{25}$$

with respect to strategies $\pi_t \in \Pi$ under risk constraint

$$-\mathrm{AVaR}_p^{\nu}(E^{\theta}(\tilde{X}_t^{\pi})) \le \delta \tag{26}$$

for time $t = 1, 2, \cdots, T$.

In (25) and (26), risk spectra λ and ν are different in general, however we can select same risk spectrum, i.e. $\lambda = \nu$. Hence from (22) the expectation and the standard deviation of reward $\tilde{X}_t^{\pi}$ are

$$E(E^{\theta}(\tilde{X}_t^{\pi})) = \sum_{i=1}^{n} \pi_t^i \mu_t^i \tag{27}$$

and

$$\sigma(E^{\theta}(\tilde{X}_t^{\pi})) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_t^i \pi_t^j \sigma_t^{ij}}. \tag{28}$$

Together with (7), we also have weighted average value-at-risk

$$\mathrm{AVaR}_p^{\nu}(E^{\theta}(\tilde{X}_t^{\pi})) = \sum_{i=1}^{n} \pi_t^i \mu_t^i + \kappa^{\nu}(p) \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_t^i \pi_t^j \sigma_t^{ij}}, \tag{29}$$

where

$$\kappa^{\nu}(p) = \int_0^p \kappa(q)\nu(q)\, dq \Big/ \int_0^p \nu(q)\, dq. \tag{30}$$

In this paper we assume $\kappa^{\lambda}(1) \le 0$ and $\kappa^{\nu}(p) < 0$. Let $\Pi_t(\delta)$ be the collection of strategies π_t satisfying risk constraint (26), and let $\Pi_t = \sup_{\delta>0} \Pi_t(\delta)$. In the rest of this section we investigate the lower bound of $-\mathrm{AVaR}_p^{\nu}(E^{\theta}(\tilde{X}_t^{\pi}))$ for the feasibility of constraint (26) in Problem (P2), i.e. $\Pi_t(\delta) \neq \emptyset$. From (29), we firstly discuss the following maximization problem for $\mathrm{AVaR}_p^{\nu}(E^{\theta}(\tilde{X}_t^{\pi}))$.

Problem (P3). Maximize weighted average value-at-risk

$$\mathrm{AVaR}_p^{\nu}(E^{\theta}(\tilde{X}_t^{\pi})) = \sum_{i=1}^{n} \pi_t^i \mu_t^i + \kappa^{\nu}(p) \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_t^i \pi_t^j \sigma_t^{ij}} \tag{31}$$

with respect to strategies $\pi_t = (\pi_t^1, \pi_t^2, \cdots, \pi_t^n)$.

Let $\gamma \in \mathbf{R}$. From (27), under a constraint

$$E(E^{\theta}(\tilde{X}_t^{\pi})) = \sum_{i=1}^{n} \pi_t^i \mu_t^i = \gamma, \tag{32}$$

Problem (P3) is solved by quadratic programming and then the corresponding value (31) is

$$\gamma + \kappa^{\nu}(p) \sqrt{\frac{A_t\gamma^2 - 2B_t\gamma + C_t}{\Delta_t}}, \tag{33}$$

where

$$\mu = \begin{bmatrix} \mu_t^1 \\ \mu_t^2 \\ \vdots \\ \mu_t^n \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_t^{11} & \sigma_t^{12} & \cdots & \sigma_t^{1n} \\ \sigma_t^{21} & \sigma_t^{22} & \cdots & \sigma_t^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_t^{n1} & \sigma_t^{n2} & \cdots & \sigma_t^{nn} \end{bmatrix}, \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

$A_t = \mathbf{1}^{\mathrm{T}}\Sigma_t^{-1}\mathbf{1}, B_t = \mathbf{1}^{\mathrm{T}}\Sigma_t^{-1}\mu_t, C_t = \mu_t^{\mathrm{T}}\Sigma_t^{-1}\mu_t, \Delta_t = A_tC_t - B_t^2$ and T denotes the transpose of a vector. If $A_t > 0$, $\Delta_t > 0$ and $\kappa^{\nu}(p) < -\sqrt{\Delta_t/A_t}$ are satisfied, we can easily check the real-valued function (19) of γ is concave and it has the maximum $\frac{B_t - \sqrt{A_t\kappa^{\nu}(p)^2 - \Delta_t}}{A_t}$ at $\gamma = \frac{B_t}{A_t} + \frac{\Delta_t}{A_t\sqrt{A_t\kappa^{\nu}(p)^2 - \Delta_t}}$. Since $\sup_{\pi_t \in \Pi}(31) = \sup_{\gamma}\{\sup_{\pi_t \in \Pi : \sum_{i=1}^{n} \pi_t^i \mu_t^i = \gamma}(31)\}$, we obtain the following analytical solutions for Problem (P3).

Theorem 1. *Let $A_t > 0$, $\Delta_t > 0$ and $\kappa^{\nu}(p) < -\sqrt{\Delta_t/A_t}$. Then the following (i) and (ii) hold.*

(i) *The maximum weighted average value-at-risk of Problem (P3) is*

$$\frac{B_t - \sqrt{A_t\kappa^{\nu}(p)^2 - \Delta_t}}{A_t} \tag{34}$$

at the expected reward

$$\gamma = \frac{B_t}{A_t} + \frac{\Delta_t}{A_t\sqrt{A_t\kappa^\nu(p)^2 - \Delta_t}}. \tag{35}$$

The corresponding strategy is given by

$$\pi_t^\circ = \xi_t^\circ \Sigma_t^{-1}\mathbf{1} + \eta_t^\circ \Sigma_t^{-1}\mu_t \tag{36}$$

if $\pi_t^\circ \geq \mathbf{0}$*, where* $\xi_t^\circ = \frac{C_t - B_t\gamma}{\Delta_t}$ *and* $\eta_t^\circ = \frac{A_t\gamma - B_t}{\Delta_t}$.

(ii) *If* $\Sigma_t^{-1}\mathbf{1} \geq \mathbf{0}$, $\Sigma_t^{-1}\mu_t \geq \mathbf{0}$ *and* $\kappa^\nu(p) \geq C_t$*, then the strategy (36) satisfies* $\pi_t^\circ \geq \mathbf{0}$.

5 RISK-SENSITIVE REWARD MAXIMIZATION UNDER FEASIBLE RISK CONSTRAINTS

Let $p \in (0,1)$ be a probability and let ν be a risk spectrum which are given in Section 4. From Theorem 1, we define the lower bound of $-\mathrm{AVaR}_p^\nu(E^\theta(\tilde{X}_t^\pi))$ by a constant $\underline{\delta}_t(p)$:

$$\begin{aligned} \underline{\delta}_t(p) &= \inf_{\pi_t \in \Pi_t}(-\mathrm{AVaR}_p^\nu(E^\theta(\tilde{X}_t^\pi))) \\ &= -\sup_{\pi_t\in\Pi_t} \mathrm{AVaR}_p^\nu(E^\theta(\tilde{X}_t^\pi)) \\ &= -\frac{B_t}{A_t} + \frac{\sqrt{A_t\kappa^\nu(p)^2 - \Delta_t}}{A_t}. \end{aligned} \tag{37}$$

Thus the feasible range of δ in risk constraint (26) is $\{\delta \mid \Pi_t(\delta) \neq \emptyset\} = [\underline{\delta}_t(p), \infty)$. Now we take a risk level $\delta \in [\underline{\delta}_t(p), \infty)$, and then we have $\sup_{\pi_t \in \Pi_t(\delta)}(31) = \sup_\gamma\{\sup_{\pi_t\in\Pi_t(\delta):\sum_{i=1}^n \pi_t^i\mu_t^i = \gamma}(31)\}$. Thus, from the view point of (33), Problem (P2) is reduced to the following problem with constraint (32), i.e. $\sum_{i=1}^n \pi_t^i\mu_t^i = \gamma$.

Problem (P4). Maximize the risk-sensitive estimation

$$\gamma_t + \kappa^\lambda(1)\sqrt{\frac{A_t\gamma_t^2 - 2B_t\gamma_t + C_t}{\Delta_t}} \tag{38}$$

with respect to $\gamma \in \mathbf{R}$ under risk constraint

$$\gamma_t + \kappa^\nu(p)\sqrt{\frac{A_t\gamma_t^2 - 2B_t\gamma_t + C_t}{\Delta_t}} \geq -\delta. \tag{39}$$

Hence (39) is equivalent to $\gamma_t \in [\gamma_t^-, \gamma_t^+]$, where

$$\begin{aligned} \gamma_t^\pm &= \frac{B_t\kappa^\nu(p)^2 + \Delta_t\delta}{A_t\kappa^\nu(p)^2 - \Delta_t} \\ &\mp \frac{\sqrt{\Delta_t}\kappa^\nu(p)\sqrt{A_t\delta^2 + 2B_t\delta + C_t - \kappa^\nu(p)^2}}{A_t\kappa^\nu(p)^2 - \Delta_t}. \end{aligned} \tag{40}$$

By solving concave maximization (38) within constraint $[\gamma_t^-, \gamma_t^+]$ in Problem (P4), we easily obtain the following results for Problem (P2).

Theorem 2. *Let* $A_t > 0$, $\Delta_t > 0$, $\kappa^\nu(p) \leq \kappa^\lambda(1) \leq 0$ *and* $\kappa^\nu(p) < -\sqrt{\Delta_t/A_t}$*. Then the maximum risk-sensitive estimation in Problem (P2) is*

$$\varphi_t^* = \begin{cases} \dfrac{B_t}{A_t} - \dfrac{\sqrt{A_t\kappa^\lambda(1)^2 - \Delta_t}}{A_t} \\ \quad \text{at } \gamma_t^* = \dfrac{B_t}{A_t} + \dfrac{\Delta_t}{A_t\sqrt{A_t\kappa^\lambda(1)^2 - \Delta_t}} \\ \quad \text{if } \delta_t^+ \leq \delta \text{ and } \kappa^\lambda(1) < -\sqrt{\Delta_t/A_t}, \\ \gamma_t^+ - \dfrac{\kappa^\lambda(1)}{\kappa^\nu(p)}(\delta + \gamma_t^+) \\ \quad \text{at } \gamma_t^* = \gamma_t^+ \text{ otherwise}, \end{cases} \tag{41}$$

where $\delta_t^+ = -\dfrac{B_t}{A_t} + \dfrac{A_t\kappa^\lambda(1)\kappa^\nu(p) - \Delta_t}{A_t\sqrt{A_t\kappa^\lambda(1)^2 - \Delta_t}}$.

6 DYNAMIC MAXIMUM RISK-SENSITIVE REWARD UNDER FEASIBLE RISK CONSTRAINTS

Let the initial state be a real number $X_0^\pi = x_0$. Then $E(X_0) = \gamma_0 = x_0$ and $\sigma(X_0)^2 = 0$. For a Markov policy $\pi = \{\pi_t\}_{t=1}^T \in \Pi$, the expectation and the standard deviation of terminal rewards $X_T^\pi = x_0 + \sum_{t=1}^T R_t^\pi$ are

Problem (P5). Maximize the *total risk-sensitive expected immediate reward*

$$\sum_{t=1}^T \beta^{t-1}\left(\gamma_t + \kappa^\lambda(1)\sqrt{\frac{A_t\gamma_t^2 - 2B_t\gamma_t + C_t}{\Delta_t}}\right) \tag{42}$$

with respect to $(\gamma_1, \gamma_2, \cdots \gamma_T) \in \mathbf{R}^T$ under risk constraint

$$\gamma_t \in [\gamma_t^-, \gamma_t^+] \tag{43}$$

for all $t = 1, 2, \cdots, T$.

Lemma 6. *Let* $\{v_t\}$ *be a sequence given by the following optimality equations*

$$\begin{aligned} v_t &= \sup_{\gamma_t \in [\gamma_t^-, \gamma_t^+]}\left\{\gamma_t + \kappa^\lambda(1)\sqrt{\frac{A_t\gamma_t^2 - 2B_t\gamma_t + C_t}{\Delta_t}}\right\} \\ &\quad + \beta v_{t+1} \end{aligned} \tag{44}$$

for $t = 1, 2, \cdots, T$ *and* $v_{T+1} = 0$*. Then* v_1 *is the maximum total risk-sensitive expected immediate reward for Problem (P5).*

From Theorem 2, we have the following results.

Theorem 3. *Let* $A_t > 0$, $\Delta_t > 0$, $\kappa^{\nu}(p) \le \kappa^{\lambda}(1) \le 0$ *and* $\kappa^{\nu}(p) < -\sqrt{\Delta_t/A_t}$ *for* $t = 1, 2, \cdots, T$.

(i) *Let* $\{v_t\}$ *be a sequence given by the following optimality equations*

$$v_t = \varphi_t^* + \beta v_{t+1} \tag{45}$$

for $t = 1, 2, \cdots, T$ *and* $v_{T+1} = 0$. *Then* v_1 *is the maximum of the total risk-sensitive expected rewards in Problem (P5).*

(ii) *Further the optimal portfolios of (44) in Lemma 6 are given by*

$$w_t^* = \xi_t^* \Sigma_t^{-1} \mathbf{1} + \eta_t^* \Sigma_t^{-1} \mu_t \tag{46}$$

for $t = 1, 2, \cdots, T$, *where* γ_t^* *is given by (41),* $\xi_t^* = \frac{C_t - B_t \gamma_t^*}{\Delta_t}$ *and* $\eta_t^* = \frac{A_t \gamma_t^* - B_t}{\Delta_t}$.

(iii) *Further, one of sufficient condition for* $w_t^* \ge \mathbf{0}$ *is the followings:* $\kappa^{\lambda}(1)^2 \ge C_t$, $C_t \kappa^{\nu}(p)^2 \ge (C_t + B_t \delta)^2$, $A_t \kappa^{\nu}(p)^2 \ge (A_t \delta + B_t)^2$, $\Sigma_t^{-1} \mathbf{1} \ge \mathbf{0}$ *and* $\Sigma_t^{-1} \mu_t \ge \mathbf{0}$ *for* $t = 1, 2, \ldots, T$.

7 NUMERICAL EXAMPLES

We give a few examples to understand the results in the previous sections.

Example 1. Let a domain $I = \mathbf{R}$ and let f be a *risk neutral utility function* $f(x) = ax + b$ for $x \in \mathbf{R}$ with constants $a (> 0)$ and $b (\in \mathbf{R})$. Then its risk spectrum in Lemma 2 is given by $\lambda(p) = 1$. The corresponding weighted average value-at-risk (4) is reduced to the *average value-at-risk* (3), and we have

$$f^{-1}(E(f(X))) = E(X) = \mathrm{AVaR}_1(X) \tag{47}$$

for $X \in \mathcal{X}$ (Yoshida, 2018).

Example 2. Let a domain $I = \mathbf{R}$ and let a *risk averse exponential utility function*

$$f(x) = \frac{1 - e^{-\tau x}}{\tau} \tag{48}$$

for $x \in \mathbf{R}$ with a positive constant τ. Then $-\frac{f''}{f'} = \tau$ is the degree of decision maker's absolute risk aversity (Arrow, 1971). Fig.1 illustrates utility functions $f(x)$. Let $\mathcal{X}$ be a family of random variables X which have normal distribution functions. Define the cumulative distribution function $G : \mathbf{R} \to (0,1)$ of the standard normal distribution by

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}}\, dz \tag{49}$$

for $x \in \mathbf{R}$, and define an increasing function $\kappa : (0,1) \mapsto \mathbf{R}$ by its inverse function $\kappa(p) = G^{-1}(p)$ for probabilities $p \in (0,1)$. Then we have value-at-risk $\mathrm{VaR}_p(X) = \mu + \kappa(p) \cdot \sigma$ for $X \in \mathcal{X}$ with mean μ and standard deviation σ. Suppose there exists a distribution $\psi : \mathbf{R} \times (0,\infty) \mapsto [0,\infty)$ such that $\psi(\mu,\sigma) = \phi(\mu) \cdot \frac{2^{1-n/2}}{\Gamma(n/2)} \sigma^{n-1} e^{-\frac{\sigma^2}{2}}$ for $(\mu,\sigma) \in \mathbf{R} \times [0,\infty)$, where $\phi(\mu)$ is some probability distribution, $\Gamma(\cdot)$ is a gamma function and $\frac{2^{1-n/2}}{\Gamma(n/2)} \sigma^{n-1} e^{-\frac{\sigma^2}{2}}$ is a chi distribution with degree of freedom n. We take a utility $f(x) = \frac{1-e^{-0.05x}}{0.05}$ with $\tau = 0.05$ in (48), and by Lemma 2 there exists a risk spectrum λ satisfying $f^{-1}(E(f(\cdot))) \approx \mathrm{AVaR}_1^{\lambda}(\cdot)$. Then, by (Yoshida, 2018), the best risk spectrum in Lemma 2 is given by

$$\lambda(p) = e^{-\int_p^1 C(q)\, dq} C(p) \tag{50}$$

for $p \in (0,1]$ with the component function

$$C(p) = \frac{1}{p} \cdot \frac{\displaystyle\int_0^{\infty} \left(1 - \frac{1}{\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\, dq}\right) \sigma^n e^{-\frac{\sigma^2}{2}}\, d\sigma}{\displaystyle\int_0^{\infty} \log\left(\frac{1}{p}\int_0^p e^{\tau\sigma(\kappa(p)-\kappa(q))}\, dq\right) \sigma^n e^{-\frac{\sigma^2}{2}}\, d\sigma}. \tag{51}$$

with $\tau = 0.05$. From (49) and (50), we have $\kappa^{\lambda}(1) = \int_0^1 \kappa(q)\lambda(q)\, dq \Big/ \int_0^1 \lambda(q)\, dq = -0.03$. On the other hand for risk measures ρ we use another utility $g(x) = 1 - e^{-x}$ with $\tau = 1$ in (48). Then by Lemma 1 there exists a risk spectrum ν such that $\rho(\cdot) = -\mathrm{AVaR}_p^{\nu}(\cdot)$. We discuss a case of risk probability 5%, i.e. $p = 0.05$, in the normal distribution, and then similarly we can calculate $\kappa^{\nu}(0.05) = \int_0^{0.05} \kappa(q)\nu(q)\, dq \Big/ \int_0^{0.05} \nu(q)\, dq = -2.29701$. We give fuzzy rewards by fuzzy random variables $\tilde{X}_t^i \in \tilde{\mathcal{X}}_a$ $(i = 1, 2, \ldots, n)$ as follows:

$$\tilde{X}_t^i(\omega)(\cdot) = 1_{\{X_t^i(\omega)\}}(\cdot) + \tilde{a}_t^i(\cdot) \tag{52}$$

for $\omega \in \Omega$, where $X_t^i (\in \mathcal{X})$ and $\tilde{a}_t^i$ is a fuzzy number $\tilde{a}_t^i(x) = \max\{1 - |x|/c_t^i, 0\}$ for $x \in \mathbf{R}$ with a positive number c_t^i, which is a fuzzy factor. Let $n = 4$ be the number of assets. Hence we put the expectations μ_t^i of rewards X_t^i and fuzzy factors c_t^i by Table 1, and we let the covariances σ_t^{ij} of rewards X_t^i by Table 2. We deal with an optimistic and possibility case, i.e. $\theta = 0$ and $w(\alpha) = 1$ for $\alpha \in [0,1]$. Hence we have $A_t = 15.1405 > 0$ and $\Delta_t = 0.003449 > 0$. And we can easily check $\kappa^{\nu}(0.05) < \kappa^{\lambda}(1) < -\sqrt{\Delta_t/A_t} = -0.0150931$. From (37) we also have $\underline{\delta}_t(p) = 0.495737$. Therefore now we take a risk level $\delta = 0.55$ in the feasible range $[0.495737, \infty)$. From Theorem 2, we obtain the maximum risk-sensitive estimation $\varphi_t^* =$

Table 1: Expectations μ_t^i and fuzzy factors c_t^i.

	μ_t^i	c_t^i
$i=1$	0.098	0.008
$i=2$	0.084	0.008
$i=3$	0.091	0.007
$i=4$	0.088	0.006

Table 2: Variance-covariances σ_t^{ij}.

σ_t^{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	0.38	-0.09	-0.07	0.05
$i=2$	-0.09	0.39	-0.08	0.06
$i=3$	-0.07	-0.08	0.38	-0.06
$i=4$	0.05	0.06	-0.06	0.37

0.0879143 at the expected reward $\gamma_t^* = 0.0968356$ for Problem (P2) with an optimal strategy $\pi_t^* = (0.453443, 0.159892, 0.313565, 0.0731002)$. Hence the difference between the real expected reward $\gamma_t^* = 0.0968356$ and the decision maker's maximum risk-sensitive estimation $\varphi_t^* = 0.0879143$ comes from decision maker's risk averse feeling.

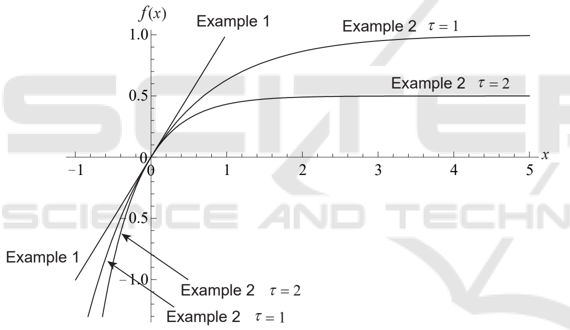

Figure 1: Utility functions $f(x)$.

We can also use a pessimistic and necessity case, i.e. $\theta = 1$ and $w(\alpha) = 1-\alpha$ for $\alpha \in [0,1]$. From Tables 1 and 2, we find the maximum risk-sensitive estimation is in $0.0785810 \le \varphi_t^* \le 0.0874494$, and the expected reward is in $0.0875022 \le \gamma_t^* \le 0.0968356$.

Fig.2 illustrates the maximum risk-sensitive estimation φ_t^* for δ in Theorem 2, and we see the two lines are cut and connected at δ_t^+. Fig.3 illustrates the maximum risk-sensitive estimation φ_t^* and the expected reward γ_t^* for δ. We see φ_t^* is smaller than γ_t^* because γ_t^* implies actual expected rewards and φ_t^* contains decision maker's risk aversity under his utility. Fig.4 also shows the feasible range $\{(p,\delta) \mid \Pi_t(\delta) \neq \emptyset\} = \{(p,\delta) \mid \underline{\delta}_t(p) \le \delta\}$ in Example 2 ($\tau = 1$).

We discuss a dynamic case with an expiration date $T = 20$ and a discount rate $\beta = 0.95$. Then by Theorem 3 we obtain the optimal total weighted average value-at-risk $v_1 = 1.67037$ for Problem (P4) and we can observe the sequence $\{v_t\}$ of sub-total weighted average value-at-risks after time t in Theorem 3.

Table 3: Risk-sensitive estimation φ_t^* and expected reward γ_t^*.

	Pess. & Nec.	Opti. & Poss.
γ_t^*	0.0875022	0.0968356
φ_t^*	0.0785810	0.0879143

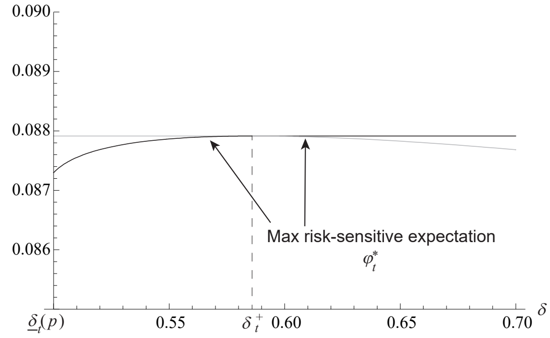

Figure 2: The max. risk-sensitive estimation φ_t^*.

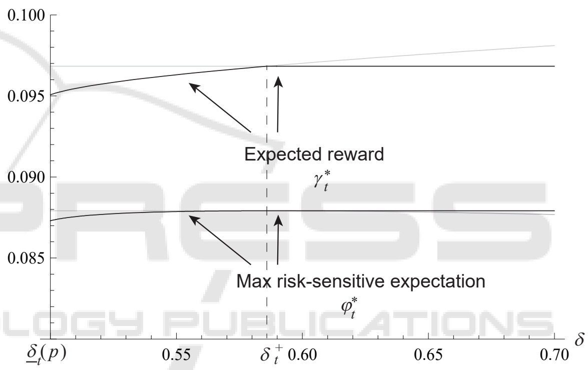

Figure 3: The max. risk-sensitive estimation φ_t^* and the expected reward γ_t^*.

Concluding Remarks. Using Lemma 2, we can incorporate the decision maker's risk averse attitude into coherent risk measures as weighting for average value-at-risks. As we have seen in Examples 1 and 2, risk-sensitive estimations with utility f are approximated by weighted average risks with a spectrum λ in (50) and (51), and the coherent risk measures ρ with fuzzy factors is given by weighted average risks with a spectrum ν. The proposed method brings us reasonable and computable risk-sensitive optimization models under risk constraints, and it is useful for other subjective optimization in management sciences. We can reply immediately risk values $\rho = -\mathrm{AVaR}_p^\lambda$ from (7) when we prepare constants (8), and this approach

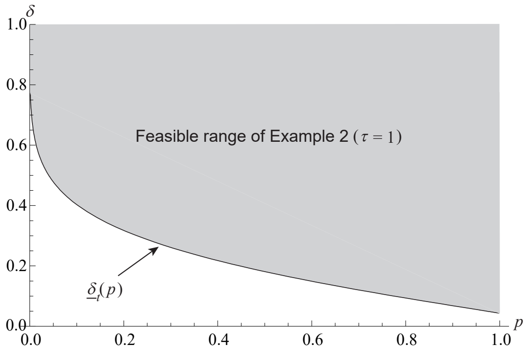

Figure 4: Feasible region for risk levels δ.

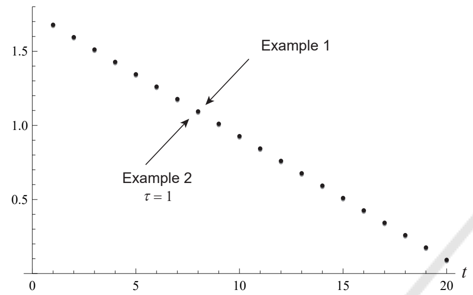

Figure 5: Sequences $\{v_t\}$ for Example 1 and Example 2 ($\tau = 1$) ($p = 0.05$).

will be applicable to timely and quick risk-sensitive decision making together with AI computing, for example, stock trading, auto driving and so on (Yoshida, to appear).

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