Merging Partial Fuzzy Rule-bases

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Abstract: We propose two basic ways of merging various partial fuzzy rule-bases containing knowledge related to the same process or dependency in general. The knowledge that is not at the disposal is considered undefined and encoded using some dummy value. For simplicity, we use only one code for undefined membership value, and we handle the undefined membership values using operations of variable-domain fuzzy set theory, i.e., the theory that allows fuzzy sets to have undefined membership values. Moreover, we study one of the essential properties in fuzzy modeling–a graded property of functionality. We provide estimations for degrees of the functionality of input models and merged models of partial fuzzy rule-bases.

1 INTRODUCTION

In this contribution, we focus on fuzzy models of IF–THEN rules due to their natural ability to handle the gradualness of expert knowledge. Models of various fuzzy IF–THEN rules (Mamdani and Assilian, 1975; Hájek, 1998) are closely related to the domain of definition. For example, we often deal with many rule bases in hierarchical systems such as in (Delgado et al., 2003; Kóczy et al., 2003; Magdalena, 2019; Nolasco et al., 2019), or in ensemble techniques (Scherer, 2012; Štěpnička et al., 2016; Fletcher et al., 2020), which can generally differ in the domains. It means that each particular rule base (a set of fuzzy IF–THEN rules) has its domain, and outside of it is considered to be undefined.

The problem of merging rule bases from different domains (Lughofer, 2011; Casillas et al., 2013; Latkowski and Mikołajczyk, 2004; Peters et al., 2004) arose from the practical need to join rule-bases based on various expert knowledge, given data, or fact base etc. Roughly speaking, it consists in unifying the domains into a common one and extending the membership functions of particular fuzzy sets by filling in an appropriately chosen membership degree such that an output merged rule-base keeps non-conflicting knowledge and preserves properties owned by both input rule-bases.

Since in some cases, we want to carry information about undefined parts of input models; consequently, none of the standard membership degrees would be applicable for filling in. Here, variable-domain fuzzy set theory (VFST) introduced in (Běhounek and Daňková, 2020a) is applicable. It provides essential tools for handling fuzzy sets with undefined membership values.

Note that there are also other approaches to handle undefined membership values, such as in (Štěpnička et al., 2019; d'Allonnes and Lesot, 2017). An advantage of an inference system involving operations of VFST is robustness in the sense of being able to handle some exceptions automatically. Generally, exceptions have various sources, e.g., an integer divide by zero or using too much memory. In our case, exceptions related to undefined values are relevant.

In VFST, various ways of merging rule-bases from two different domains are definable as an alternative to the know approaches. In the following, we will investigate which avoids artifacts and behaves as expected. Moreover, we will present the notion of functionality and study its properties w.r.t. merged rule-bases.

The paper is organized as follows. In Section 2, we recall notions of partial fuzzy sets and relations, two essential extensions of residuated lattice operations, and other necessary notions mostly from (Běhounek and Novák, 2015; Běhounek and Daňková, 2020a). Conjunctive and disjunctive models of partial fuzzy rule-bases are presented in Section 3. Next, in Section 4, we introduce basic merging techniques that use partial fuzzy set operations. All presented notions are together with illustrative ex-

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amples that should help a reader with understanding. The study of the transmission of functionality property by merged models is in Section 5. Finally, the main results and features of the used formalism are summarized in Section 6.

2 BASIC NOTATIONS

Consider a complete residuated lattice

$$L = \langle L, \vee, \wedge, \odot, \Rightarrow, 0, 1 \rangle, \tag{1}$$

and a single dummy element $* \notin L$ representing an error code related to undefined truth degree. We extend the operations of L to $*$ by the following truth tables:

- The *Bochvar-style* operations

$$\begin{array}{c|cc} c_{\mathrm{B}} & \beta & * \\ \hline \alpha & \alpha c \beta & * \\ * & * & * \end{array} \tag{2}$$

  where $c \in \{\wedge, \vee, \odot, \Rightarrow\}$. Here, $*$ behaves as the annihilator and it can be viewed as the representative of a fatal error.

- The *Sobociński-style* operations

$$\begin{array}{c|cc} c_{\mathrm{S}} & \beta & * \\ \hline \alpha & \alpha c \beta & \alpha \\ * & \beta & * \end{array} \tag{3}$$

  $c \in \{\wedge, \vee, \odot\}$ and

$$\begin{array}{c|cc} \Rightarrow_{\mathrm{S}} & \beta & * \\ \hline \alpha & \alpha \Rightarrow \beta & \alpha \Rightarrow 0 \\ * & \beta & * \end{array} \tag{4}$$

  These operations treat $*$ as the neutral element, which means that they ignore $*$ as far as possible. The only suspicious operation is $\Rightarrow_{\mathrm{S}}$, where $\Rightarrow$ is monotone in the second argument and antitone in the fist one. These properties of $\Rightarrow$ lead to inputting 0 instead of $*$ to compute $\alpha \Rightarrow *$ in (4).

Both names of classes of extensions are inspired by $\{0, 1, *\}$-valued connectives in (Ciucci and Dubois, 2013).

Additionally, we introduce two basic bivalent orderings of truth values:

- The following ordering treats $*$ as a bottom element:

$$\begin{array}{c|cc} \leq_* & \beta & * \\ \hline \alpha & \alpha \leq \beta & 0 \\ * & 1 & 1 \end{array} \tag{5}$$

- And dually, we treat $*$ as a top element:

$$\begin{array}{c|cc} \leq^* & \beta & * \\ \hline \alpha & \alpha \leq \beta & 1 \\ * & 0 & 1 \end{array} \tag{6}$$

Let $L^* = L \cup \{*\}$ and $\alpha_i \in L^*$ for each $i \in I$ (where I is an arbitrary index set). Then we define:

- The *Bochvar infimum*

$$\bigwedge_{i \in I}{}_{\mathrm{B}}\, \alpha_i = \begin{cases} \bigwedge_{i \in I} \alpha_i & \text{if } \alpha_i \neq * \text{ for each } i \in I; \\ * & \text{otherwise.} \end{cases}$$

- The *Bochvar supremum*

$$\bigvee_{i \in I}{}_{\mathrm{B}}\, \alpha_i = \begin{cases} \bigvee_{i \in I} \alpha_i & \text{if } \alpha_i \neq * \text{ for each } i \in I; \\ * & \text{otherwise.} \end{cases}$$

- The *Sobociński infimum*

$$\bigwedge_{i \in I}{}_{\mathrm{S}}\, \alpha_i = \begin{cases} \bigwedge_{\substack{i \in I \\ \alpha_i \neq *}} \alpha_i & \text{if } \alpha_i \neq * \text{ for some } i \in I; \\ * & \text{otherwise.} \end{cases}$$

- The *Sobociński supremum*

$$\bigvee_{i \in I}{}_{\mathrm{S}}\, \alpha_i = \begin{cases} \bigvee_{\substack{i \in I \\ \alpha_i \neq *}} \alpha_i & \text{if } \alpha_i \neq * \text{ for some } i \in I; \\ * & \text{otherwise.} \end{cases}$$

2.1 Partial Fuzzy Sets and Relations

Ordinary fuzzy sets are identified with their L-valued membership functions, while the *partial fuzzy sets* introduced by (Běhounek and Novák, 2015) allow membership functions to be undefined. The membership functions of partial fuzzy sets are total functions from some universe of discourse to $L \cup \{*\}$. The additional dummy value $* \notin L$ allows us to capture the domain of partial fuzzy set.

Definition 2.1. Let U be a universe of discourse, $X_A \subseteq U$, $* \notin L$, $L^* = L \cup \{*\}$ and $\mu_A \colon X_A \to L$ be a membership function from X_A to a suitable structure L of membership degrees.

1. A *partial fuzzy set* is a pair $A = (X_A, \mu_A)$. This fact is denoted by $A \underset{\sim}{\subset} X_A$, X_A is called the crisp domain of A and we often write $\mathrm{Dom}(A)$ instead of X_A.
2. A partial fuzzy set $A = (X_A, \mu_A)$ in a universe $U \supseteq X_A$ is *represented* by a L^*-valued membership function A on U, defined for each $x \in U$ as:

$$A(x) = \begin{cases} \mu_A(x) & \text{if } x \in X_A; \\ * & \text{if } x \in U \setminus X_A. \end{cases} \tag{7}$$

Fuzzy relations between two crisp sets A and B are fuzzy sets on $A \times B$, while the domain of a partial fuzzy relation is intended to be a subset of $A \times B$. It is defined as follows:

Definition 2.2. Let $A, B \neq \emptyset$.

We say that a partial fuzzy set $R = (X_R, \mu_R)$ is a *partial fuzzy relation between A and B* if $X_R \subseteq A \times B$ and $\mu_R \colon X_R \to L$.

If $A = B$, we speak of partial fuzzy relations *on* A.

Consider $A, B \subset U$. Due to Definition 2.1, L^*-valued membership function of a partial fuzzy relation R between A and B is defined for each $(x, y) \in U \times U$ as:

$$R(x,y) = \begin{cases} \mu_R(x) & \text{if } x \in X_R; \\ * & \text{if } x \in (U \times U) \setminus X_R. \end{cases} \tag{8}$$

Example 2.3. Let $U = \langle 0, 1 \rangle$, $X_A = X_B \subseteq U$, $A = (X_A, \mu_A)$, $B = (X_B, \mu_B)$ due to Figure 1. Then we define partial fuzzy relations $(A \times_{\mathrm{B}} B)$, $(A \times_{\mathrm{S}} B)$, $(A \rtimes_{\mathrm{B}} B)$ and $(A \rtimes_{\mathrm{S}} B)$ with the following representations on U:

$$(A \times_{\mathrm{B}} B)(x,y) =_{\mathrm{df}} A(x) \odot_{\mathrm{B}} B(y),\ x, y \in U, \tag{9}$$
$$(A \times_{\mathrm{S}} B)(x,y) =_{\mathrm{df}} A(x) \odot_{\mathrm{S}} B(y),\ x, y \in U, \tag{10}$$
$$(A \rtimes_{\mathrm{B}} B)(x,y) =_{\mathrm{df}} A(x) \Rightarrow_{\mathrm{B}} B(y),\ x, y \in U, \tag{11}$$
$$(A \rtimes_{\mathrm{S}} B)(x,y) =_{\mathrm{df}} A(x) \Rightarrow_{\mathrm{S}} B(y),\ x, y \in U, \tag{12}$$

depicted on Figures 2–5, respectively. Observe that the domain of $A \times_{\mathrm{S}} B$ is $(X_A \times U) \cup (U \times X_B)$.

Partial fuzzy relations $A \times_{\mathrm{B}} B$ and $A \times_{\mathrm{S}} B$ can be viewed as two variants of Cartesian product of partial fuzzy sets A and B.

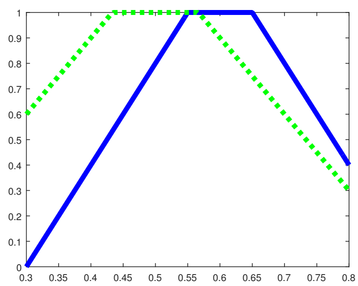

Figure 1: Fuzzy sets A (solid line) and B (dotted line) on $X_A = X_B = \langle 0.3, 0.8 \rangle$.

2.2 Partial Fuzzy Relational Operations

We have the following two main options to define relational operations based on a various treatments of undefined values:

- *Bochvar intersection*:

$$(R \sqcap_{\mathrm{B}} S)(x,y) =_{\mathrm{df}} R(x,y) \wedge_{\mathrm{B}} S(x,y)$$

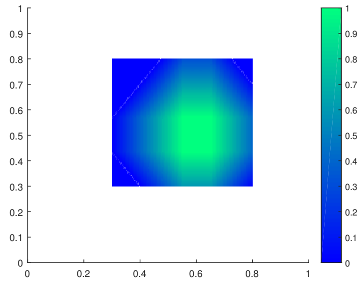

Figure 2: $A \times_{\mathrm{B}} B$ from Example 2.3.

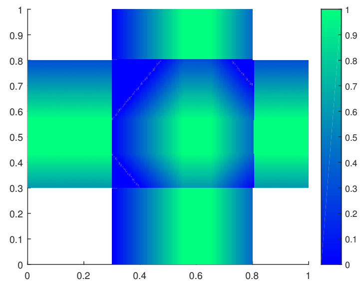

Figure 3: $A \times_{\mathrm{S}} B$ from Example 2.3.

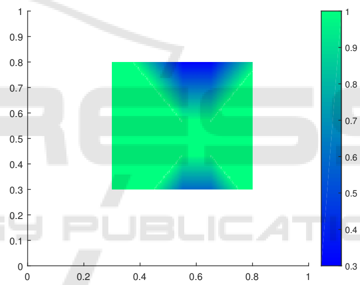

Figure 4: $A \rtimes_{\mathrm{B}} B$ from Example 2.3.

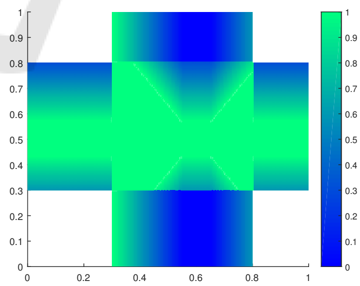

Figure 5: $A \rtimes_{\mathrm{S}} B$ from Example 2.3.

- *Sobociński intersection*:

$$(R \sqcap_{\mathrm{S}} S)(x,y) =_{\mathrm{df}} R(x,y) \wedge_{\mathrm{S}} S(x,y)$$

- *Bochvar strong-intersection*:

$$(R \cap_{\mathrm{B}} S)(x,y) =_{\mathrm{df}} R(x,y) \odot_{\mathrm{B}} S(x,y)$$

- *Sobociński strong-intersection*:

$$(R \cap_{\mathrm{S}} S)(x,y) =_{\mathrm{df}} R(x,y) \odot_{\mathrm{S}} S(x,y)$$

- *Bochvar union*:

$$(R \sqcup_{\mathrm{B}} S)(x,y) =_{\mathrm{df}} R(x,y) \vee_{\mathrm{B}} S(x,y)$$

- *Sobociński union*:

$$(R \sqcup_{\mathrm{S}} S)(x,y) =_{\mathrm{df}} R(x,y) \vee_{\mathrm{S}} S(x,y)$$

3 MODELS OF PARTIAL FUZZY IF-THEN RULES

Let I be some index set. A collection $\mathcal{R}$ of rules

"IF $x \in A_i$ THEN $y \in B_i$",

where each $A_i, B_i, i \in I$ is modeled by a partial fuzzy set, is called *partial fuzzy rules* or *partial fuzzy rule-base*. Fuzzy rules have two main models; we call them disjunctive and conjunctive models of IF-THEN rules due to a type of connective used to join rules. We can write formally

$$D^{\mathcal{R}} =_{\mathrm{df}} \bigsqcup_{i \in I} (A_i \times B_i), \tag{13}$$

$$C^{\mathcal{R}} =_{\mathrm{df}} \bigsqcap_{i \in I} (A_i \rtimes B_i), \tag{14}$$

where $A_i, B_i \mathrel{\underset{\sim}{\subsetneq}} U$,

$$(A \times B)(x,y) =_{\mathrm{df}} A(x) \odot B(y),$$
$$(A \rtimes B)(x,y) =_{\mathrm{df}} A(x) \Rightarrow B(y), \text{ for all } x,y \in U,$$

$\bigsqcup$ and $\bigsqcap$ are the usual union and intersection of fuzzy sets based on lattice operations $\vee$ and $\wedge$, respectively.

In case of partial fuzzy rules, disjunctive and conjunctive models are defined as

$$D_{\mathrm{SB}}^{\mathcal{R}} =_{\mathrm{df}} \bigsqcup_{i \in I}{}_{\mathrm{S}} (A_i \times_{\mathrm{B}} B_i), \tag{15}$$

$$C_{\mathrm{SB}}^{\mathcal{R}} =_{\mathrm{df}} \bigsqcap_{i \in I}{}_{\mathrm{S}} (A_i \rtimes_{\mathrm{B}} B_i), \tag{16}$$

respectively. These models are partial fuzzy sets with the same domain $\bigcup_{i \in I}(\mathrm{Dom}(A_i) \times \mathrm{Dom}(B_i))$. In the sequel, we will simply write $D, C, D_{\mathrm{SB}}, C_{\mathrm{SB}}$ instead of $D^{\mathcal{R}}, C^{\mathcal{R}}, D_{\mathrm{SB}}^{\mathcal{R}}, C_{\mathrm{SB}}^{\mathcal{R}}$, respectively, provided that $\mathcal{R}$ is clear from the context.

Notice that the introduced models use Bochvar and Sobociński extensions that are very similar to extended operations that form the so-called Dragonfly algebra (Štěpnička et al., 2019). They differ only at values 0 and 1 because in Dragonfly algebra $*$ is supposed to represent some degree lying between these two values.

Example 3.1. Let $U = \langle 0,1 \rangle$, A_1, A_2, B_1, B_2 be partial fuzzy sets with membership functions as on Figure 6 and

$$\mathrm{Dom}(A_1) = \mathrm{Dom}(B_1) = [0.2, 0.7] \tag{17}$$
$$\mathrm{Dom}(A_2) = \mathrm{Dom}(B_2) = [0.4, 0.9] \tag{18}$$

Then, the partial fuzzy relations D_{SB} and C_{SB} are depicted on Figures 7 and 8, respectively.

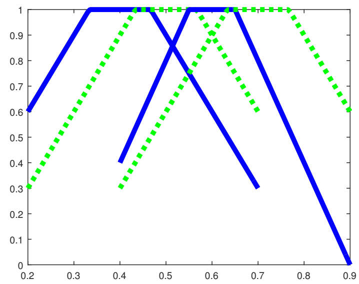

Figure 6: Partial fuzzy sets A_1, A_2 (solid line) and B_1, B_2 (dotted line).

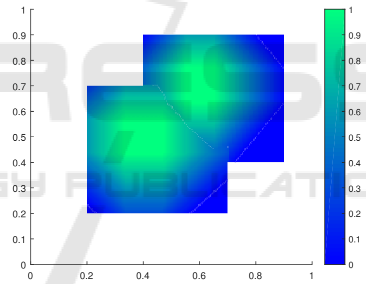

Figure 7: D_{SB} model of partial fuzzy rules.

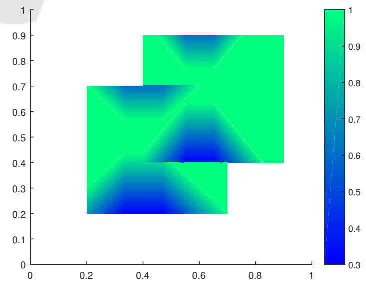

Figure 8: C_{SB} model of partial fuzzy rules.

In the following examples, we will overview other possible combinations of extensions of operations than in (15) and (16).

Example 3.2. Consider the same setting as in the previous example and define the following partial fuzzy relations:

$$D_{\mathrm{BS}} =_{\mathrm{df}} \bigsqcup_{i\in I}{}_{\mathrm{B}} (A_i \times_{\mathrm{S}} B_i) \qquad (19)$$

$$C_{\mathrm{BS}} =_{\mathrm{df}} \bigsqcap_{i\in I}{}_{\mathrm{B}} (A_i \rtimes_{\mathrm{S}} B_i) \qquad (20)$$

In D_{BS} and C_{BS}, Sobociński and Bochvar extensions of operations are reversed comparing with D_{SB} and C_{SB}. Their domain is

$$\bigcap_{i=1}^{2} (\mathrm{Dom}(A_i) \times U) \cup (U \times \mathrm{Dom}(B_i)) \qquad (21)$$

D_{SB} is drawn on Figures 7 for partial fuzzy sets from Figure 6. As seen from this figure, D_{BS} and C_{BS} cannot represent a functional dependence in appropriate way. Outside $\bigcup_{i=1}^{2}(\mathrm{Dom}(A_i) \times \mathrm{Dom}(B_i))$, Sobociński-operations add relational dependencies by means of partial fuzzy sets $A_i, B_i, i = 1,2,$ which changes significantly meaning and understandability. Consequently, we do not include them in models of partial fuzzy rules. Observe that use of only Sobociński-extension of operations leads to the same problem as described above.

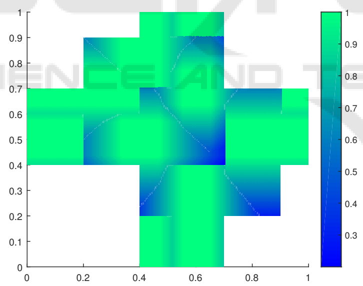

Figure 9: D_{BS} from Example.

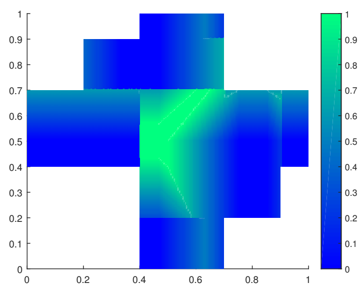

Figure 10: C_{BS} from Example.

Example 3.3. Consider the same setting as in Example 3.1 and define the following partial fuzzy relations:

$$D_{\mathrm{BB}} =_{\mathrm{df}} \bigsqcup_{i\in I}{}_{\mathrm{B}} (A_i \times_{\mathrm{B}} B_i) \qquad (22)$$

$$C_{\mathrm{BB}} =_{\mathrm{df}} \bigsqcap_{i\in I}{}_{\mathrm{B}} (A_i \rtimes_{\mathrm{B}} B_i) \qquad (23)$$

In D_{BB} and C_{BB}, only Bochvar extensions of operations are used, which leads to the following domain

$$\bigcap_{i=1}^{2} (\mathrm{Dom}(A_i) \times \mathrm{Dom}(B_i)) \qquad (24)$$

Hence, these partial fuzzy relations can be viewed as minimalist models of partial fuzzy rules, where information outside of a common domain is refused. Such information can be viewed as unreliable and therefore it do not propagate into the models.

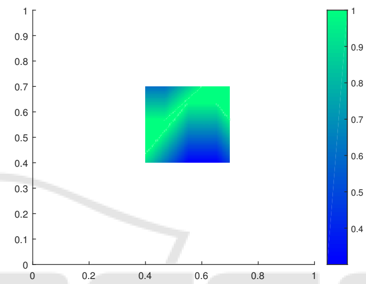

Figure 11: C_{BB} model of partial fuzzy rules.

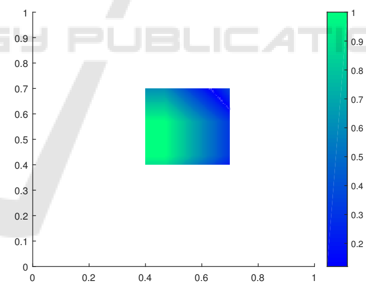

Figure 12: D_{BB} model of partial fuzzy rules.

Due to deficiencies explained in the above examples, we focus only on the models given by (15) and (16).

We define a sup-T composition of partial fuzzy relations using Bochvar and Sobociński operations as follows:

- *Sobociński–Bochvar sup-T composition*

$$(R \circ_{\mathrm{SB}} S)(x,y) =_{\mathrm{df}} \bigvee_{z\in U}{}_{\mathrm{S}} \big(R(x,z) \odot_{\mathrm{B}} S(z,y)\big),$$

Notice that $\mathrm{Dom}(R \circ_{\mathrm{SB}} S) = \mathrm{Dom}(R) \circ \mathrm{Dom}(S)$ and a sup-T composition of partial fuzzy relations is as usual on the composition of the domains and otherwise, it remains undefined.

4 MERGING MODELS OF PARTIAL FUZZY RULE-BASES

In this section, we will deal with disjunctive and conjunctive models of partial fuzzy rule-bases given by (15) and (16), respectively. Recall that these models are partial fuzzy relations. Therefore, we can apply partial fuzzy operations for merging several partial fuzzy rule-bases describing the same dependency into the single one. This process is visualized on Figure 13. The used operation with its corresponding extension determines the merging uniquely.

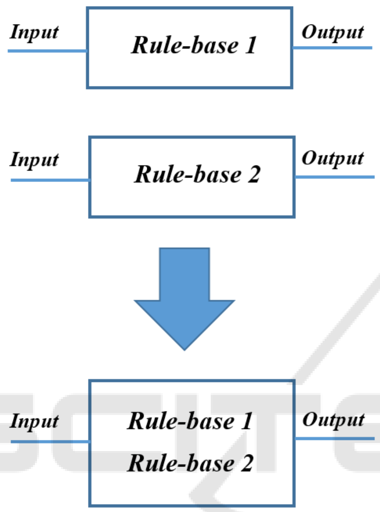

Figure 13: Merging of two rule-bases describing the same dependency.

Sobociński extensions of operations ignores undefined inputs represented by $*$. Hence, using these operations we unite various information from different sources. We apply this fact and define a family of unifying merging operations.

Definition 4.1. Let $\mathcal{R},\mathcal{S}$ be partial fuzzy rule-bases and R,S be their models, respectively. Moreover, let $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$ and $\circ \in \{\sqcap,\sqcup,\cap,\cup\}$.

Then, we say that a *unifying merging* of $\mathcal{R}$ and $\mathcal{S}$ using operation $\circ$ is $\mathcal{M}$ with a model

$$M =_{\mathrm{df}} R \circ_{\mathrm{S}} S. \tag{25}$$

Analogously, we employ Bochvar extensions of operations for incorporating information from various sources, which is assumed to be unreliable outside a domain of common knowledge.

Definition 4.2. Under the assumptions of Definition 4.1, we say that a *strict merging* of $\mathcal{R}$ and $\mathcal{S}$ using operation $\circ$ is $\mathcal{M}$ with a model

$$M =_{\mathrm{df}} R \circ_{\mathrm{B}} S. \tag{26}$$

An extension to n models is straightforward.

Example 4.3. Assume two models $R = (X_R,\mu_R)$ and $S = (X_S,\mu_S)$ of some partial fuzzy rule-bases $\mathcal{R}$ and $\mathcal{S}$, respectively. For a simplicity, assume $\mathcal{R}$ consists of 3 rules, where $\mathrm{Dom}(A_i) = [0.3,0.8]$ and $\mathrm{Dom}(B_i) = [0.3,0.9]$ for all $i \in I = \{1,2,3\}$, and moreover, $\mathcal{S}$ consists of 3 rules, where $\mathrm{Dom}(A'_j) = [0,0.6]$ and $\mathrm{Dom}(B'_j) = [0.2,0.95]$ for all $j \in J = \{1,2,3,4\}$. Their disjunctive models are depicted on Figures 14 and 15.

Figure 16 shows a model of unifying merging of $\mathcal{R}$ and $\mathcal{S}$ using $\sqcap$, i.e., $R \sqcap_{\mathrm{S}} S$. Similarly, Figure 17 depicts $R \sqcup_{\mathrm{S}} S$. A model of strict merging of $\mathcal{R}$ and $\mathcal{S}$ using $\sqcap$ ($\sqcup$) is the same as $R \sqcap_{\mathrm{S}} S$ ($R \sqcup_{\mathrm{S}} S$) on the intersection of domains $X_R \cap X_S$, and otherwise it remains undefined. Hence, graphs of $R \sqcap_{\mathrm{B}} S$ and $R \sqcup_{\mathrm{B}} S$ can be easily derived from Figure 16 and Figure 17, respectively.

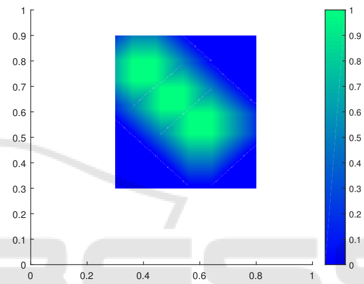

Figure 14: R from Example 4.3.

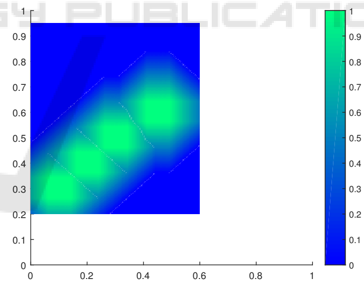

Figure 15: S from Example 4.3.

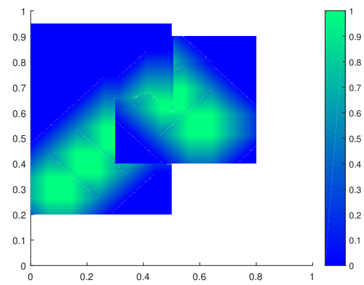

Figure 16: $R \sqcap_{\mathrm{S}} S$.

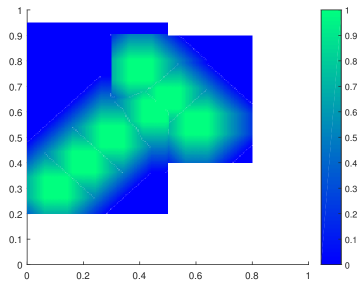

Figure 17: $R \sqcup_{\mathrm{S}} S$.

5 PRESERVATION OF GRADED FUNCTIONALITY BY MERGED MODELS

In this section, we will focus on a special kind of rule-bases, i.e., the one that describes a functional dependency. A functional dependency of a rule-base means functionality property of the related model (fuzzy relation). Provided that we work with partial fuzzy sets, we have to introduce functionality into this framework.

Definition 5.1. Let $U \neq \emptyset$ be a universe of discourse, $X, Y \subset U$, and R be a partial fuzzy relation between X and Y. The *functionality* property of R w.r.t. $\approx_1, \approx_2$ is defined as

$$\mathrm{Func}_{\approx_1,\approx_2}(R) \equiv_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{xx'yy' \in U} [(x \approx_1 x' \odot_{\mathrm{B}} R(x,y) \odot_{\mathrm{B}} R(x',y')) \Rightarrow_{\mathrm{B}} y \approx_2 y'] \quad (27)$$

We say that R is functional to degree d if $d = \mathrm{Func}_{\approx_1,\approx_2}(R)$.

This definition stems from graded notion of functionality for fuzzy sets by (Demirci, 2001; Daňková, 2018). $\mathrm{Func}_{\approx_1,\approx_2}(R)$ is designed in agreement with our intuitive expectations, i.e., its degree is computed for all elements from the respective domains of partial fuzzy relations $\approx_1, \approx_2$, and R.

Provided that we deal with a partial fuzzy rule-base R, it is necessary to construct partial fuzzy relations $\approx_1$ and $\approx_2$ from partial fuzzy sets $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$, respectively, using Valverde's representation theorem (Valverde, 1985) modified for partial fuzzy sets.

Theorem 5.2. *Let I be some index set, $\{A_i\}_{i\in I}$ be a family of partial fuzzy sets, and $\{c_i\}_{i\in I}$ be a family of elements such that $c_i \in X_{A_i}$ and $A_i(c_i) = 1$ for each $i \in I$, U be a common universe set such that $X_{A_i} \subseteq U$ for all $i \in I$. Moreover, let*

$$x \approx y =_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{i\in I} (A_i(x) \Leftrightarrow_{\mathrm{B}} A_i(y)). \quad (28)$$

Then the following two statements are equivalent.

(1) A binary partial fuzzy relation $\approx$ on $\bigcup_{i\in I} X_{A_i}$ is such that

$$A_i(x) = x \approx c_i, \ \text{for all } x \in U. \quad (29)$$

(2) For all $i, j \in I$

$$\bigvee\nolimits_{\mathrm{S}}\limits_{x\in U} (A_i(x) \odot_{\mathrm{B}} A_j(x)) \leq \bigwedge\nolimits_{\mathrm{S}}\limits_{y\in U} (A_i(y) \Leftrightarrow_{\mathrm{B}} A_j(y)) \quad (30)$$

where $\leq \in \{\leq_*, \leq^*\}$.

Proof. We provide only a sketch of the proof. (1) follows from (2) by replacing undefined parts of $\{A_i\}_{i\in I}$ by filling in the membership value 0. Then $\{A_i\}_{i\in I}$ become total on U and we can apply Valverde's representation theorem for fuzzy sets. The resulting similarity (reflexive, symmetric and transitive) relation $\approx'$ is total on U. Let us define

$$\mathrm{Dom}^*(A)(x) =_{\mathrm{df}} \begin{cases} 1 & \text{if } x \in X_A; \\ * & \text{if } x \in U \setminus X_A. \end{cases} \quad (31)$$

Then $A_i(x) = (x \approx c_i) = (x \approx' c_i) \odot_{\mathrm{B}} \mathrm{Dom}^*(A_i)(x)$ for all $x \in U$, where $*$ is either top or bottom element w.r.t. $\leq$.

The reverse implication follows from the fact that $\approx'$ is similarity relation for which (30) is valid. For the partial fuzzy sets $\{A_i\}_{i\in I}$, the expressions on both sides of (30) are computed over the same sets $X_{A_i} \cap X_{A_j} \subseteq U$ for all $i, j \in I$. Hence, (30) is valid for partial fuzzy sets $\{A_i\}_{i\in I}$ defined by (28). □

In the sequel, we will study translation of functionality property of merged partial fuzzy rule-bases. Let us consider two partial fuzzy rule-bases $\mathcal{R}$ and $\mathcal{S}$ with partial fuzzy sets $\{A_i, B_i\}_{i\in I}$ and $\{C_j, D_j\}_{j\in J}$, respectively. Their models $D^{\mathcal{R}}, C^{\mathcal{R}}, D^{\mathcal{S}}, C^{\mathcal{S}}$ are given by (15) and (16). Moreover, let us define

$$x \approx_1 y =_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{i\in I} (A_i(x) \Leftrightarrow_{\mathrm{B}} A_i(y)) \quad (32)$$

$$x \approx_2 y =_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{i\in I} (B_i(x) \Leftrightarrow_{\mathrm{B}} B_i(y)) \quad (33)$$

$$x \sim_1 y =_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{i\in J} (C_i(x) \Leftrightarrow_{\mathrm{B}} C_i(y)) \quad (34)$$

$$x \sim_2 y =_{\mathrm{df}} \bigwedge\nolimits_{\mathrm{S}}\limits_{i\in J} (D_i(x) \Leftrightarrow_{\mathrm{B}} D_i(y)) \quad (35)$$

for all $x, y \in U$, $k = 1, 2$, where $U \supseteq X_{A_i}, X_{B_i}, X_{C_j}, X_{D_j}$, $i \in I$, $j \in J$.

Let F be one of the models of $\mathcal{R}$ and G be one of the models of $\mathcal{S}$. Then, we can prove the following boundaries for graded functionality in the case of strict merging of $\mathcal{R}$ and S using $\sqcap$ and also $\cap$:

$$\mathrm{Func}_{\approx_1,\approx_2}(F) \wedge_{\mathrm{B}} \mathrm{Func}_{\sim_1,\sim_2}(G) \leq^* \mathrm{Func}_{\simeq_1,\simeq_2}(F \sqcap_{\mathrm{B}} G), \quad (36)$$

where $\simeq_k =_{\text{df}} (\approx_k) \sqcap_{\text{B}} (\sim_k)$, $k = 1,2$.

$$\text{Func}_{\approx_1,\approx_2}(F) \odot_{\text{B}} \text{Func}_{\sim_1,\sim_2}(G) \leq^* \text{Func}_{\simeq_1,\simeq_2}(F \cap_{\text{B}} G), \quad (37)$$

where $\simeq_k =_{\text{df}} (\approx_k) \cap_{\text{B}} (\sim_k)$, $k = 1,2$.

The union operator is not applicable in the case of functionality property provided that the domains of partial fuzzy sets in the models are arbitrary. Easily, we can find two functional fuzzy relations such that their union is not functional. It can be proved for both orderings $\leq \in \{\leq_*, \leq^*\}$ that if $\bigcup_{i\in I} X_{A_i} \cap \bigcup_{j\in J} X_{C_j} = \emptyset$ then

$$\text{Func}_{\approx_1,\approx_2}(F) \wedge_{\text{B}} \text{Func}_{\sim_1,\sim_2}(G) \leq \text{Func}_{\simeq_1,\simeq_2}(F \sqcup_{\text{B}} G), \quad (38)$$

is trivially valid, because $(F \sqcup_{\text{B}} G)(x,y) = *$ for all $x,y \in U$. However, a unifying merging via $\sqcup$ operator leads to a non-trivial estimation based on the same requirement as above, i.e., if $\bigcup_{i\in I} X_{A_i} \cap \bigcup_{j\in J} X_{C_j} = \emptyset$ then

$$\text{Func}_{\approx_1,\approx_2}(F) \wedge_{\text{S}} \text{Func}_{\sim_1,\sim_2}(G) \leq \text{Func}_{\simeq_1,\simeq_2}(F \sqcup_{\text{S}} G). \quad (39)$$

Recall that the domain of Sobociński operations $\cap_{\text{S}}, \sqcap_{\text{S}}, \sqcup_{\text{S}}$ is the union of domains of the input partial fuzzy sets. Therefore, if $\bigcup_{i\in I} X_{A_i} \cap \bigcup_{j\in J} X_{C_j} = \emptyset$ then the following inequalities are trivially valid for both orderings:

$$\text{Func}_{\approx_1,\approx_2}(F) \wedge_{\text{S}} \text{Func}_{\sim_1,\sim_2}(G) \leq \text{Func}_{\simeq_1,\simeq_2}(F \sqcap_{\text{S}} G), \quad (40)$$

where $\simeq_k =_{\text{df}} (\approx_k) \sqcap_{\text{S}} (\sim_k)$, $k = 1,2$,

$$\text{Func}_{\approx_1,\approx_2}(F) \odot_{\text{S}} \text{Func}_{\sim_1,\sim_2}(G) \leq \text{Func}_{\simeq_1,\simeq_2}(F \cap_{\text{S}} G) \quad (41)$$

where where $\simeq_k =_{\text{df}} (\approx_k) \cap_{\text{S}} (\sim_k)$, $k = 1,2$, and $\leq \in \{\leq_*, \leq^*\}$.

As can be observed in Figure 16, if $\bigcup_{i\in I} X_{A_i} \cap \bigcup_{j\in J} X_{C_j} \neq \emptyset$ then it can happen that the output of unifying merging is not functional anymore. Though functionality comes with degrees, the functionality of merged rule-basis can be of zero. There are several ways to overcome this deficiency: we can specify a particular rule base's reliability and then incorporate it into the merging process; or erasing all knowledge lowering the degree of functionality using, e.g., a newly designed merging operator.

Observe that in Definitions 4.1 and 4.2, we used only partial fuzzy set operations. Consequently, it is suitable for merging rule-bases describing the same dependency between sets X and Y. Now, let us consider that our rule-bases describe dependencies between sets X and Y in the case of $\mathcal{R}$, and between sets Y and Z in the case of $\mathcal{S}$. Such rule-bases can be joined as "serial" using sup-T composition of partial fuzzy relations for which we can prove the following estimation:

$$\text{Func}_{\approx_1,\approx_2}(F) \odot_{\text{B}} \text{Func}_{\sim_1,\sim_2}(G) \leq \text{Func}_{\simeq'_1,\simeq'_2}(F \circ_{\text{SB}} G), \quad (42)$$

where $\leq \in \{\leq_*, \leq^*\}$, and $\simeq'_k =_{\text{df}} (\approx_k) \circ_{\text{SB}} (\sim_k)$ for $k = 1,2$. For simplicity of exposition, merging using $\circ$ was not included in formal definitions of merging methods because it does not have strict either unifying character. It can be viewed as a reasonable merging operator. Moreover, we have proposed only one combination of Bochvar and Sobociński extensions for sup-T composition. It is not in the scope of this paper to investigate other ways of extensions. For properties of $\circ_{\text{SB}}$, we refer to (Běhounek and Daňková, 2020b), and also consult (Štěpnička and Cao, 2018) for other extensions of $\circ$ to partial fuzzy relations.

6 CONCLUSIONS

We have proposed two basic ways of merging rule-bases describing the same dependency. Moreover, we have studied a graded property of functionality, which is one of the essential properties in fuzzy modeling. We extended the so-called Valverde's representation theorem for partial fuzzy sets to build a bridge between partial fuzzy rule-bases and functional partial fuzzy relations. Using this theorem, we were able to move from partial fuzzy rule-bases to functional partial fuzzy relations and then to provide estimations for degrees of the functionality of input models and merged models. Proofs of these estimations are left for a full paper.

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